Models in Finance - Part 5

Master in Actuarial Science

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Ordinary differential equations

- Deterministic ordinary diff. eqs.:

$$f\left(t, x\left(t\right), x'\left(t\right), x''\left(t\right), \ldots\right) = 0, \qquad 0 \leq t \leq T.$$

- 1st order ordinary diff. eq.:

$$\frac{dx\left(t\right)}{dt} = \mu\left(t, x\left(t\right)\right)$$

or

$$dx\left(t\right) = \mu\left(t, x\left(t\right)\right) dt$$

- Example:

$$\frac{dx\left(t\right)}{dt} = cx\left(t\right)$$

has solution

$$x\left(t\right) = x\left(0\right) e^{ct}.$$

Stochastic Differential Equations

- SDE in differential form

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t, \qquad (1)$$
$$X_0 = X_0$$

- $\mu(t, X_t)$ is the drift coefficient, $\sigma(t, X_t)$ is the diffusion coefficient.
- SDE in integral form

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s. \qquad (2)$$

- To prevent the "explosion" of the solution process (hitting $\pm\infty$ in finite time), a sufficient condition is the linear growth property:

$$|\mu(t, x)| + |\sigma(t, x)| \leq C(1 + |x|), \; x \in \mathbb{R}, \; t \in [0, T]$$

- A sufficient condition to ensure uniqueness of solutions is the Lipschitz property:

$$|\mu(t, x) - \mu(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D|x - y|, \; x, y \in \mathbb{R}.$$

Stochastic Differential Equations

Definition

A solution of SDE (1) or (2) is a stochastic process $\{X_t\}$ which satisfies:

1. $\{X_t\}$ is an adapted process (to Bm) and has continuous sample paths.
2. The integrals in (2) are well defined
3. $\{X_t\}$ satisfies the SDE (1) or (2)

- The solutions of SDE's are called diffusions or "diffusion processes".
- A diffusion is "locally" like Brownian motion with drift, but with a variable drift coefficient $\mu(x)$ and diffusion coefficient $\sigma(x)$.

Solving an SDE by Itô formula

- **Example**: Standard model for risky asset price (SDE):

$$dS_t = \alpha S_t dt + \sigma S_t dB_t \tag{3}$$

or

$$S_t = S_0 + \alpha \int_0^t S_s ds + \sigma \int_0^t S_s dB_s \tag{4}$$

- How to solve this SDE?
- Assume that $S_t = f(t, B_t)$ with $f \in C^{1,2}$.
  By Itô formula:

$$S_t = f(t, B_t) = S_0 + \int_0^t \left(\frac{\partial f}{\partial t}(s, B_s) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, B_s) \right) ds + \tag{5}$$
$$+ \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s.$$

- Comparing (4) with (5) then (uniqueness of representation as an itô process)

$$\frac{\partial f}{\partial s}(s, B_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, B_s) = \alpha f(s, B_s), \qquad (6)$$

$$\frac{\partial f}{\partial x}(s, B_s) = \sigma f(s, B_s). \qquad (7)$$

- Differentiating (7) we get

$$\frac{\partial^2 f}{\partial x^2}(s, x) = \sigma \frac{\partial f}{\partial x}(s, x) = \sigma^2 f(s, x)$$

and replacing in (6) we have

$$\left(\alpha - \frac{1}{2}\sigma^2\right) f(s, x) = \frac{\partial f}{\partial s}(s, x)$$

- Separating the variables: $f(s,x) = g(s)\,h(x)$, we get

$$\frac{\partial f}{\partial s}(s,x) = g'(s)\,h(x)$$

and

$$g'(s) = \left(\alpha - \frac{1}{2}\sigma^2\right) g(s)$$

wich is a linear ODE, with solution:

$$g(s) = g(0)\exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right) s\right]$$

- Using (7), we get $h'(x) = \sigma h(x)$ and

$$f(s,x) = f(0,0)\exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right) s + \sigma x\right].$$

The Geometric Brownian motion

- Conclusion:

$$S_t = f(t, B_t) = S_0 \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right] \tag{8}$$

which is the geometric Brownian motion. Therefore $\frac{S_t}{S_0}$ has lognormal distribution with parameters $\left(\alpha - \frac{1}{2}\sigma^2\right) t$ and $\sigma^2 t$.

- Remark: Note that the solution of the SDE was obtained by solving a deterministic PDE (partial differential equation).
- Moreover

$$E\left[\frac{S_t}{S_0}\right] = e^{\alpha t}, \text{ var}\left[\frac{S_t}{S_0}\right] = e^{2\alpha t}\left(e^{\sigma^2 t} - 1\right).$$

The Geometric Brownian motion

- Let us verify that (8) satisfies SDE (3) or (4).
- Apllying the Itô formula to $S_t = f(t, B_t)$ with

$$f(t,x) = S_0 \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma x\right],$$

we obtain

$$\begin{aligned} S_t &= S_0 + \int_0^t \left[\left(\alpha - \frac{1}{2}\sigma^2\right) S_s + \frac{1}{2}\sigma^2 S_s\right] ds + \int_0^t \sigma S_s dB_s \\ &= S_0 + \alpha \int_0^t S_s ds + \sigma \int_0^t S_s dB_s \end{aligned}$$

- or:

$$dS_t = \alpha S_t dt + \sigma S_t dB_t.$$

Ornstein-Uhlenbeck process (or Langevin equation)

- $$dX_t = \mu X_t dt + \sigma dB_t$$
  or
  $$X_t = X_0 + \mu \int_0^t X_s ds + \sigma \int_0^t dB_s.$$
- Let $Y_t = e^{-\mu t} X_t$ or $Y_t = f(t, X_t)$ with $f(t, x) = e^{-\mu t} x$. By Itô formula,
  $$Y_t = Y_0 + \int_0^t \left(-\mu e^{-\mu s} X_s + \mu e^{-\mu s} X_s + \frac{1}{2}\sigma^2 \times 0 \right) ds + \int_0^t \sigma e^{-\mu s} dB_s.$$
- Therefore,
  $$X_t = e^{\mu t} X_0 + e^{\mu t} \int_0^t \sigma e^{-\mu s} dB_s.$$
- If $X_0 =$cte., this process is called the Ornstein-Uhlenbeck process.

The geometric Brownian motion (again)

- Let

$$dS_t = \alpha S_t dt + \sigma S_t dB_t \tag{9}$$

or

$$S_t = S_0 + \alpha \int_0^t S_s ds + \sigma \int_0^t S_s dB_s. \tag{10}$$

- Assumption

$$S_t = e^{Z_t}.$$

or

$$Z_t = \ln\left(S_t\right).$$

- By the Itô formula, with $f(x) = \ln(x)$, we have

$$\begin{aligned} dZ_t &= \frac{1}{S_t} dS_t + \frac{1}{2}\left(\frac{-1}{S_t^2}\right)\left(dS_t\right)^2 \\ &= \left(\alpha - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t. \end{aligned}$$

The geometric Brownian motion (again)

- That is $Z_t = Z_0 + \left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma B_t$ (is a Brownian motion with drift or arithmetic Brownian motion) and

$$S_t = S_0 \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right].$$

- In general, the solution of the homogeneous linear SDE

$$dX_t = \mu(t) X_t dt + \sigma(t) X_t dB_t$$

is

$$X_t = X_0 \exp\left[\int_0^t \left(\mu(s) - \frac{1}{2}\sigma(s)^2\right) ds + \int_0^t \sigma(s)\, dB_s\right].$$

Ornstein-Uhlenbeck process with mean reversion

$$dX_t = a\left(m - X_t\right)dt + \sigma dB_t,$$
$$X_0 = x.$$

$a, \sigma > 0$ and $m \in \mathbb{R}$.

- Solution of the associated ODE $dx_t = -ax_t dt$ is $x_t = xe^{-at}$.
- Consider the variable change $X_t = Y_t e^{-at}$ or $Y_t = X_t e^{at}$.
- By the Itô foemula applied to $f\left(t, x\right) = xe^{at}$, we have

$$Y_t = x + m\left(e^{at} - 1\right) + \sigma \int_0^t e^{as} dB_s.$$

Ornstein-Uhlenbeck process with mean reversion

- Therefore

$$X_t = m + (x - m)\, e^{-at} + \sigma e^{-at} \int_0^t e^{as} dB_s.$$

- This is a Gaussian process, since the random part is $\int_0^t f(s)\, dB_s$, where f is deterministic, so it is a Gaussian process.
- Mean:

$$E[X_t] = m + (x - m)\, e^{-at}$$

Ornstein-Uhlenbeck process with mean reversion

- Covariance: By Itô isometry

$$\begin{aligned}\operatorname{Cov}[X_t, X_s] &= \sigma^2 e^{-a(t+s)} E\left(\int_0^t e^{ar} dB_r\right)\left(\int_0^s e^{ar} dB_r\right) \\ &= \sigma^2 e^{-a(t+s)} \int_0^{t\wedge s} e^{2ar} dr \\ &= \frac{\sigma^2}{2a}\left(e^{-a|t-s|} - e^{-a(t+s)}\right).\end{aligned}$$

- Note that

$$X_t \sim N\left[m + (x-m) e^{-at}, \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)\right].$$

Ornstein-Uhlenbeck process with mean reversion

- When $t \to \infty$, the distribution of X_t converges to

$$\nu := N\left[m, \frac{\sigma^2}{2a}\right].$$

  which is the invariant or stationary distribution.
- Note that if X_0 has distribution ν then the distribution of X_t will be ν for all t.

Financial applications of the Ornstein-Uhlenbeck process with mean reversion

- Vasicek model for interest rate:

$$dr_t = a\left(b - r_t\right)dt + \sigma dB_t,$$

  with a, b, σ real constants.

- Solution:

$$r_t = b + \left(r_0 - b\right)e^{-at} + \sigma e^{-at}\int_0^t e^{as}dB_s.$$

Financial applications of the Ornstein-Uhlenbeck process with mean reversion

- Black-Scholes model with stochastic volatility: assume that volatility $\sigma(t) = f(Y_t)$ is a function of anOrnstein-Uhlenbeck process with mean reversion :

$$dY_t = a(m - Y_t)\,dt + \beta dW_t,$$

where $\{W_t, 0 \leq t \leq T\}$ is a sBm.

- The SDE which models the asset price evolution is

$$dS_t = \alpha S_t dt + f(Y_t)\, S_t dB_t$$

where $\{B_t, 0 \leq t \leq T\}$ is a sBm
and the sBm's W_t and B_t may be correlated, i.e.,

$$E[B_t W_s] = \rho\,(s \wedge t)\,.$$

Important theoretical result

- Useful theoretical result:

Let $f : [0, +\infty) \to \mathbb{R}$ be a deterministic function.
Then

1. $M_t = \exp\left(\int_0^t f(s) dB_s - \frac{1}{2}\int_0^t (f(s))^2 ds\right)$ is a martingale
2. $\int_0^t f(s) dB_s$ has a normal distribution with mean 0 and variance $\int_0^t (f(s))^2 ds$.

- Part 1 is a simple generalization of the fact that $\exp\left(\lambda B_t - \frac{1}{2}\lambda^2 t\right)$ is a martingale.
- Part 2 follows from 1, because martingales have constant mean and $E[M_0] = 1$ and $E\left[\exp\left(\lambda \int_0^t f(s) dB_s\right)\right] = \exp\left(\frac{1}{2}\lambda^2 \int_0^t (f(s))^2 ds\right)$, which is the moment generating function of the $N\left(0, \int_0^t (f(s))^2 ds\right)$ distribution.

Exam-style problem

- A derivatives trader is modelling the volatility of an equity index using the following time-discrete model (model 1):

$$\sigma_t = 0.12 + 0.4\sigma_{t-1} + 0.05\varepsilon_t, \quad t = 1, 2, 3, \ldots$$

where σ_t is the volatility at time t years and $\varepsilon_1, \varepsilon_2, \ldots$are a sequence of i.i.d. random variables with standard normal distribution. The initial volatility is $\sigma_0 = 0.15$ (that is, 15%). The trader is developing a related continuous-time model for use in derivative pricing. The model is defined by the following SDE (model 2):

$$d\sigma_t = -\alpha\left(\sigma_t - \mu\right) dt + \beta dB_t,$$

where σ_t is the volatility at time t years, B_t is the standard Brownian motion (sBm) and the parameters α, β and μ all take positive values.
(a) Determine the long-term distribution of σ_t for model 1.
(b) Show that for model 2 (solve the SDE), we have that

$$\sigma_t = \sigma_0 e^{-\alpha t} + \mu\left(1 - e^{-\alpha t}\right) + \int^t \beta e^{-\alpha(t-s)} dB_s.$$