Exercises of Lévy Processes and Applications

João Guerra

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Contents

This list of exercises was prepared for the course “Lévy Processes and applications”, of the Master in Mathematical Finance in ISEG, University of Lisbon, in the academic year 2020/2021.

Chapter 1

Exercises

Exercise 1.1 *Explain what are the main limitations of the Black-Scholes model and how can these limitations be surpassed by using Lévy processes to model asset returns.*

Exercise 1.2 *If* ϕ_μ *is a characteristic function, show that* $|\phi_\mu(u)| \leq 1$.

Exercise 1.3 *Let* $\alpha > 0$, $\beta > 0$. *Show that the gamma-*(α, β) *distribution*

$$\mu_{\alpha,\beta}(dx) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx, \quad \text{with } x > 0,$$

with characteristic function $\left(1 - \frac{iu}{\beta}\right)^{-\alpha}$ *is an infinitely-divisible distribution.*

Exercise 1.4 *Let* $d = 1$. *Show that if* $X \sim Po(\lambda)$ *then* $\phi_X(u) = \exp\left[\lambda\left(e^{iu} - 1\right)\right]$.

Exercise 1.5 *Let* X *and* Y *be independent standard normal random variables (with mean 0). Show that* Z *has a Cauchy distribution, where* $Z = X/Y$ *if* $Y \neq 0$ *and* $Z = 0$ *if* $Y = 0$.

Exercise 1.6 *Consider the Cauchy distribution and its probability density function*

$$f(x) = \frac{1}{\pi(1+x^2)}, \ x \in \mathbb{R}.$$

(a) Derive the characteristic function associated to the Cauchy distribution and prove that it is an infinitely divisible distribution. Hint: you can use the integral formula

$$\int_0^{+\infty} \frac{\cos(ux)}{1+x^2} dx = \frac{\pi}{2} e^{-|u|}.$$

(b) Discuss the simmetry, the existence of mean, variance and higher moments, the "fat tails" property of the Cauchy distribution and the advantages/drawbacks of using this distribution to model the returns of financial assets like stocks or stock indexes.

Exercise 1.7 *Prove that if X is stochastically continuous, then the map $t \to \phi_{X(t)}(u)$ is continuous for each $u \in \mathbb{R}^d$.*

Exercise 1.8 *Let N_t be a standard Poisson proces.*

a) *Show that the number of jumps of N is finite a.s. in every compact interval [0,t] (or that $N_t < \infty$ a.s. for every t).*

b) *Show that the Poisson process N is a Lévy process.*

Exercise 1.9 *Define what is a Lévy measure and what are the conditions that the parameters α, β and δ must satisfy in order that the measure*

$$\nu(x) = e^{\alpha(x+1)}\mathbf{1}_{\{x\leq -1\}} + x^{\beta}\mathbf{1}_{\{x\geq 1\}} + |x|^{\delta}\,\mathbf{1}_{\{|x|<1\}}$$

is a Lévy measure.

Exercise 1.10 *Let ν be a Lévy measure. Show that $\nu[(-\varepsilon,\varepsilon)^c] < \infty$, for all $\varepsilon > 0$.*

Exercise 1.11 *Show that the condition that defines a Lévy measure ν is equivalent to $\int_{\mathbb{R}^d-\{0\}} \frac{|x|^2}{1+|x|^2}\nu(dx) < \infty$.*

Exercise 1.12 *Consider an infinitely divisible distribution with characteristic triplet (b, c, ν). Show that*

$$\lim_{u\to 0}\left|\int_{\mathbb{R}}\left(e^{iux} - 1 + \left(\frac{u^2x^2}{2} - iux\right)\mathbf{1}_{\{|x|<1\}}\right)\nu(dx)\right| = 0.$$

Hint: For $|x| < 1$, you can develop e^{iux} using the second order Taylor formula and then use the definition of the Lévy measure.

Exercise 1.13 *Discuss the definition and the main properties of the following Lévy processes: the jump-diffusion process of Merton (with jumps sizes given by a normal distribution), and the (1/2)-stable subordinator.*

Exercise 1.14 *Prove that if T is a gamma subordinator and B is a Brownian motion then $Z(t) = B(T(t))$ is a Lévy process with characteristic function*

$$\Phi_{Z(t)}(u) = E\left[e^{uiZ(t)}\right] = \left(1 + \frac{u^2}{2b}\right)^{-at}.$$

Exercise 1.15 *Consider a Lévy process X_t with characteristic triplet (b, c, ν).*

(a) Present what are the conditions that the Lévy measure ν, the drift coeeficient b and the diffusion coefficient c must satisfy in order to have a process such that:

(i) has trajectories with finite variation and has finite variance (or second moment).

(ii) is a subordinator with jumps of size larger than 0.01.

(iii) has only jumps of negative size, it has infinite activity and has a deterministic component which is strictly increasing.

(b) Present the definition of subordinator and discuss for what values of the parameters α, β and γ can we say that the measure

$$\nu(dx) = \alpha |x|^{-3/2} \mathbf{1}_{\{x<0\}} + x^{\beta}\mathbf{1}_{\{0<x<1\}} + e^{\gamma x}\mathbf{1}_{\{x>1\}}$$

is the Lévy measure of a subordinator.

Exercise 1.16 *Consider a Lévy process X_t.*

(a) Assume that the Lévy process at time $t = 1$ has a characteristic exponent $\eta_X(u) = -\sigma^{\alpha} |u|^{\alpha}$ with $0 < \alpha \leq 2$

(i) What kind of process is the process X and what are the associated distributions?

(ii) Show that this process is self-similar and calculate the Hurst parameter.

(iii) Explain in a simple way (by words) what is a "self-similar" process.

(b) Give one concrete example of a subordinator process X_t such that this process has jumps of all positive sizes and is strictly increasing in t, for all $t \geq 0$. Specify what are the parameters in the characteristic triplet (b, c, ν) for this example and show that the Lévy measure satisfies all the appropriate conditions.

Exercise 1.17 *If X is a Lévy process, show that* $\exp\{i(u, X(t)) - t\eta(u)\}$ *is a martingale.*

Exercise 1.18 *Consider a Lévy process Z defined by*

$$Z_t = mt + 4B_t + Y_1 + Y_2 + \cdots + Y_{N(t)},$$

where B_t is a Brownian motion, the r.v. $Y_1, ..., Y_j, ...$ are i.i.d. with mean value $\mathbb{E}[Y_j] = \mu$ and $N(t)$ is a Poisson process with intensity λ.

(a) What kind of process is Z and what should be the relationship between m, μ and λ in order to be sure that the process Z is a martingale? Explain.

(b) Assuming that the random variables Y_j have a normal distribution with mean 5 and variance 2, $\lambda = 1$ and that the process is a martingale, present the characteristic function of Z_t and show that the distribution of Z_t is infinitely divisible.

Exercise 1.19 *Discuss the definition and main properties of the following Lévy processes: Cauchy process, Gamma subordinator and Variance-Gamma.*

Exercise 1.20 *Consider the Lévy measure*

$$\nu(x) = \frac{1}{x^{\frac{3}{2}}}\mathbf{1}_{\{x>0\}}(x)$$

associated to the Lévy process $X(t)$.

(a) Calculate

$$\mathbb{E}\left[\int_{\varepsilon}^{1} x^2 N(t, dx)\right], \quad \text{and}$$
$$\text{Var}\left[\int_{\varepsilon}^{1} x^2 N(t, dx)\right].$$

(b) Show that

$$\sum_{\{0\leq s\leq t:\Delta X(s)\in[\varepsilon,1]\}} (\Delta X(s))^2 - \frac{2}{3}t\left(1-\varepsilon^{\frac{3}{2}}\right)$$

is a martingale .

Exercise 1.21 *Consider the Poisson integral*

$$\int_{1}^{+\infty} x^{\frac{1}{4}} N(t, dx)$$

and the associated Lévy measure $\nu(x) = \frac{1}{x^{\frac{7}{4}}}\mathbf{1}_{\{x>0\}}(x)$. Calculate

$$\text{Var}\left[\int_{1}^{+\infty} x^{\frac{1}{4}} N(t, dx)\right].$$

Exercise 1.22 *Let X be a Lévy process*

(a) Consider that the Lévy measure associated to X is given by $\nu(dx) = x^{-2}\mathbf{1}_{\left\{\frac{1}{10}<x<1\right\}} + x^{-4}\mathbf{1}_{\{1\leq x<+\infty\}}$. (i) Calculate the expected value and the variance of the Poisson integral

$$\int_{1/10}^{+\infty} x^2 N(t, dx).$$

(ii) Give an interpretation for the meaning of this Poisson integral and say if the kurtosis or moment of order 4 of X_t is finite or not? Explain.

Exercise 1.23 *Let X be a Lévy process. Consider that the Lévy measure is*

$$\nu(dx) = \frac{1}{x^\alpha}\mathbf{1}_{\{x>0\}}.$$

(a) For what values of α is ν a Lévy measure?
(b) In the case $\alpha = 2$, calculate the expected value and the variance of

$$\int_\varepsilon^1 x^2 \widetilde{N}(t,dx),$$

and then calculate the limit of these quantities when $\varepsilon \to 0$, and give an interpretation for the meaning of this quantity.

Exercise 1.24 *Let L be a Lévy process with associated Lévy measure $\nu(dx) = e^{-|x|^5}$ and $N(t,dx)$ be the Poisson random measure associated to this Lévy process and Lévy measure.*
(a) Deduce what should be the function $g(t)$ such that the process

$$\int_{\mathbb{R}\setminus[-1,1]} x^4 N(t,dx) - g(t)$$

is a martingale and interpret the meaning of the integral $\int_{\mathbb{R}\setminus[-1,1]}^{+\infty} x^4 N(t,dx)$.

Exercise 1.25 *Prove that if X is a one-dimensional Brownian motion then the OU process $Y(t)$ is a Gaussian process with mean $e^{-\lambda t}y_0$ and variance $\frac{1}{2\lambda}\left(1-e^{-2\lambda t}\right)$.*

Exercise 1.26 *Prove that $d\mathcal{E}_Y(t) = \mathcal{E}_Y(t)\,dY(t)$, by apllying the Itô formula to*

$$\begin{aligned} dS_Y(t) = {} & F(t)\,dB(t) + \left(G(t) - \frac{1}{2}F(t)^2\right)dt \\ & + \int_{|x|\geq 1} \log(1+K(t,x))\,N(dt,dx) + \int_{|x|<1} \log(1+H(t,x))\,\widetilde{N}(dt,dx) \\ & + \int_{|x|<1} (\log(1+H(t,x)) - H(t,x))\,\nu(dx)\,dt, \end{aligned} \tag{1.1}$$

and recall that $\mathcal{E}_Y(t) = e^{S_Y(t)}$.

Exercise 1.27 *Let X be a Lévy-type of integral of the form*

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dB(t) + \int_{|x|>1} \gamma(t,x)\,N(dt,dx) + \int_{|x|\leq 1} \gamma(t,x)\,\widetilde{N}(dt,dx),$$

with $\mu(t) + \frac{(\sigma(t))^2}{2} + \int_{\mathbb{R}} \left(e^{\gamma(t,x)} - 1 - \gamma(t,x) \mathbf{1}_{\{|x|\leq 1\}}(x)\right) \nu(dx) = 0$ *a.s. for all* t.

(a) Show that

$$e^{X(t)} = e^{X_0} + \int_0^t F(s) dB(s) + \int_0^t \int_{\mathbb{R}} H(s,x) \widetilde{N}(ds, dx)$$

and find expressions for the processes $F(s)$ *and* $H(s,x)$.

(b) Assume that $|\sigma(t)|$ *and* $\left|\int_{\mathbb{R}} \left(e^{\gamma(t,x)} - 1\right)^2 \nu(dx)\right|$ *are bounded by a constant* C. *Use Gronwall's Lemma in order to show that* $e^{X(t)}$ *is a square-integrable martingale.*

(Note: Gronwall's Lemma: Let ϕ *be a positive and locally bounded function on* $\mathbb{R}_0^+$ *such that* $\phi(t) \leq a + b\int_0^t \phi(s) ds$ *for all* t, *with* $a, b \geq 0$. *Then* $\phi(t) \leq ae^{bt}$.*)*

Exercise 1.28 *Consider the stochastic differential equation (of the so-called Geometric Lévy process):*

$$dX(t) = X(t-) \left[bdt + \sigma dB(t) + \int_{|x|<1} H(t,x) \widetilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x) N(dt,dx)\right],$$

where b, σ *are constants,* $H(t,x) \geq -1$ *for all* t *and* x *and* $H(t,x)$ *and* $K(t,x)$ *are processes such that the Poisson integrals above are well defined. Determine the solution of this equation*

Exercise 1.29 *Let* $X(t)$ *be a Lévy-type of integral of the form*

$$dX(t) = \alpha(t) dt + \sigma(t) dB(t) + \int_{\mathbb{R}\setminus\{0\}} \gamma(t,x) \widetilde{N}(dt,dx).$$

Consider the process $Y(t) = f(X(t))$ *and determine the processes* $a(t)$, $b(t)$ *and* $c(t,x)$ *such that*

$$dX(t) = a(t) dt + b(t) dB(t) + \int_{\mathbb{R}\setminus\{0\}} c(t,x) \widetilde{N}(dt,dx),$$

in the following cases:

(a) $Y(t) = (X(t))^2$

(b) $Y(t) = \cos(X(t))$

Exercise 1.30 *Consider that the process* X *is a solution of the stochastic differential equation*

$$dX_t = 4X_{t-}dt - X_{t-}dB_t + 3X_{t-} \int_{|x|<1} \left(e^{\frac{x}{4}} - 1\right) \widetilde{N}(dt,dx),$$

with the initial condition $X_0 = 2$ *and the random Poisson measure has a Lévy measure associated given by* $\nu(dx) = x^{-2}\mathbf{1}_{\left\{\frac{1}{10}<x<1\right\}} + x^{-4}\mathbf{1}_{\{1\leq x<+\infty\}}$. *Solve this stochastic differential equation and present the explicit solution.*

Exercise 1.31 *Consider the Lévy measure associated to a Poisson random measure* $N(dt, dx)$

$$\nu(dx) = \frac{1}{x^2}\mathbf{1}_{\{x>0\}}.$$

. Let the process X *be the solution of the stochastic differential equation*

$$dX_t = (\mu - X_{t-})\,dt + \sigma dB_t + \int_{|x|<1} x^2 \widetilde{N}(dt, dx), \qquad \text{with } X(0) = 5.$$

Solve this stochastic differential equation (Hint: Consider the process $e^t X_t$ *and apply the Itô formula).*

Exercise 1.32 *Consider a Lévy-type stochastic integral:*

$$dY(t) = G(t)\,dt + F(t)\,dB(t) + \int_{|x|<1} H(t,x)\,\widetilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\,N(dt,dx).$$

Assume that:

- $\mathbb{E}\left[\int_0^t \int_{|x|\geq 1} |K(s,x)|^2 \nu(dx)\,ds\right] < \infty$ *for each* $t > 0$,
- $\int_0^t \mathbb{E}[|G(s)|]\,ds < \infty$ *for each* $t > 0$.

Prove that Y *is a martingale if and only if*

$$G(t) + \int_{|x|\geq 1} K(t,x)\,\nu(dx) = 0 \quad \text{(a.s.) for a.a. } t \geq 0.$$

Exercise 1.33 *Consider a financial market with one risky asset with price process* S_t *given as the solution of the stochastic differential equation*

$$dS(t) = S(t-)\,dX(t),$$

where $X(t)$ *is a Lévy process with decomposition*

$$X(t) = mt + B(t) + \int_c^{+\infty} x\widetilde{N}(t, dx),$$

where $m \geq 0$ *and* $c := -1$. *Assume that the riskless interest rate is* $r > 0$.

State the condition that m and r must satisfy in order for the discounted price process $\widetilde{S}$ to be a martingale with respect to an equivalent martingale measure Q and discuss this condition, the completeness of the market and the existence and uniqueness of this martingale measure, when:

(i) $X(t) = \frac{1}{5}t + B(t) + \widetilde{N}(t)$, where $\widetilde{N}(t)$ is an independent compensated Poisson process with intensity $\lambda = 2$ and jump size $J = 1$.

(ii) in the context of the Esscher transform measure Q_u, with $F(t) = -ku$, $H(t,x) = -ux$. Discuss the existence and uniqueness of the Esscher transform measure.

Exercise 1.34 *Consider a financial market with one risky asset with price process S_t given as the solution of the stochastic differential equation*

$$dS(t) = S\left(t-\right) dZ\left(t\right),$$

where $Z\left(t\right) = \sigma X\left(t\right) + \mu t$ and $X\left(t\right)$ is a Lévy process with decomposition

$$X\left(t\right) = mt + kB\left(t\right) + \int_{c}^{+\infty} x\widetilde{N}(t, dx),$$

where $k \geq 0$, $m \geq 0$ and $c := -\sigma^{-1}$. Assume that the riskless interest rate is $r > 0$.

State the condition that σ, μ, k, m and r must satisfy in order for the discounted price process $\widetilde{S}$ to be a martingale with respect to an equivalent martingale measure Q and discuss the completeness of the financial market when:

(i) $X(t) = B(t) + 2t$, where $B\left(t\right)$ is a standard Brownian motion.

(ii) $X(t)=\sum_{i=1}^{n} \widetilde{N}_i(t)$, where the processes $\widetilde{N}_i(t)$ are independent standard Poisson processes (with jumps of size 1) and with intensities λ_i.

(iii) $X(t)=\widetilde{N}_1(t) + B(t) + 3t$, where the processes $B(t)$ and $\widetilde{N}_1(t)$ are independent.

Exercise 1.35 *Consider a financial market with one riskless asset and one risky asset with discounted price process $\widetilde{S}(t)$ modeled as an ordinary exponential of a Lévy process $X(t)$, that is*

$$\widetilde{S}(t) = S_0 \exp(X\left(t\right)).$$

What is the general condition that allows us to conclude that the market model is arbitrage free? In which of the following cases can we ensure that the market model is arbitrage free (explain why in each case) ? :

(i) $X(t)$ is a compound Poisson process $X(t) = \sum_{k=1}^{N(t)} J_k$, where the random variables J_k have normal distribution $N(4,1)$.

(ii) $X(t)$ *is a jump process with drift* $b = 2$ *and with Lévy measure* $\nu(dx) = x^{-1/2}\mathbf{1}_{\{0<x<1\}}$.

(iii) $X(t)$ *is a jump-diffusion process* $X(t) = 3t + 2B(t) + \sum_{k=1}^{N(t)} J_k$, *where the random variables* J_k *have an exponential distribution.*

(iv) $X(t)$ *is a pure jump process with Lévy measure* $\nu(dx) = x^{-2}\mathbf{1}_{\{0<x<1\}}$.

Chapter 2

Exam problems

Exercise 2.1 *Discuss the main drawbacks of the classical Black-Scholes model (in finance) and explain how the Lévy processes can be used to overcome these drawbacks.*

Exercise 2.2 *State the Lévy-Itô decomposition for a general Lévy process and present a financial interpretation for the jump terms of this decomposition.*

Exercise 2.3 *Consider a "jump-diffusion" model without compensation term for the jumps.*

(a) Define the Lévy process associated to the Merton "jump-diffusion" model, give an interpretation of each term in the definition, present the characteristic function of the Lévy process at time $t = 1$ and present the characteristic triplet of the process.

(b) Consider the "jump-diffusion" process where the distribution for the jump sizes has a probability density function given by:

$$f_J(x) = p\theta_1 e^{\theta_1 x}\mathbf{1}_{\{x<0\}} + (1-p)\,\theta_2 e^{-\theta_2 x}\mathbf{1}_{\{x>0\}}.$$

Calculate the characteristic function for the "jump-diffusion" process at time $t = 1$ and show that the measure ν associated to the process satisfies the conditions of a Lévy measure.

Exercise 2.4 *Consider a distribution with characteristic function*

$$\phi(u) = \exp\left(imu - \sigma|u|\left[1 + i\beta\frac{2}{\pi}\mathrm{sgn(u)}\log|u|\right]\right),$$

where $\sigma > 0$, $-1 \leq \beta \leq 1$ and $m \in \mathbb{R}$.

(a) Show that this distribution is infinitely divisible

(b) Let X be a random variable with the distribution associated to $\phi(u)$ in the case $\beta = 0$. State the usual name given to the distribution of X, present the probability density function of X, present the value of $\mathbb{E}[|X|]$ and discuss how is the decay of the distribution tail of X when $x \to +\infty$, i.e., how is the decay of $\mathbb{P}[X > x]$ as a function of x when $x \to +\infty$.

Exercise 2.5 *Let X be a Lévy process with Lévy measure $\nu(dx) = \frac{\exp(-x)}{x^2}\mathbf{1}_{\{x>0\}}$.*

(a) Calculate the expected value of the Poisson integral

$$\mathbb{E}\left[\int_{\varepsilon}^{+\infty} e^x N(t, dx)\right].$$

where $\varepsilon > 0$ and deduce how is the function $h(t, \varepsilon)$ such that

$$\int_{\varepsilon}^{+\infty} e^x N(t, dx) - h(t, \varepsilon)$$

is a martingale.

(b) Consider that the process X is the solution of the stochastic differential equation

$$dX_t = 2X_{t-}dt + X_{t-}dB_t + X_{t-}\int_{|x|\geq 1}\left(e^{\frac{x}{2}} - 1\right)N(dt, dx).$$

Solve this equation.

Exercise 2.6 *Consider a financial market with one risky asset with price process S_t given as the solution of the stochastic differential equation*

$$dS(t) = S(t-)\,dZ(t),$$

where $Z(t) = \sigma X(t) + \mu t$ and $X(t)$ is a Lévy process with decomposition

$$X(t) = mt + kB(t) + \int_{c}^{+\infty} x\widetilde{N}(t, dx),$$

where $k \geq 0$, $m \geq 0$ and $c < -1$. Assume that the riskless interest rate is $r > 0$.

(a) State the condition that σ, μ, k, m and r must satisfy in order for the discounted price process $\widetilde{S}$ to be a martingale with respect to an equivalent martingale measure Q and discuss the completeness of the market in the following cases:

(i) $X(t) = mt + \widetilde{N}(t)$, where $\widetilde{N}(t)$ is a compensated Poisson process

(ii) $X = mt + \widetilde{N}_1(t) + \widetilde{N}_2(t)$, *where* $\widetilde{N}_1$ *and* $\widetilde{N}_2$ *are compensated Poisson processes with intensities* λ_1 *and* λ_2 *and with constant jump sizes* c_1 *e* c_2*, respectively (*$\widetilde{N}_1$ *and* $\widetilde{N}_2$ *are also independent).*

(b) Show that the general condition that must be satisfied in order for the discounted price process $\widetilde{S}$ *to be a martingale with respect to an equivalent martingale measure* Q *is an equation that has, in general, an infinite number of solutions* (F, H).

Hint: Consider that $\frac{dP}{dQ} = e^{Y(T)}$ *where*

$$dY(t) = G(t)dt + F(t)dB(t) + \int_{\mathbb{R}\backslash\{0\}} H(t,x)\widetilde{N}(dt,dx)$$

and show that if the pair (F, H) *is a solution then the pair* $(F + \int_{\mathbb{R}\backslash\{0\}} f(x)\nu(dx)$, $\log\left(e^H - \frac{kf(x)}{x}\right)$ *is also a solution for any* $f \in L^1(\mathbb{R}\backslash\{0\}, \nu)$.

Chapter 3

More Exam problems

Exercise 3.1 *Present the definitions of infinitely divisible random variable and stable random variable and present 3 examples of infinitely divisible distributions and 3 examples of stable distributions.*

Exercise 3.2 *Let* $X = (X_1, X_2, X_3)$ *be a Gaussian random vectorial process with distribution* $N(m, A)$*, where* $A = I$ *is the covariance matrix and* $m = (1, -1, 0)$*. Knowing that the characteristic function for the one dimensional Gaussian random variable* $N(\mu, \sigma^2)$ *is*

$$\phi(u) = \exp\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right),$$

derive the characteristic function of the random vector $X = (X_1, X_2, X_3)$ *and show that the distribution of* X *is infinitely divisible.*

Exercise 3.3 *Consider a Lévy process* $U(t)$ *with characteristic triplet* (b, c, ν)*.*

(a) What conditions should the parameters b *and* c *and the Lévy measure* ν *satisfy in order to ensure that* U *is a subordinator? Present also the general form of the characteristic function of* $U(t)$ *when* U *is a subordinator.*

(b) Let U *be a* $(1/2)$*-stable subordinator and* X *a stochastic process such that* $X(t) = \sqrt{2}B(t)$*, where* $B(t)$ *is a standard Brownian motion independent of* $U(t)$*. What type of process is the one that one obtains by* $V(t) = X(U(t))$ *and which is the distribution of* $V(1)$*?*

Exercise 3.4 *Consider a Lévy process* X_t *with characteristics triplet* $(b, 0, \nu)$ *and Lévy measure* $\nu(dx) = x^{\alpha}\mathbf{1}_{\{0<x<1\}} + x^{\beta}\mathbf{1}_{\{x>1\}}$*. What should be the conditions in the parameters* α *and* β*, such that* ν *is a Lévy Maasure and*

(i) the process has finite activity

(ii) the process has infinite activity

(iii) the process has paths with finite variation

Exercise 3.5 *Let X be a Lévy process with characteristics triplet (b, c, ν) and Lévy measure*

$$\nu(dx) = x\mathbf{1}_{\{0<x<1\}} + \frac{1}{x^6}\mathbf{1}_{\{x\geq 1\}},$$

(a) Calculate the varance of the following Poisson integral

$$\int_{\frac{1}{4}}^{+\infty} x^2 N(5, dx),$$

and interpret the meaning of this integral in terms of the jumps of the Lévy process.

(b) Consider that the process X is a solution of the stochastic differential equation

$$dX_t = (m - X_{t-})\, dt + \sigma dB_t + \int_{|x|\geq 1} xN(dt, dx), \qquad \text{with } X(0) = 1.$$

Solve this equation (hint: Consider process $e^t X_t$ and apply the Itô formula).

Exercise 3.6 *Consider a stochastic integral of the Lévy type*

$$dY(t) = G(t)\, dt + F(t)\, dB(t) + \int_{|x|<1} H(t, x)\, \widetilde{N}(dt, dx) + \int_{|x|\geq 1} K(t, x)\, N(dt, dx).$$

(a) Derive the condition (equation) that the processes $G(t)$, $F(t)$, $H(t, x)$ and $K(t, x)$ must satisfy in order to ensure that the process $e^{Y(t)}$ is a martingale.

(b) Using the condition in (a), verify that $e^{Y(t)}$ is a martingale for the following processes

(i) $Y(t) = -2t + 4B(t)$.

(ii) $Y(t) = \int_0^t G(s)ds + \int_0^t K(s)dN(s)$, where N is a Poisson process with intensity λ and Lévy measure $\nu(dx) = \lambda\delta_1(x)$.

Exercise 3.7 *Consider a Lévy process Z_t with characteristics triplet (μ, A, ν).*

(a) Present the condition that ν must satisfy in general and the conditions that must be satisfied in order to ensure that the process (i) has paths with finite variation

(ii) has paths with infinite variation

(iii) has infinite activity

(iv) is a subordinator
(v) has finite moments of all orders
(b) Consider that $\mu = 0$, $A = 0$ *and*

$$\nu(dx) = \frac{1}{2x^{\frac{3}{2}}} \mathbf{1}_{\{x>0\}}.$$

In this case, is Z_t *a Lévy process? Justify. If the answer is yes, what type of Lévy process? And what is its characteristic exponent and what is the probability density function of* Z_1*? Discuss also the "decay law" for the distribution tails.*

Exercise 3.8 *Consider the "Variance-Gamma" process* L_t

(a) (i) Define the process using an appropriate subordinator and assuming that the associated "Variance Gamma" distribution $V(\sigma, b, \theta)$ *is such that* $\sigma = 1$ *and* $\theta = 0$. *(ii) Define also the same process using two appropriate subordinators (iii) Present the Lévy measure associated to the process, describing its decay when* $x \to \infty$.

(b) Show that the characteristic function of the process is

$$\Phi_{L(t)}(u) = \mathbb{E}\left[e^{iuL(t)}\right] = \left(1 + \frac{u^2}{2b}\right)^{-at},$$

where a *and* b *are the parameters associated to the subordinator in part (a).*

Exercise 3.9 *Consider an infinitely divisible distribution with characteristics* (m, A, ν).

(a) Show that

$$\lim_{u\to 0} \left| \int_{-1}^{1} \left(e^{iyu} - 1 - iyu\right) \nu(dy) + \int_{\mathbb{R}\setminus(-1,1)} \left(e^{iyu} - 1\right) \nu(dy) \right| = 0.$$

(b) Show that the exponential distribution with probability density function

$$f(x) = \theta e^{-\theta x} \mathbf{1}_{\{x\geq 0\}}$$

and characteristic function

$$\phi(u) = \frac{\theta}{\theta - iu}$$

is infinitely divisible.

Hint: Note that the characteristic function of a distribution Gamma-(α, β) *is* $\phi(u) = \frac{1}{(1-iu/\alpha)^{\beta}}$.

Exercise 3.10 *Present 3 different examples of Lévy processes that are martingales and given a Lévy process $Y(t)$ with characteristic exponent $\eta(u)$, show that the process*

$$\exp\{iuY(t) - t\eta(u)\}$$

is also a martingale.

Exercise 3.11 *Consider a market in which the price of the risky asset S_t is modeled by the SDE*

$$dS(t) = S(t-)\, dZ(t),$$

where $Z(t) = \sigma X(t) + \mu t$ e $X(t)$ is a Lévy process with decomposition

$$X(t) = mt + kB(t) + \int_c^{+\infty} x\widetilde{N}(t, dx),$$

with $k \geq 0$ and $m \geq 0$. Assume that the riskless interest rate is $r > 0$.

(a) What condition the parameters σ, μ, k, m and r should satisfy in order to ensure that the discounted price of the risky asset is a martingale with respect to the equivalent martingale probability measure $\mathbb{Q}$? Explain also why is this market model incomplete in most cases. (b) Discuss the previous condition and the market completeness in the following cases:

(i) Process X is a Brownian motion $B(t)$;

(ii) $X = mt + B(t) + \widetilde{N}_t$, where N is a compensated Poisson process with intensity λ and independent of B.