Models in Finance - Part 7

Master in Actuarial Science

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Price processes

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- Suppose also that we have a risk-free cash bond B which has a value at time t of B_t (we assume that the risk-free rate of interest is constant: that is, $B_t = B_0 e^{rt}$).

Self-financing strategies

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- Consider the instantaneous pure investment gain in the value of this portfolio over the period t up to $t + dt$ (assuming that there is no inflow or outflow of cash during the period $[t, t + dt]$). This is equal to $\varphi_t dS_t + \psi_t dB_t$.

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- The instantaneous change in the value of the portfolio, allowing for cash inflows and outflows, is given by $dV(t) = \varphi_t dS_t + \psi_t dB_t + S_t d\varphi_t + d\varphi_t . dS_t + B_t d\psi_t + dB_t d\psi_t$.

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- The portfolio strategy is self-financing if $dV(t) = \varphi_t dS_t + \psi_t dB_t$.

Complete markets

- Let X be any derivative payment (payoff) contingent upon $\mathcal{F}_U$ where U is the time of payment of X (i.e. X is $\mathcal{F}_U$-measurable and, therefore, depends upon the path of S_t up to time U). The time U may be fixed or it may be a random stopping time.
- A replicating strategy is a self-financing strategy (φ_t, ψ_t), define for $0 \leq t < U$, such that $V(U) = \varphi_U S_U + \psi_U B_U = X$.
- In other words, for an initial investment of $V(0)$ at time 0, if we follow the self-financing portfolio strategy (φ_t, ψ_t) we will be able to reproduce the derivative payment without risk.
- The market is said to be complete if for any such contingent claim X there is a replicating strategy (φ_t, ψ_t).

Forward contracts

- Forward contract: agreement to buy or sell an asset at a certain future time for a certain price.
- A forward contract is the most simple form of derivative contract.
- It is also the most simple to price since the forward price can be established without reference to a model for the underlying share price.
- Assume thar we can also invest in a cash account which earns interest at the continuously compounding rate of r per annum.
- Forward price K: price that the holder agrees to pay at expiry date T (corresponds to the exercise price of an option).
- When you purchase a forward contract there is no initial premium to pay.

Forward contracts

- Pricing idea: the forward price K is such that the value of the contract at time 0 is zero.
- S_0: Price of the asset underlying the forward contract today.
- **Proposition**: The fair or economic forward forward price is

$$K = S_0 e^{rT}.$$

- **Proof** (a): (1) Assume $K = S_0 e^{rT}$. We can borrow the value of S_0 in cash (interest rate is r) and buy 1 share. The net cost at time 0 is zero. The forward contract costs 0.
- At time T we will have a share with value S_T, a debt of $S_0 e^{rT}$ in cash and a contract to sell the share by K.
- Therefore we hand over the one share to the holder of the forward contract and receive $K = S_0 e^{rT}$, we repay the loan $S_0 e^{rT}$.
- There is no chance of losing money or making a profit (risk-free trading strategy).

Forward contracts

- **Proof** (cont.): (2) Suppose that $K > S_0 e^{rT}$. We can enter a forward contract in a short position (sell the underlying at T), borrow S_0 in cash and buy one share. The net cost at time 0 is zero.
- At time T, we will have a share with value S_T, a debt of $S_0 e^{rT}$ and a contract to sell the share by K.
  Since $K > S_0 e^{rT}$ we have made a positive profit with no risk: arbitrage opportunities. By the principle of no arbitrage, this cannot occur.
- (3) Suppose that $K < S_0 e^{rT}$. We can enter a forward contract in a long position (buy forward contract), invest amount S_0 in cash and sell one share. at price S_0. The net cost at time 0 is zero.
- At time T, we will have a cash amount of $S_0 e^{rT}$ and a contract to buy the share by K.
  Since $K < S_0 e^{rT}$ we have made a positive profit with no risk: arbitrage opportunities. By the principle of no arbitrage, this cannot occur.

Forward contracts

- Note: in the arbitrage situation, why not trade lots of forward contracts and make a fortune?
- In practice a flood of sellers (or buyers) would come in immediately, pushing down (or up) the forward price to be equal to S_0e^{rT}.
- The arbitrage could exist briefly but it would disappear quickly before substantial arbitrage profits could be made.

Forward contracts

- **Proof** (b - alternative): Consider the 2 portfolios:
- A: one long position in the forward contract (that gives you a share at time T by the price K)
- B: borrow Ke^{-rT} in cash and buy one share by S_0.
- At time T both portfolios have a value of $S_T - K$.
  By the principle of no arbitrage, these portfolios must have the same value at time 0.
  Since at time 0 portfolio B value is $S_0 - Ke^{-rT}$,
  then the value of portfolio A at time 0, which is the forward price, is $S_0 - Ke^{-rT}$ and must be zero (the value of the forward contract at time 0 must be zero)
  and therefore $K = S_0 e^{rT}$.

Forward contracts

- Exercise: Consider a 6-month forward contract on a share with current price of 25.50 Eur. If the forward price is 26.25 Eur, calculate the (continuously compounded) risk-free interest rate.