Models in Finance - Part 4

Master in Actuarial Science

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One-dimensional Itô's formula or Itô's lemma

- Itô's formula or Itô's lemma is a stochastic version of the chain rule.
- Suppose we have a function of a function $f(b_t)$ and we consider f is a C^2 class function. We want to find $\frac{d}{dt} f(b_t)$.
  Then by Taylor's theorem (2nd order expansion):

$$\delta f(b_t) = f'(b_t)\,\delta b_t + \frac{1}{2} f''(b_t)\,(\delta b_t)^2 + \cdots$$

  Dividing by δt and letting $\delta t \to 0$, we obtain the classical chain rule:

$$\frac{d}{dt} f(b_t) = f'(b_t)\,\frac{db_t}{dt} + \frac{1}{2} f''(b_t)\,\frac{db_t}{dt} \lim_{\delta t \to 0} (\delta b_t) = f'(b_t)\,\frac{db_t}{dt}$$

  or

$$df(b_t) = f'(b_t)\,db_t.$$

One-dimensional Itô's formula or Itô's lemma

- What if we replace b_t (deterministic) by the sBm B_t?
  Then, the 2nd order term $\frac{1}{2} f''(B_t)(\delta B_t)^2$ cannot be ignored because $(\delta B_t)^2 \approx (dB_t)^2 \approx dt$ is not of the order $(dt)^2$
  , that is (Itô formula):
$$df(B_t) = f'(B_t)\, dB_t + \frac{1}{2} f''(B_t)\, dt. \tag{1}$$
- Example: Compute the stochastic differential of B_t^2 and represent this process using a stochastic integral.
- We have $B_t^2 = f(B_t)$ with $f(x) = x^2$.
  Therefore, by (1)
$$\begin{aligned} d\left(B_t^2\right) &= 2B_t dB_t + \frac{1}{2} 2 (dB_t)^2 \\ &= 2B_t dB_t + dt. \end{aligned}$$
  (Taylor expansion of B_t^2 as a function of B_t and assuming that $(dB_t)^2 = dt$).

Itô process

- If f is a C^2 function then

$$\begin{aligned} f(B_t) &= \text{stochastic integral+process with differentiable paths} \\ &= \text{Itô process} \end{aligned}$$

- An adapted and continuous process $X = \{X_t, 0 \leq t \leq T\}$ is called an Itô process if it satisfies the decomposition:

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds, \tag{2}$$

  where u and v are adapted and measurable stochastic processes such that the integrals are well defined.
- In differential form, X is an Itô process if

$$dX_t = u_t dB_t + v_t dt, \tag{3}$$

One-dimensional Itô's formula or Itô's lemma

Theorem

(One-dimensional Itô's formula or Itô's lemma): Let $X = \{X_t, 0 \leq t \leq T\}$ *a Itô process of type (2). Let* $f(t,x)$ *be a* $C^{1,2}$ *function. Then* $Y_t = f(t, X_t)$ *is an Itô process and we have:*

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, u_s dB_s + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, v_s ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)\, u_s^2 ds.$$

- In the differential form, the Itô formula is:

$$df(t,X_t)=\frac{\partial f}{\partial t}(t,X_t)\,dt+\frac{\partial f}{\partial x}(t,X_t)\,dX_t$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\,(dX_t)^2\,.$$

where $(dX_t)^2$ can be computed using (3) and the table of products

$\times$	dB_t	dt
dB_t	dt	0
dt	0	0

- Itô's formula for $f(t,x)$ and $X_t = B_t$, or $Y_t = f(t, B_t)$.

$$f(t, B_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)\, ds.$$

$$df(t, B_t) = \frac{\partial f}{\partial t}(t, B_t)\, dt + \frac{\partial f}{\partial x}(t, B_t)\, dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)\, dt.$$

- Itô's formula for $f(x)$ and $X_t = B_t$, or $Y_t = f(B_t)$.

$$df(B_t) = \frac{\partial f}{\partial x}(B_t)\, dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(B_t)\, dt.$$

Multidimensional Itô's formula or Itô's lemma

- Assume that $B_t := \left(B_t^1, B_t^2, \ldots, B_t^m\right)$ is an m-dimensional standard Brownian motion, that is, components B_t^k, $k = 1, \ldots, m$ are one-dimensional independent sBm.
- Consider a Itô process of dimension n, defined by

$$\begin{aligned}
X_t^1 &= X_0^1 + \int_0^t u_s^{11} dB_s^1 + \cdots + \int_0^t u_s^{1m} dB_s^m + \int_0^t v_s^1 ds, \\
X_t^2 &= X_0^2 + \int_0^t u_s^{21} dB_s^1 + \cdots + \int_0^t u_s^{2m} dB_s^m + \int_0^t v_s^2 ds, \\
&\vdots \\
X_t^n &= X_0^n + \int_0^t u_s^{n1} dB_s^1 + \cdots + \int_0^t u_s^{nm} dB_s^m + \int_0^t v_s^n ds.
\end{aligned}$$

Multidimensional Itô's formula

- In differential form:

$$dX_t^i = \sum_{j=1}^{m} u_t^{ij} dB_t^j + v_t^i dt,$$

with $i = 1, 2, \ldots, n$.

- Or, in compact form:

$$dX_t = u_t dB_t + v_t dt,$$

where v_t is n-dimensional, u_t is a $n \times m$ matrix of processes.

- We assume that the components of u and the components of v are adapted and measurable stochastic processes such that all the integrals are well defined.

Multidimensional Itô's formula

- If $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^p$ is a $C^{1,2}$ function, then $Y_t = f(t, X_t)$ is a Itô process and we have the Itô formula or Itô lemma:

$$
\begin{aligned}
dY_t^k = {} & \frac{\partial f_k}{\partial t}(t, X_t)\, dt + \sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i}(t, X_t)\, dX_t^i \\
& + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f_k}{\partial x_i \partial x_j}(t, X_t)\, dX_t^i dX_t^j.
\end{aligned}
$$

Multidimensional Itô's formula

- The product of the differentials $dX_t^i dX_t^j$ is computed following the product rules:

$$
\begin{aligned}
dB_t^i dB_t^j &= \begin{cases} 0 & \text{se } i \neq j \\ dt & \text{se } i = j \end{cases}, \\
dB_t^i dt &= 0, \\
(dt)^2 &= 0.
\end{aligned}
$$

Multidimensional Itô's formula

- If B_t is a n-dimensional sBm and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function with $Y_t = f(B_t)$ then:

$$f(B_t) = f(B_0) + \sum_{i=1}^{n} \int_0^t \frac{\partial f}{\partial x_i}(B_t)\, dB_s^i + \frac{1}{2}\int_0^t \left(\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(B_t) \right) ds$$

Integration by parts formula

- Integration by parts formula: If X_t^1 and X_t^2 are Itô processes and $Y_t = X_t^1 X_t^2$, then by Itô's formula applied to $f(x) = f(x_1, x_2) = x_1 x_2$, we get

$$d\left(X_t^1 X_t^2\right) = X_t^2 dX_t^1 + X_t^1 dX_t^2 + dX_t^1 dX_t^2.$$

That is:

$$X_t^1 X_t^2 = X_0^1 X_0^2 + \int_0^t X_s^2 dX_s^1 + \int_0^t X_s^1 dX_s^2 + \int_0^t dX_s^1 dX_s^2.$$

Example

- Consider the process

$$Y_t = \left(B_t^1\right)^2 + \left(B_t^2\right)^2 + \cdots + \left(B_t^n\right)^2.$$

Represent this process in terms of Itô stochastic integrals with respect to n-dimensional sBm.

- By n-dimens. Itô formula applied to
$f(x) = f(x_1, x_2, \ldots, x_n) = x_1^2 + \cdots + x_n^2$
, we obtain

$$dY_t = 2B_t^1 dB_t^1 + \cdots + 2B_t^n dB_t^n \\ + ndt.$$

That is:

$$Y_t = 2\int_0^t B_s^1 dB_s^1 + \cdots + 2\int_0^t B_s^n dB_s^n + nt.$$

Exercise

- Exercise: Let $B_t := \left(B_t^1, B_t^2\right)$ be a two dimensional Bm
  Represent the following process as an an Itô process:
$$Y_t = \left(B_t^1 t, \left(B_t^2\right)^2 - B_t^1 B_t^2\right).$$
- By the multidimensional Itô's formula applied to
  $f(t,x) = f(t,x_1,x_2) = \left(x_1 t, x_2^2 - x_1 x_2\right)$, we obtain: (homework)
$$\begin{aligned} dY_t^1 &= B_t^1 dt + t dB_t^1, \\ dY_t^2 &= -B_t^2 dB_t^1 + \left(2B_t^2 - B_t^1\right) dB_t^2 + dt \end{aligned}$$

  that is
$$\begin{aligned} Y_t^1 &= \int_0^t B_s^1 ds + \int_0^t s dB_s^1, \\ Y_t^2 &= -\int_0^t B_s^2 dB_s^1 + \int_0^t \left(2B_s^2 - B_s^1\right) dB_s^2 + t. \end{aligned}$$

- Exercise: Assume that a process X_t satisfies the SDE

$$dX_t = \sigma(X_t)\, dB_t + \mu(X_t)\, dt.$$

Compute the stochastic differential of the process $Y_t = X_t^3$ and represent this process as an Itô process.