Models in Finance - Exercises

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These exercises (and solutions) were prepared for the course ”Models in Finance”, of the MSc. degree in Actuarial Science in ISEG, University of Lisbon, in the academic year 2020/2021.

Chapter 1

Exercises

Exercise 1 *Prove that if* B *is a standard Brownian motion, then* $W_t = W_0 + \mu t + \sigma B_t$ *is a Brownian motion with drift* μ *and diffusion coefficient* σ.

Exercise 2 *Prove the time inversion property (property 7) of the Brownian motion by computing the expectation and the covariance function of* B_2.

Exercise 3 *Prove that if* $M = \{M_n; n \geq 0\}$ *is a martingale, then*

1. $E[M_n] = E[M_0]$ for all $n \geq 1$.
2. $E[M_n | \mathcal{F}_k] = M_k$ for all $n \geq k$.

Exercise 4 *Let* S_t *be a geometric Brownian motion defined by* $S_t = \exp(\mu t + \sigma B_t)$, *where* B_t *is a standard Brownian motion (sBm) and* μ *and* σ *are constants.*

(a) Write down the SDE satisfied by $X_t = \ln(S_t)$.

(b) By apllying Ito's Lemma (Itô formula), write down the SDE satisfied by S_t.

(c) The price of a share follows a geometric Brownian motion motion with $\mu = 0.06$ *and* $\sigma = 0.25$ *(both expressed in annual units). Find the probability that, over a given 1-year period, the share price will fall.*

Exercise 5 *Prove that the process* $X_t = \exp\left(aB_t - \frac{a^2 t}{2}\right)$ *is a* $\left\{\mathcal{F}_t^B, t \geq 0\right\}$*-martingale.*

Exercise 6 *Prove that* $E\left[\int_0^T u_t dB_t\right] = 0$ *if* u *is a simple process.*

Exercise 7 *Compute* $\int_0^5 f(s)\, dB_s$ *with* $f(s) = 1$ *if* $0 \leq s \leq 2$ *and* $f(s) = 4$ *if* $2 < s \leq 5$. *What is the distribution of the resulting random variable?*

Exercise 8 *Let* $B_t := (B_t^1, B_t^2)$ *be a two dimensional Brownian motion. Represent the process*

$$Y_t = \left(B_t^1 t, \left(B_t^2\right)^2 - B_t^1 B_t^2\right)$$

as an Itô process.

Exercise 9 *Assume that a process* X_t *satisfies the SDE*

$$dX_t = \sigma\left(X_t\right) dB_t + \mu\left(X_t\right) dt.$$

Compute the stochastic differential of the process $Y_t = X_t^3$ *and represent this process as an Itô process.*

Exercise 10 *Let* $F = B_T^3$. *What is the Itô integral representation of this random variable?*

Exercise 11 *What is the process* u *such that* $\int_0^T tB_t^2 dt - \frac{T^2}{2} B_T^2 = -\frac{T^3}{6} + \int_0^T u_t dB_t$ *?*

Exercise 12 *(Integration by parts): Assume that* $f(s)$ *is a deterministic function of class* C^1. *Prove that*

$$\int_0^t f(s)\, dB_s = f(t)\, B_t - \int_0^t f'(s)\, B_s ds.$$

Exercise 13 *(Exam style problem): A derivatives trader is modelling the volatility of an equity index using the following time-discrete model (model 1):*

$$\sigma_t = 0.12 + 0.4\sigma_{t-1} + 0.05\varepsilon_t, \quad t = 1, 2, 3, \ldots$$

where σ_t *is the volatility at time t years and* $\varepsilon_1, \varepsilon_2, \ldots$ *are a sequence of i.i.d. random variables with standard normal distribution. The initial volatility is* $\sigma_0 = 0.15$ *(that is, 15%). The trader is developing a related continuous-time model for use in derivative pricing. The model is defined by the following SDE (model 2):*

$$d\sigma_t = -\alpha\left(\sigma_t - \mu\right) dt + \beta dB_t,$$

where σ_t *is the volatility at time t years,* B_t *is the standard Brownian motion and the parameters* α, β *and* μ *all take positive values.*

(a) Determine the long-term distribution of σ_t *for model 1.*

(b) Show that for model 2 (solve the SDE), we have that

$$\sigma_t = \sigma_0 e^{-\alpha t} + \mu\left(1 - e^{-\alpha t}\right) + \int_0^t \beta e^{-\alpha(t-s)} dB_s.$$

(c) *Determine the numerical value of μ and a relationship between parameters α and β if it is required that σ_t has the same long-term mean and variance under each model (models 1 and 2)*

(d) *State another consistency property between the two models that could be used to determine precise numerical values for α and β.*

(e) *The derivative pricing formula used by the trader involves the squared volatility $V_t = \sigma_t^2$, which represents the variance of the returns on the index. Determine the SDE for V_t in terms of the parameters α, β and μ.*

Exercise 14 *Prove that the stochastic process $\{L_t, t \in [0,T]\}$, given by $L_t = exp\left(-\lambda B_t - \frac{\lambda^2}{2}t\right)$, is a positive martingale with expected value 1 and satisfies the stochastic differential equation*

$$dL_t = -\lambda L_t dB_t,$$
$$L_0 = 1.$$

Exercise 15 *Consider that the share price of a non-dividend paying security is given by a stochastic process S_t which is the solution of the Stochastic Differential Equation (SDE)*

$$dS_t = \mu S_t dt + h(t, S_t)\, dB_t,$$

where B_t is a standard Brownian motion, μ is a constant, $h(t,x)$ is a bounded function with continuous and bounded partial derivatives and t is the time from now measured in years.

(a) Consider the process $Y_t = g(t, S_t)$, where $g : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a function of class $C^{1,2}\left(\mathbb{R}_0^+ \times \mathbb{R}\right)$ with bounded partial derivatives and such that

$$\frac{\partial g}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x)\left(h(t,x)\right)^2 + \mu x \frac{\partial g}{\partial x}(t,x) = 0,$$

for all $t \geq 0$ and $x \in \mathbb{R}$. Show that the process Y_t is a martingale with the respect to the filtration generated by the Brownian motion.

(b) Let $\mu = 0.1$ and $h(t, S_t) = 0.25 S_t$. Calculate the probability that the 2-year return will be less than 15%

Exercise 16 *The longer the time to expiry, the greater the chance that the underlying share price can move significantly in favor of the holder of the call option or put option before expiry. Why?*

Exercise 17 *List the 6 factors that determine the price of an american put option and, for each factor, state whether an increase in its value produces an increase or a decrease in the value of the option, explaining why.*

Exercise 18 *(a) Use no arbitrage arguments in order to deduce the formula for the fair forward price for a forward contract.*

(b) Consider a forward contract on a non-dividend paying share, with expiry date two years from now. Calculate the forward price, if the current share price is 10 Eur and the (continuously compounded) risk-free interest rate is 6% *p.a.*

Exercise 19 *Consider a 6-month forward contract on a share with current price of* 25.50 *Euros. If the forward price is* 26.25 *Euros, calculate the (continuously compounded) risk-free interest rate.*

Exercise 20 *Consider a share with price process S_t and a forward contract written on S with maturity date T. Assume that the share pays dividends at a constant rate (continuously compounded) q. The risk-free interest rate is assumed to be r (constant).*

(a) Deduce a formula for the (fair) forward price K of the forward contract, at time $t = 0$.

(b) If the forward price is 30€*, the current price is* 25€*, the risk-free interest rate is* 10% *p.a. (continuously compounded) and the time to expiry is* 30 *months, calculate the dividend rate of the share.*

Exercise 21 *What is the lower bound for a 3-month European put option on a share X if the share price is 95 EUR, the exercise price is 100 EUR and the (continuously compounded) risk-free rate is 12% p.a.*

Exercise 22 *(a) Explain how and why do the strike price and the interest rate affect the price of the European call and put options.*

(b) Considering an appropriate portfolio, derive a lower bound for the price of the call option.

(c) Calculate the price of a put option (at the money), knowing that a call option (also at the money) and with the same expiry date has a price $c_t = 1$€, the current price of the share is 15€*, the time to expiry is* 18 *months and the continuously compounded interest rate is* 4%*.*

Exercise 23 *(a) State what is meant by put-call parity.*

(b) Consider a European call and a European put option on a non-dividend paying share with the same time to expiry (6 months from now) and the same strike price 10.5 *Eur. Assume that the current share price is* 10 *Eur and the*

(continuously compounded) risk-free interest rate is 8% *p.a. If the call option price is* 0.5 *Eur, calculate the price of the put option.*

(c) By constructing two portfolios with identical payoffs at the exercise date of the options, derive an expression for the put-call parity of European options on non-dividend paying shares.

(d) By constructing two portfolios with identical payoffs at the exercise date of the options, derive an expression for the put-call parity of European options on a dividend paying share, where the dividend d *is known to be payable at some date* t_1 *with* $t < t_1 < T$*.*

(e) If the put-call parity in (b) does not hold, explain how an arbitrageur can make a riskless profit.

Exercise 24 *Consider a 3-period recombining binomial model for the non-dividend paying share with price process* S_t *such that the price at time* $t+1$ *is either* $S_t u$ *or* $S_t d$ *with* $d = \frac{1}{u}$*. Assume that* $u = 1.12$ *and that the current price of the share is* 10€*. Assuming that the risk-free interest rate is* 5% *per year, construct the binomial tree and calculate the price of a derivative composed of a sum of an European put option and an European call option. The put option has strike* $K_p = 8.5$ *and the call option has a strike* $K_c = 12$*. The time to maturity of both options is* 3 *years.*

Exercise 25 *Consider a 3-period recombining binomial model for the non-dividend paying share with price process* S_t *such that over each time period the stock price can move down by a factor* $d = 0.92$ *or up by a factor* $u = \frac{1}{d}$*. Assume that the (continuously compounded) risk-free interest rate is* 5% *per period and that* $S_0 = 10$€*.*

(a) Construct the binomial tree for the 3-period model and calculate the price of a European financial derivative with payoff given by $max\left\{exp\left(\frac{S_T}{5}\right) -, 0\right\}$ *with maturity date in 3 periods.*€*.*

(b) Consider the binomial model under the risk-neutral measure Q *and the lognormal model (also under* Q*) such that*

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left(\left(r - \frac{1}{2}\sigma^2\right)(t - t_0), \sigma^2 (t - t_0)\right),$$

where r *is the interest rate per year. Consider that the binomial model has a time step given by* δt*. If we calibrate the binomial model in a way that the return and the variance of the log-return over the time interval* δt *of the binomial model and the lognormal model are equal, deduce that (for small* δt*)*

$$q = \frac{e^{r\delta t} - d}{u - d}$$

and

$$u = exp\left(\sigma\sqrt{\delta t}\right).$$

(Hint: assume that $\left\{\mathbb{E}_Q\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right]\right\}^2 \approx 0$ *when* δt *is small.)*

Exercise 26 *Consider call options and put options written on the same underlying share (we assume that the share pays no dividends), with the same maturity and strike price.*

(a) Compare the price of American call options (respect. American put options) and European call options (respect. European put options) with the same strike and maturity. Explain your reasons.

(b) Calculate the price of an American put option, using a Binomial model with 2 periods and assuming that the interest rate (for each period) is 5%, the volatility is 20% per period and the time until maturity is 2 periods. Consider that the current price of the underlying share is 10 *Euros and that the strike price is* 11 *Euros.*

Exercise 27 *Consider a 3-period recombining binomial model for the non-dividend paying share with price process* S_t *such that the price at time* $t+1$ *is either* $S_t u$ *or* $S_t d$ *with* $d = \frac{1}{u}$*. Assume that* $u = 1.08$ *and that the current price of the share is* 10€*.*

(a) Discuss if the model is free of arbitrage (if it is not, prove that there is an arbitrage strategy) in the following cases:

1. *if the risk-free interest rate is* 0.02% *per period (continuously compounded).*
2. *if the risk-free interest rate is* 10% *per period (continuously compounded).*

(b) Compare, from the computational effort point of view, a recombining binomial model of the type considered before with n periods (n large) with the general (non-recombining) binomial model with the same n periods. Moreover, explain what is the relationship between the form of the volatility and the recombining model and how you can calculate the factor u from the volatility parameter associated to the lognormal continuous model.

Exercise 28 *Consider a 2-period recombining binomial model for the non-dividend paying share with price process* S_t *such that the price at time* $t+1$ *is either* $S_t u$ *or* $S_t d$*. Assume that* $u = 1.1$ *and that the current price of the share is* 10€*.*

(a) If the continuously compounded risk-free interest rate is 3% *per period, construct the binomial tree and calculate the price of a derivative with maturity* $T = 2$ *periods and with payoff*

$$\max\left\{\left(\frac{S_T}{5}\right)^4 - 10, 0\right\}.$$

(b) Calculate the replicating portfolio of the derivative with payoff given in part (a) for the two periods.

Exercise 29 *Evaluate* $E_Q[S_1]$ *and hence suggest a reason why* Q *is called the risk neutral probability measure.*

Exercise 30 *Starting from the price of the call option given by the BS formula, use put-call parity to derive the formula for the price of the put option.*

Exercise 31 *A forward contract is arranged where an investor agrees to buy a share at time* T *for an amount* K*. It is proposed that the fair price of this contract is*

$$f(t, S_t) = S_t - Ke^{-r(T-t)}.$$

Show that this:
(i) Satisfies the appropriate boundary condition.
(ii) Satisfies the Black-Scholes PDE.

Exercise 32 *Prove the Black-Scholes formula for the put option.*

Exercise 33 *You are trying to replicate a 6-month European call option with strike price* 500*, which you purchased* 4 *months ago. If* $r = 0.05$*,* $\sigma = 0.2$*, and the current share price is* 475*, what portfolio should you be holding (assuming no dividends) ?*

Exercise 34 *An investor A has* 10000 *Euros invested in a portfolio of* 1000 *shares. Investor B has* 10000 *Euros invested in a portfolio of* 5000 *call options on the share and the delta is* 0.5*. If the share price increases by* 10% *what will be the value of each portfolio?*

Exercise 35 *For each of the Greeks* ν, Θ, ρ *and* λ *discuss wether its value will be positive or negative in case of:*
(i) a call option.
(ii) a put option.

Exercise 36 *Consider a call option and a put option on a dividend-paying security with the same maturity and exercise price. By considering the put-call parity for this case:*

(i) prove that $\Delta_c = \Delta_p + e^{-q(T-t)}$.

(ii) prove that $\Gamma_c = \Gamma_p$.

Exercise 37 *Consider an investor with a portfolio of* N *put options and* 50000 *shares. Assume that the delta of an individual option is* -0.25 *and that its gamma is* 0.1.

(a) Calculate the number N *of put options in the portfolio such that the portfolio has zero delta.*

(b) Consider that the investor can invest in two other financial derivatives: the derivative X *and the derivative* Y*. These financial derivatives have delta and derivative of delta with respect to the underlying price satisfying*

$$\Delta_X = 0.3, \quad \frac{\partial \Delta_X}{\partial S} = 0.15,$$
$$\Delta_Y = 0.4, \quad \frac{\partial \Delta_Y}{\partial S} = 0.25.$$

Calculate the number of derivatives X *and* Y *that should be added to the portfolio in order to obtain a total portfolio with both delta and gamma equal to zero.*

Exercise 38 *Consider the Black-Scholes model and a European call option written on a non-dividend paying share with expiry date 15 months from now, strike price* 10€ *and current price* 9€*. Assume that the (continuously compounded) free-risk interest rate is* 5% *p.a. and that the volatility is* $\sigma = 0.2$.

(a) Consider that an investor has 10000 call options as defined above. Calculate the corresponding hedging portfolio in shares and cash.

(b) Consider a financial derivative Φ *that has the following payoff at expiry date* T *depending on the price of the underlying non-dividend paying share at maturity* T*:*

$$Payoff = \begin{cases} K \text{ if } S_T > e^K, \\ \ln\left[S(T)\right] \text{ if } S_T \leq e^K, \end{cases}$$

where K *is positive constant. Show that the price of the derivative at time* t *is given by (for* $t < T$*)*

$$e^{-r(T-t)} \int_{-\infty}^{K^*} \left(\ln\left(S_t\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma z \sqrt{T-t} \right) f(z) dz$$
$$+ K e^{-r(T-t)} \left[1 - \Phi\left(K^*\right)\right],$$

where K^* is an appropriate constant, $f(z)$ is the p.d.f. and $\Phi(z)$ is the cumulative distribution function of a certain distribution.

Exercise 39 *Consider the Black-Scholes model.*

(a) List the assumptions underlying the Black-Scholes model and state the general risk-neutral valuation formula for the price, at time $t < T$, of a derivative security with payoff X at the expiry date T.

(b) Consider a financial derivative with the payoff

$$Y = \frac{1}{T - t_0} \int_{t_0}^{T} S_u du,$$

where T is the maturity date, $t_0 < T$, and S_u is the price of the underlying non-dividend paying share at time u. Show that the price of this financial derivative at time $t < t_0$ is given by

$$V_t = \frac{S_t}{r(T - t_0)} \left[1 - \exp\left(-r(T - t_0)\right)\right].$$

Exercise 40 *Consider a portfolio of* 100000 *European put options written on a share and N shares. Assume that the Delta of an individual option is* -0.25.

(a) (i) Define the greeks Δ, Γ, λ for a general derivative; (ii) calculate the number N of shares in the portfolio such that the portfolio Δ is zero; (iii) Comment on the importance of keeping a Γ close to zero, from the portfolio management perspective.

(b) Consider the Black-Scholes model and a put option written on a dividend paying share with expiry date 15 months from now, strike price 20€ *and current price* 18€. *Assume that the (continuously compounded) free-risk interest rate is* 5% *p.a., the volatility is* 0.2 *and the dividends are payable continuously at the constant rate of* 3% *p.a. Calculate the price of this option.*

Exercise 41 *Suppose that $t = 5$ and that the force of interest has been a constant of* 4% *p.a. over the last* 5 *years. Suppose also that the force of interest implied by current market prices is a constant* 4% *p.a. for the next two years and a constant* 6% *p.a. thereafter. If $T = 10$ and $S = 15$, write down or calculate each of the six quantities: $B(t,T)$, $r(t)$, $C(t)$, $F(t,T,S)$, $f(t,T)$ and $R(t,T)$.*

Exercise 42 *Deduce the formula $B(t,T) = \exp\left[-\int_t^T f(t,u)du\right]$ from $f(t,T) = \lim_{h \to 0} F(t,T,T+h)$.*

Exercise 43 *Under the term structure model*

$$f(t,T) = 0.03e^{-0.1(T-t)} + 0.06(1 - e^{-0.1(T-t)}).$$

Sketch a graph of $f(t,T)$ as a function of T and derive expressions for $B(t,T)$ and $R(t,T)$.

Exercise 44 *Show that the instantaneous forward rate for the Vasicek model can be expressed as:*

$$f(t,T) = r(t)e^{-\alpha\tau} + \left[\mu - \frac{\sigma^2}{2\alpha^2}\right]\left(1 - e^{-\alpha\tau}\right) + \frac{\sigma^2}{2\alpha^2}\left(e^{-\alpha\tau} - e^{-2\alpha\tau}\right).$$

Exercise 45 *Consider the zero-coupon bond market.*

(a) Discuss the limitations of one factor interest rate models.

(b) Present the stochastic differential equations (SDE) for the short rate in the Vacisek and CIR models and discuss the differnces between the two models.

Exercise 46 *Consider the zero-coupon bond market for zero-coupon bonds paying 1€ at time T. Under the real-world probability measure $\mathbb{P}$, the price of a zero-coupon bond with maturity T is*

$$B(t,T) = \exp\left\{-(T-t)r(t) + \frac{1}{k}(T-t)^{\frac{5}{2}}\right\},$$

where $r(t)$ is the short rate of interest at time t and k is a positive constant. Derive formulas for the instantaneous forward rate $f(t,T)$, the spot rate $R(t,T)$ and the market price of risk $\gamma(t,T)$ in terms of $r(t)$. In order to derive the formula for $\gamma(t,T)$, assume that

$$\begin{aligned} dr(t) &= \alpha r(t)\,dt + \sigma dZ_t, \\ dB(t,T) &= B(t,T)\left[m(t,T)dt + S(t,T)dZ_t\right], \\ \gamma(t,T) &= \frac{m(t,T) - r(t)}{S(t,T)}, \end{aligned}$$

where $\alpha > 0$ and Z_t is a standard Brownian motion under $\mathbb{P}$.

Exercise 47 *Consider a two-state model for credit rating with deterministic transition intensity, where the recovery rate is δ and the zero coupon bond price is given by*

$$B(t,T) = e^{-r(T-t)}\left[1 - (1-\delta)\left(1 - \exp\left(t^{3/2} - T^{3/2}\right)\right)\right].$$

(a) Discuss how are defined the transition probabilities in the two-state model for credit rating with deterministic transition intensity and in the Jarrow-Lando-Turnbull model.

(b) State the general formula for the zero coupon bond prices in a two state model for credit ratings and then deduce the implied risk-neutral transition intensity $\widetilde{\lambda}(s)$ *for our particular two-state model.*

Exercise 48 *Consider the two-state model for credit risk with deterministic intensity.*

(a) Let X *be the state at time* t*. Describe the two possible states* N *and* D *and present the formulas for the transition probabilities* $P\left[(X(t+dt)=N|X(t)=N)\right]$, $P\left[(X(t+dt)=D|X(t)=N)\right]$ *in terms of the intensity* $\lambda(t)$.

(b) Consider that the corporation XY *has issued 5-year zero-coupon bonds. Assume that risk-neutral transition rate for default is given by*

$$\lambda\left(t\right)=0.025t^{2},$$

where t *is the time in years. Assume also that the 5-year risk-free spot yield is 2% per year (continuously compounded interest rate) and in the event of default, assume a recovery rate for a payment due at time* t *is given by* $\delta\left(t\right)=1-0.01t^{2}$*. Calculate the risk-neutral probability that* XY *will default by the end of the 5-year period and the price of a 500 Euros (nominal) bond.*

Chapter 2

Exercises with solutions

2.1 Part 1

1. Consider the standard Brownian motion $B = \{B_t, t \in [0,T]\}$.

   (a) Let X be the stochastic process defined by $X_t := B_{2t} - B_t$, $t \in \left[0, \frac{T}{2}\right]$. Calculate the mean and the variance of X_t. Is X a Gaussian process? And is it a standard Brownian motion? Justify your answers.

   (b) Show that the stochastic process Y defined by $Y_t = \exp(t/2)\sin(B_t)$ is a martingale with respect to the filtration generated by B.

2. Consider the standard Brownian motion $B = \{B_t, t \in [0,T]\}$.

   (a) Find explicitly the stochastic process $u = \{u_t, t \in [0,T]\}$ such that

   $$B_T^3 = \int_0^T u_s dB_s.$$

   (Hint: you can use Itô's formula in order to show that $\int_0^T B_t dt = \int_0^T (T-t) dB_t$).

   (b) Let a and b be positive constants and consider the stochastic differential equation (SDE)

   $$dY_t = \frac{b - Y_t}{T - t} dt + dB_t, \quad 0 \leq t < T,$$
   $$Y_0 = a.$$

Show that

$$Y_t = a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + (T-t)\int_0^t \frac{dB_s}{T-s}, \qquad 0 \le t < T$$

is a solution of this SDE.

3. Consider European put and call options with the same strike and maturity.

   (a) Use the standard put-call parity in order to calculate the delta of a European put option on a non-dividend paying stock in terms of the delta of the call option with the same strike and maturity.

   (b) Consider European put and call options on a dividend paying stock, with the same strike and maturity. Assume that the stock pays a dividend D at the time t_D, with $t_D < T$, where T is the option maturity. Prove the put-call parity in this context. (Hint: try to adapt the proof of standard put-call parity of the non-dividend paying stock case).

4. Consider the Black-Scholes model with a risky asset with price S_t and a riskless asset with price B_t. Consider a contingent claim (derivative) with payoff $\chi = \Phi(S_T) = \ln(S_T)$.

   (a) What are the main assumptions underlying the Black-Scholes model?

   (b) State the stochastic differential equations (SDE's) satisfied by S_t and $Y_t := \ln(S_t)$, under the risk neutral measure Q.

   (c) Deduce the price of the contingent claim (derivative) with payoff $\chi = \Phi(S_T) = \ln(S_T)$.

5. Consider a recombining binomial model for a non-dividend paying stock with two periods. Let S_t be the price of the stock (with $t = 0, 1, 2$). Assume that the state-price deflator after one period is

$$A_1 = \begin{cases} 0.6 & \text{if } S_1 = S_0 u \\ 1.4 & \text{if } S_1 = S_0 d \end{cases},$$

   where the real world dynamics is

$$S_{t+1} = \begin{cases} S_t u & \text{with probability } p \\ S_t d & \text{with probability } 1-p \end{cases},$$

   and $0 < d < u$. Consider also that exists a risk-free instrument that offers a continuously compounded rate of return of 3% per period.

(a) Calculate the value of p and the risk neutral probability measure.

(b) Calculate the price at time $t = 0$ of a derivative that pays 1 at time $t = 2$ if $S_2 < S_0$ and pays 0 if $S_2 > S_0$.

(a) State the stochastic differential equations for the short rates under the risk-neutral measure for the Vasicek and Cox-Ingersoll-Ross (CIR) models. Outline the main properties of these models.

(b) Discuss the main disadvantages and limitations of using one factor interest-rate models.

(a) Discuss the two-state model for credit-ratings.

(b) The company QT has issued 10-year zero-coupon bonds. In order to model the company status and calculate the price of bonds, assume a continuous-time two-state model with risk-neutral transition rate for failure

$$\lambda(t) = 0.005t,$$

where t is the time in years. Assume also that the 10-year risk-free spot yield is 4% (annual efective rate) and in the event of failure, assume a recovery rate for a payment due at time t given by $\delta(t) = 1 - 0.08t$.

Calculate the risk-neutral probability that the company will have failed by the end of 10 years and the fair price of a 1000 Euros (nominal) bond, considering the risk of company failure.

2.2 Solutions of Part 1

1 (a) We know that $B_t \sim N(0, t)$ and $E[B_t B_s] = \min(s, t)$. Therefore

$$\begin{aligned} E[X_t] &= E[B_{2t}] - E[B_t] = 0, \\ Var[X_t] &= E\left[(B_{2t} - B_t)^2\right] = \\ &= E\left[B_{2t}^2 + B_t^2 - 2B_t B_{2t}\right] = 2t + t - 2t = t. \end{aligned}$$

The process X_t is clearly Gaussian since their finite-dimensional distributions are Gaussian. However, it is not a Brownian motion, since its covariance function (for $s < t < 2s$)

$$\begin{aligned} E[X_t X_s] &= E[B_{2t}B_{2s} - B_{2t}B_s - B_t B_{2s} + B_t B_s] \\ &= 2s - s - t + s = 2s - t \end{aligned}$$

is different from the Brownian motion covariance function. We could also show that the increments $X_t - X_s$ and $X_s - X_0$ are not independent.

(b) Let $Y_t = f(t, B_t)$ With $f(t,x) = e^{\frac{t}{2}} \sin(x)$. This is a $C^{1,2}$ function. By Ito's lemma:

$$\begin{aligned} dY_t &= \frac{1}{2} e^{\frac{t}{2}} \sin(B_t)\, dt + e^{\frac{t}{2}} \cos(B_t)\, dB_t - \frac{1}{2} e^{\frac{t}{2}} \sin(B_t)\, dt \\ &= e^{\frac{t}{2}} \cos(B_t)\, dB_t. \end{aligned}$$

or

$$Y_t = \int_0^t e^{\frac{s}{2}} \cos(B_s)\, dB_s.$$

Since $e^{\frac{s}{2}} \cos(B_t)$ is an adapted process and $E\left[\int_0^T \left(e^{\frac{s}{2}} \cos(B_s)\right)^2 ds\right] \le E\left[\int_0^T e^s \cos^2(B_s)\, ds\right] \le Te^T < \infty$, the stochastic integral process $Y_t = \int_0^t e^{\frac{s}{2}} \cos(B_s)\, dB_s$.is a martingale (one of the main properties of the stochastic integral process).

2. (a) Let $Y_t = f(B_t) = B_t^3$ with $f(x) = x^3$. By Ito's lemma

$$dY_t = 3B_t^2 dB_t + 3B_t dt$$

or

$$Y_T = B_T^3 = 3\int_0^T B_t^2 dB_t + 3\int_0^T B_t dt.$$

Applying Ito's lemma to $Z_t = tB_t$, or $Z_t = g(t, B_t)$ with $g(t,x) = tx$, we get:

$$dZ_t = B_t dt + t dB_t,$$

or

$$Z_T = TB_T = \int_0^T B_s ds + \int_0^T s dB_s$$

and

$$\int_0^T B_s ds = TB_T - \int_0^T s dB_s$$

Therefore

$$\begin{aligned} B_T^3 &= 3\int_0^T B_t^2 dB_t + 3TB_T - 3\int_0^T t dB_t \\ &= \int_0^T 3B_t^2 dB_t + \int_0^T 3T dBt - \int_0^T 3t dB_t \\ &= \int_0^T 3(B_t^2 + T - t) dB_t. \end{aligned}$$

and $u_t = 3(B_t^2 + T - t)$.

(b) We can represent $Y_t = a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + (T-t)\,X_t$, where $X_t = \int_0^t \frac{1}{T-s} dB_s$. Or, $Y_t = f\,(t, X_t)$ with $f(t,x) = a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + (T-t)\,x$. By Ito's lemma, with $dX_t = \frac{1}{T-t} dB_t$, we obtain:

$$\begin{aligned} dY_t &= \left(-\frac{a}{T} + \frac{b}{T} - X_t\right) dt + (T-t)\,dX_t \\ &= \frac{1}{T-t}\left(b - \left(a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + (T-t)\,X_t\right)\right) dt + \frac{T-t}{T-t} dB_t \\ &= \frac{b - Y_t}{T-t} dt + dB_t. \end{aligned}$$

or

$$\begin{aligned} dY_t &= \frac{b - Y_t}{T-t} dt + dB_t, \\ Y_0 &= a. \end{aligned}$$

4. (a) The put-call parity for European call and put options with strike K and maturity T is

$$C_t + K \exp\left(-r(T-t)\right) = P_t + S_t$$

Therefore, with $\Delta_c = \frac{\partial C_t}{\partial S_t}$ and $\Delta_P = \frac{\partial P_t}{\partial S_t}$, we have that

$$\Delta_c = \Delta_P + 1$$

or

$$\Delta_P = \Delta_C - 1.$$

(b) Let $t < t_D < T$. Consider a portfolio A: at time t buy a call and sell a put. The value of the portfolio is:

$$V_A\,(t) = C_t - P_t, \qquad V_A\,(T) = S_T - K.$$

Consider a portfolio B: at time t buy the underlying asset and borrow money such that at maturity the value of the portfolio is $S_T - K$. The value os this portfolio is

$$V_B\,(t) = S_t - Ke^{-r(T-t)} - De^{-r(t_D - t)}, \qquad V_B\,(T) = S_T - K.$$

At time t_D, the value of the dividend, D, is paid and is added to the portfolio D.

By the absence of arbitrage opportunities, the value os portfolios A and B is the same at time t:

$$C_t - P_t = S_t - Ke^{-r(T-t)} - De^{-r(t_D-t)}.$$

5 (a) The assumptions underlying the Black-Scholes model are as follows:

1. The price of the underlying share follows a geometric Brownian motion.
2. There are no risk-free arbitrage opportunities.
3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending.
4. Unlimited short selling (that is, negative holdings) is allowed.
5. There are no taxes or transaction costs.
6. The underlying asset can be traded continuously and in infinitesimally small numbers

(b) The dynamics of the stock prices is given by the SDE $dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_t$, which is equivalent to

$$\begin{aligned} dS_t &= r\, S_t\, dt + \sigma\, S_t \left(\frac{\mu - r}{\sigma} dt + dW_t \right) \\ &= r\, S_t\, dt + \sigma\, S_t\, d\overline{W}_t, \end{aligned}$$

where, by Girsanov's theorem, $\overline{W}_t := \frac{\mu-r}{\sigma} dt + dW_t$ is a standard Brownian motion with respect to the risk-neutral measure Q. The dynamics of S_t under Q is given by the SDE

$$dS_t = r\, S_t\, dt + \sigma\, S_t\, d\overline{W}_t.$$

By Ito's lemma applied to $X_t = \ln(S_t)$, we have

$$dX_t = \left(r - \frac{\sigma^2}{2} \right) dt + \sigma d\overline{W}_t$$

or

$$X_t = \ln(S_0) + \left(r - \frac{\sigma^2}{2} \right) t + \sigma \overline{W}_t.$$

(c) The price of the derivative is given by

$$\begin{aligned} F(t, S_t) &= e^{-r(T-t)} E^Q_{t,s} [\Phi(S_T)] \\ &= e^{-r(T-t)} E^Q_{t,s} [\ln(S_T)] \end{aligned}$$

where

$$\begin{aligned} dS_u &= r\, S_u\, du + \sigma\, S_u\, d\overline{W}_u, \\ S_t &= s \end{aligned}$$

By part (a), we have

$$X_T = \ln(S_T) = \ln(s) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\overline{W}_T - \overline{W}_t\right).$$

Therefore

$$E^Q_{t,s}\left[\ln(S_T)\right] = \ln(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t)$$

and

$$F(t, S_t) = e^{-r(T-t)}\left[\ln(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right].$$

6 (a) By the state price deflator approach, we have that

$$A_1 = \begin{cases} e^{-r}\frac{q}{p} = 0.6 & \text{if } S_1 = S_0 u \\ e^{-r}\frac{1-q}{1-p} = 1.4 & \text{if } S_1 = S_0 d \end{cases}.$$

Therefore $\begin{matrix} e^{-0.03}\frac{q}{p} = 0.6 \\ e^{-0.03}\frac{1-q}{1-p} = 1.4 \end{matrix}$ with solution: $p = 0.536943$ and the risk neutral probability is $q = 0.331977$.

(b) If we assume that $ud = 1$ then the price is

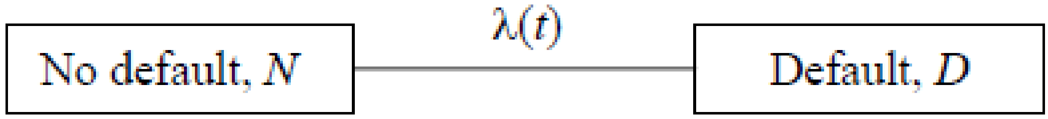
$$\begin{aligned} e^{-2r}E_Q[P_2] &= e^{-2r} \times (1-q)^2 \times 1 \\ &= e^{-0.06}(1 - 0.331977)^2 \\ &= 0.420267. \end{aligned}$$

If we assume that $ud < 1$ then the price is

$$\begin{aligned} e^{-2r}E_Q[P_2] &= e^{-2r} \times \left[2q(1-q) \times 1 + (1-q)^2 \times 1\right] \\ &= 0.837974. \end{aligned}$$

7. (a) The dynamics of the Vasicek model:

$$dr_t = \alpha(\mu - r_t)\, dt + \sigma dZ_t.$$



$\alpha, \mu, \sigma > 0$ are constants and Z is a Brownian motion under the risk-neutral measure Q. The dynamics of the CIR model:

$$dr_t = \alpha\left(\mu - r_t\right)dt + \sigma\sqrt{r_t}dZ_t.$$

Main properties of the Vasicek model: mean reversion, is arbitrage free, allows negative interest rates. Main properties of the CIR model: mean reversion, is arbitrage free, negative interest rates not allowed, volatility is high when rates are high and volatility is low when rates are low, it is more difficult to implement than Vasicek model.

(b) If we look at historical interest rate data we can see that changes in the prices of bonds with different terms to maturity are not perfectly correlated as one would expect to see if a one-factor model was correct. Sometimes we even see, for example, that short-dated bonds fall in price while long-dated bonds go up. Recent research has suggested that around three factors, rather than one, are required to capture most of the randomness in bonds of different durations.

Second, if we look at the long run of historical data we find that there have been sustained periods of both high and low interest rates with periods of both high and low volatility. Again these are features which are difficult to capture without introducing more random factors into a model.

Third, we need more complex models to deal effectively with derivative contracts which are more complex than, say, standard European call options. For example, any contract which makes reference to more than one interest rate should allow these rates to be less than perfectly correlated.

8. (a) The two-state model for credit ratings is a Markov model in continuous time, with two states N (not previously defaulted) and D (previously defaulted). Under this simple model it is assumed that the default-free interest rate term structure is deterministic with $r(t) = r$ for all t. If the transition intensity, under the real-world measure P, from N to D at time t is denoted by $\lambda(t)$, this model can be represented as: and D is an absorbing state. If $X(t)$ is the state at time t.

The transition intensity, $\lambda(t)$, can be interpreted as:

$P\left[(X(t+dt) = N|X(t) = N)\right] = 1 - \lambda(t)\,dt + o(dt) \quad \text{as} \quad dt \to 0,$

$$P\left[(X(t+dt)=D|X(t)=N)\right]=\lambda\left(t\right)dt+o(dt) \quad \text{as} \quad dt\to 0.$$

(b) The risk neutral probability of company failure (by time n) is

$$p\left(n\right)=1-\exp\left(-\int_0^n \lambda\left(t\right)dt\right)$$

In our case,

$$p\left(n\right)=1-\exp\left(-0.0025n^2\right)$$

and

$$p\left(10\right)=0.221199.$$

Recovery rate at time 10: $\delta\left(10\right)=1-0.08\times 10=0.2$.

Risk-neutral expected payment at maturity: $p\left(10\right)\delta\left(10\right)+\left(1-p\left(10\right)\right)\times 1=0.823041$. The fair price of the 1000 Euro Bond: $0.823041\times\left(1.04\right)^{-10}\times 1000=556,017$.

2.3 Part 2

1. Consider that the share price of a non-dividend paying security is given by a stochastic process S_t which is the solution of the Stochastic Differential Equation (SDE)

$$dS_t=\alpha S_t dt+\sigma S_t dB_t,$$

   where:

   - B_t is a standard Brownian motion,
   - $\alpha=0.08$
   - $\sigma=0.25$ (25% p.a.)
   - t is the time from now measured in years
   - $S_0=10$ €

   (a) Solve the SDE and derive the distribution of S_t.

   (b) Calculate the probability that over a two year period, the share price will fall.

2. Consider a non-dividend paying share with price process S_t and a forward contract with expiry date T.

(a) Using the no arbitrage principle, deduce the formula for the (fair) forward price for the forward contract at time t, where $0 \leq t \leq T$.

(b) Calculate the (fair) forward price for the forward contract if the current share price is $S_t = 20€$, the (continuously compounded) risk-free interest rate is 5% p.a. and the time to expiry is 15 months.

3. Consider European put and call options written on a non-dividend paying share.

    (a) Explain how and why do the strike price and the volatility affect the price of the European call and put options.

    (b) Derive the put-call parity relationship

    (c) Calculate the (continuously compounded) risk free interest rate r, knowing that a call option (at the money) and a put option (at the money) with the same expiry date have prices $c_t = 0.8€$, $p_t = 0.6€$, the current price of the share is $12€$ and the time to expiry is 9 months.

4. Consider a binomial model for the non-dividend paying share with price process S_t such that the price at time $t + 1$ is either $1.2S_t$ or $0.85S_t$ (assume that the time t is measured in years).

    (a) Explain why $d < e^r < u$, where r is the (continuously compounded) risk-free interest rate and d and u are quantities you should define and how could an investor make a certain profit if $d < u < e^r$.

    (b) If the continuously compounded risk-free interest rate is 5% p.a., calculate the risk-neutral probability measure.

    (c) Consider the two-period model and assume that $S_0 = 50€$. Calculate the price of a 2 year European call option with a strike price of $K = 60€$.

5. Consider the Black-Scholes model and a call option written on a non-dividend paying share with expiry date 18 months from now, strike price 100€ and current price 95€. Assume that the (continuously compounded) free-risk interest rate is 2% p.a. and that the volatility is $\sigma = 0.2$.

    (a) List the assumptions underlying the Black-Scholes option pricing formula.

(b) Calculate the option price

(c) Calculate the corresponding hedging portfolio in shares and cash for 50000 options on the share.

6. Consider the zero-coupon bond market.

   (a) Discuss the limitations of one factor interest rate models.

   (b) Present the stochastic differential equations (SDE) for the short rate in the Vacisek and CIR models and discuss the critical differnce between the two models.

   (c) Solve the SDE for the Vasicek model.

2.4 Part 2 - Solutions

1. (a) By Itô's lemma (or Itô's formula), with $f(x) = \log(x)$ (it is a C^2 function):

$$\begin{aligned} d(\log(S_t)) &= \frac{1}{S_t} dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(dS_t)^2 \\ &= \frac{1}{S_t}\left[\alpha S_t dt + \sigma S_t dB_t\right] dS_t \\ &+ \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\left[\alpha S_t dt + \sigma S_t dB_t\right]^2 \\ &= \left(\alpha - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t, \end{aligned}$$

using $(dB_t)^2 = dt$. In integral form, we have

$$\log(S_t) = \log(S_0) + \left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma B_t.$$

or

$$S_t = S_0 \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right].$$

Replacing the parameter values, we have

$$S_t = 10 \exp\left[0.0488t + 0.25B_t\right].$$

Since $B_t \sim N(0; t)$, then $\log(S_t) \sim N\left(\log(S_0) + \left(\alpha - \frac{1}{2}\sigma^2\right)t; \sigma^2 t\right)$ or $\log(S_t) \sim N(2.3026 + 0,0488t; 0.0625t)$.

(b)

$$P(S_2 < S_0) = P\left(\frac{S_2}{S_0} < 1\right) = P(\exp[0.0976 + 0.25B_2] < 1)$$
$$= P(Z < 0).$$

where $Z = 0.0976 + 0.25B_2 \sim N(0.0976; 0.125)$. Therefore: $P(S_2 < S_0) = 0,3913$.

3. (a) Consider two portfolios at time t :

A: one long position in the forward contract (that gives you a share at time T by the price K)

B: borrow $Ke^{-r(T-t)}$ in cash and buy one share by S_t.

At time T both portfolios have a value of $S_T - K$. By the principle of no arbitrage, these portfolios must have the same value at time $t < T$. Since at time t portfolio B value is $S_t - Ke^{-r(T-t)}$, then the value of portfolio A at time t, which is the forward price, is $S_t - Ke^{-rT}$ and must be zero. Therefore $K = S_t e^{r(T-t)}$.

(b) Applying the formula deduced for the fair price,

$$K = S_t e^{r(T-t)} = 20e^{\left(\frac{15}{12}\right)\times 0.05} = 21.290€.$$

4. (a) In the case of a call option, a higher strike price means a lower intrinsic value. A lower intrinsic value means a lower premium. For a put option, a higher strike price will mean a higher intrinsic value and a higher premium. In each case the change in the value of the option will not match precisely the change in the intrinsic value because of the later timing of the option payoff. The higher the volatility of the underlying share, the greater the chance that the underlying share price can move significantly in favour of the holder of the option before expiry. So the value of an option will increase with the volatility of the underlying share.

(b) Consider the two portfolios at time t:

A: one call + cash $Ke^{-r(T-t)}$

B: one put + one share S_t

Portfolio A: payoff at T:

$$\begin{cases} S_T - K + K = S_T & \text{if } S_T > K \text{ (call option exercised)} \\ 0 + K = K & \text{if } S_T \leq K \text{ (call expires wothless)} \end{cases}$$

Portfolio B: payoff at T:

$$\begin{cases} 0 + S_T = S_T & \text{if } S_T > K \text{ (put expires worthless)} \\ K - S_T + S_T = K & \text{if } S_T \leq K \text{ (put option exercised)} \end{cases}$$

At expiry T, both portfolios have a payoff $\max\{K, S_T\}$. Now, since the portfolios have the same value at T, and the options cannot be exercised before, the portfolios have the same value at any time $t < T$, i.e.

$$c_t + Ke^{-r(T-t)} = p_t + S_t.$$

(c) From the put-call parity (note that $K = S_t = 12$€), we have that

$$0.8 + 12e^{-r\left(\frac{9}{12}\right)} = 0.6 + 12.$$

Therefore

$$\begin{aligned} r &= \frac{-12}{9} \log\left(\frac{0.6 + 12 - 0.8}{12}\right) \\ &= 0.0224 \end{aligned}$$

and the (continuously compounded) risk-free interest rate is 2.24% p.a.

5. (a) The inequality must be satisfied in order to have an arbitrage free market. Otherwise an investor could make a guaranteed profit (arbitrage opportunity).

u and d are the proportionate changes in the price of the underlying in each period if the price goes up or down, respectively. In our case: $u = 1.2$ and $d = 0.85$.

If $d < u < e^r$ the cash investment would outperform the share investment in all circumstances. An investor could (at time 0) sell the share and invest S_0 in a cash account. At time 1 he could buy again the share and have a certain positive profit of $S_0e^r - S_0u > 0$.

(b) The risk neutral probability of an up-movement is

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.05} - 0.85}{1.2 - 0.85} = 0.5751.$$

(c) Using the usual backward procedure:

$C_2(u^2) = \max\{S_0u^2 - K, 0\} = 12$, $C_2(ud) = \max\{S_0ud - K, 0\} = 0$, $C_2(d^2) = \max\{S_0d^2 - K, 0\} = 0$ and at time 1: $C_1(u) = \exp(-r)\left[qC_2(u^2) + (1-q)C_2(ud)\right] = \exp(-0.05)\left[0.5751 \times 12\right] = 6.5646$

$C_1(d) = \exp(-r)[qC_2(ud) + (1-q)C_2(d^2)] = 0$

And the price at time 0 is $C_0 = \exp(-r)[qC_1(u) + (1-q)C_1(d)] = \exp(-0.05) \times 0.5751 \times 6.5646 = 3.5912$.

6. (a) The assumptions underlying the Black-Scholes model are as follows:

1. The price of the underlying share follows a geometric Brownian motion.
2. There are no risk-free arbitrage opportunities.
3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending.
4. Unlimited short selling (that is, negative holdings) is allowed.
5. There are no taxes or transaction costs.
6. The underlying asset can be traded continuously and in infinitesimally small numbers of units.

(b) The option price is given by:

$$c_t = S_t \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2).$$

with:

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{95}{100}\right) + \left(0.02 + \frac{0.2^2}{2}\right) \times 1.5}{0.2\sqrt{1.5}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Therefore, $d_1 = 0.0355$, $d_2 = -0.2094$ and

$$\begin{aligned} c_t &= 95\Phi(0.0355) - 100e^{-0.02\left(\frac{18}{12}\right)}\Phi(-0.2094) \\ &= 95 \times 0.5142 - 100e^{-0.02\left(\frac{18}{12}\right)} \times 0.4171 \\ &= 8.3717. \end{aligned}$$

(c) The call delta is:

$$\begin{aligned} \Delta &= \frac{\partial c_t}{\partial S_t} = \Phi(d_1) = \Phi(0.0355) \\ &= 0.5142. \end{aligned}$$

Therefore the hedging portfolio is: $\Delta \times 50000 = 0.5142 \times 50000 = 25710$ units of stock and $50000 \times 8.3717 - 25710 \times 95 = -2023900$€ in cash.

7. (a) (1) if we look at historical interest rate data we can see that changes in the prices of bonds with different terms to maturity are not perfectly

correlated as one would expect to see if a one-factor model was correct. Sometimes we even see, for example, that short-dated bonds fall in price while long-dated bonds go up.

(2) If we look at the long run of historical data we find that there have been sustained periods of both high and low interest rates with periods of both high and low volatility. Again these are features which are difficult to capture without introducing more random factors into a model. This issue is especially important for two types of problem in insurance: the pricing and hedging of long-dated insurance contracts with interest-rate guarantees; and asset-liability modelling and long-term risk-management.

(3) we need more complex models to deal effectively with derivative contracts which are more complex than, say, standard European call options. For example, any contract which makes reference to more than one interest rate should allow these rates to be less than perfectly correlated.

(b) The Vasicek model has the dynamics, under the risk-neutral measure Q:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma d\widetilde{W}(t)$$

where $\widetilde{W}$ is a standard Brownian motion under Q. The Cox-Ingersoll-Ross (CIR) model has the dynamics under Q:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}d\widetilde{W}(t).$$

The critical difference between the two models occurs in the volatility, which is increasing in line with the square-root of $r(t)$ for the CIR model. Since this diminishes to zero as $r(t)$ approaches zero, and provided σ^2 is not too large ($r(t)$ will never hit zero provided $\sigma^2 \leq 2\alpha\mu$), we can guarantee that $r(t)$ will not hit zero. Consequently all other interest rates will also remain strictly positive.

(c) Solve the SDE for the Vasicek model and deduce the form of the distribution of the zero-coupon bond price for this model

$$dr_t = \alpha\left(\mu - r_t\right)dt + \sigma d\widetilde{W}_t$$

$\alpha, \sigma > 0$ and $\mu \in \mathbb{R}$.

Solution of the associated ODE $dx_t = -\alpha x_t dt$ is $x_t = xe^{-\alpha t}$.

Consider the variable change $r_t = Y_t e^{-\alpha t}$ or $Y_t = r_t e^{\alpha t}$.

By the Itô formula applied to $f(t,x) = xe^{\alpha t}$, we have $Y_t = x+\mu\left(e^{\alpha t} - 1\right)+ \sigma\int_0^t e^{\alpha s}dB_s$.Therefore

$$r_t = \mu + (x - \mu)e^{-\alpha t} + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}dB_s.$$

2.5 Part 3

1. Let X_t be a stochastic process which is the solution of the stochastic differential equation (SDE)

$$dX_t = -\mu X_t dt + \sigma dB_t,$$

   where:

   - B_t is a standard Brownian motion,
   - μ and σ are positive parameters

   (a) Solve the SDE and derive the distribution of X_t

   (b) What is the long-term (stationary/invariant) distribution?

2. Consider European call options and European put options written on the same non-dividend paying share.

   (a) By considering an appropriate portfolio and no arbitrage arguments, prove that the call option price satisfies

$$c_t \geq S_t - Ke^{-r(T-t)},$$

   and define the terms in this inequality.

   (b) If the call option price is 1.2€, the strike price is 20€, the continuously compounded risk-free interest rate is 5% p.a. and the time to expiry is 21 months, calculate an upper bound for the current share price.

   (c) Assuming that the current share price is 18€, calculate the price of the put option, considering the data given in the previous question (3 (b)).

3. Consider a two-period binomial model for a non-dividend paying share with price process S_t such that over each period the stock price can either move up by 25% or move down by 20%, $S_0 = 10$ and the (continuously compounded) risk-free interest rate is 4% per period.

   (a) Construct the binomial tree for the two period model.

   (b) Calculate the price of an European put option with maturity date in two periods and strike price 12€.

4. Consider a portfolio of N European put options written on a non-dividend paying share and 10000 shares. Assume that the delta of an individual option is -0.16 and its gamma is 0.25.

   (a) Define the greeks $\Delta, \Gamma, \Theta, \lambda, \rho$ and vega (ν) for a general derivative.

   (b) If the portfolio has a delta of zero, calculate the number N of options in the portfolio.

   (c) Assume that two other derivatives can be traded in the market (call options with the same underlying, strike and maturity of the put options and another derivative Φ with the same underlying share and with a delta of 0.22 and gamma of 0.18). Calculate the number m of call options and the number j of derivatives Φ that should be added to the portfolio in order to obtain a total portfolio with both delta and gamma equal to zero.

5. Consider the Black-Scholes model.

   (a) Describe the risk-neutral pricing technique or martingale approach to the valuation of derivatives and state the general formula for the price, at time $t < T$, of a derivative security with payoff X at the expiry date T.

   (b) Consider a derivative Φ that has the following payoff at expiry date T depending on the price of the underlying non-dividend paying share at maturity S_T:

$$Payoff = \begin{cases} 1€ & \text{if } K_1 < \log(S_T) < K_2 \\ 0 & \text{otherwise} \end{cases},$$

   with K_1 and K_2 positive constants. Show that the price of the derivative is given by:

$$e^{-r(T-t)} P_Q \left[K_1 < \ln(s) + \left(r - \frac{\sigma^2}{2} \right)(T-t) + \sigma \left(\overline{W}_T - \overline{W}_t \right) < K_2 \right]$$

6. Consider the zero-coupon bond market.

   (a) State the formulas that relate the zero-coupon bond price at time t of a zero-coupon bond paying 1€ at time T (and denoted by $B(t,T)$) and the

   (i) Spot rate curve $R(t,T)$

   (ii) instantaneous forward rate curve $f(t,T)$

(b) Consider that the instantaneous forward rate is modelled by

$$f(t,T) = 0.04e^{-0.3(T-t)} + 0.08(1 - e^{-0.3(T-t)}).$$

Sketch the graph of $f(t,T)$ as a function of T and derive the expressions for $B(t,T)$ and for $R(t,T)$.

2.6 Part 3 - Solutions

1. (a) Let

$$Y_t = e^{\mu t} X_t$$

$Y_t = f(t, X_t)$ with $f(t,x) = e^{\mu t}x$. By Itô formula,

$$\begin{aligned} dY_t &= \mu e^{\mu t} X_t dt + e^{\mu t} dX_t \\ &= \mu e^{\mu t} X_t dt + e^{\mu t}\left(-\mu X_t dt + \sigma dB_t\right) \\ &= \sigma e^{\mu t} dB_t \end{aligned}$$

Therefore, $Y_t = X_0 + \sigma \int_0^t e^{\mu s} dB_s$ and

$$X_t = e^{-\mu t} X_0 + \sigma e^{-\mu t} \int_0^t e^{\mu s} dB_s.$$

This is a Gaussian process, since the random part is $\int_0^t f(s)\, dB_s$, where f is deterministic, so it is a Gaussian process.

The mean is (the expected value of a Itô integral is zero):

$$E[X_t] = e^{-\mu t} X_0.$$

The variance is (by Itô isometry)

$$E\left[(X_t - E[X_t])^2\right] = \sigma^2 e^{-2\mu t} \int_0^t e^{2\mu s} ds = \frac{\sigma^2}{2\mu}\left(1 - e^{-2\mu t}\right).$$

(b) The long-term distribution is obtained when $t \to +\infty$ and it is a Gaussian distribution mean 0 and variance $\frac{\sigma^2}{2\mu}$ and therefore is:

$$N\left[0, \frac{\sigma^2}{2\mu}\right].$$

3. (a) K is the strike price of the call option, T is the expiry date and r is the (continuously compounded) risk-free interest rate and S_t is the price process for the share.

At time t, consider portfolio A: one European call + cash $Ke^{-r(T-t)}$.

At time T, value of A is equal to $S_T - K + K = S_T$ if $S_T > K$. If $S_T < K$ then the payoff from portfolio A is $0 + K > S_T$.

Therefore the portfolio payoff $\geq S_T \Longrightarrow c_t + Ke^{-r(T-t)} \geq S_t$ and the lower bound for the price of European call is

$$c_t \geq S_t - Ke^{-r(T-t)}.$$

(b) Replacing the values in the inequality, we have:

$$1.2 \geq S_t - 20e^{-0.05\times\left(\frac{21}{12}\right)}$$

and

$$S_t \leq 19.524€.$$

(c) By the put-call parity:

$$c_t + Ke^{-r(T-t)} = p_t + S_t.$$

Therefore:

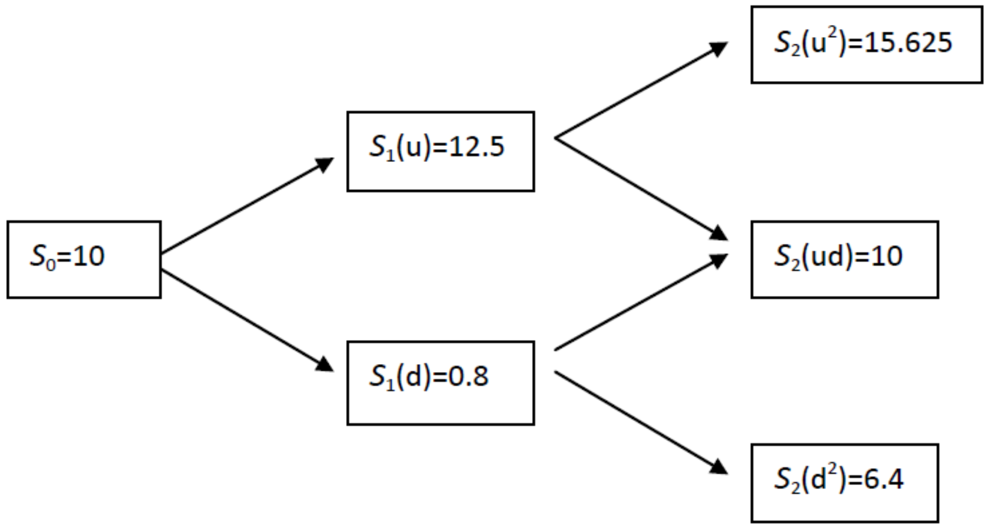
$$\begin{aligned} p_t &= c_t + Ke^{-r(T-t)} - S_t \\ &= 1.2 + 20e^{-0.05\times\left(\frac{21}{12}\right)} - 18 \\ &= 1.5244€ \end{aligned}$$

4. (a) In our case: $u = 1.25$ and $d = 0.8$. The binomial tree for 2 periods is:

(b) Since $d < e^r < u$ because $e^{0.04} = 1.0408$, the model is arbitrage free. The risk neutral probability of an up-movement is

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.04} - 0.8}{1.25 - 0.8} = 0.5351.$$

Using the usual backward procedure for a put option: $C_2\left(u^2\right) = \max\left\{K - S_0u^2, 0\right\} = 0$, $C_2(ud) = \max\left\{K - S_0ud, 0\right\} = 2$, $C_2\left(d^2\right) = \max\left\{K - S_0d^2, 0\right\} = 5.6$ and at time 1:



$C_1(u) = \exp(-r)\left[qC_2(u^2) + (1-q)C_2(ud)\right] = \exp(-0.04)\times(0.5351 \times 0 + 0.4649 \times 2) = 0.8933$.

$C_1(d) = \exp(-r)\left[qC_2(ud) + (1-q)C_2(d^2)\right] = \exp(-0.04)\times(0.5351 \times 2 + 0.4649 \times 5.6) = 3.5296$. And the price at time 0 is

$C_0 = \exp(-r)\left[qC_1(u) + (1-q)C_1(d)\right] = \exp(-0.04)\times(0.5351 \times 0.8933 + 0.4649 \times 3.5296) = 2.0358$€.

5. (a) The Greeks are the derivatives of the price of a derivative security with respect to the different parameters needed to calculate the price and and measure the sensitivity (rate of change) of the option price to changes in that parameter or variable.If F represents the value of the derivative, we define the greeks:

$$\Delta = \frac{\partial F}{\partial S}, \; \Gamma = \frac{\partial^2 F}{\partial S^2}, \; \Theta = \frac{\partial F}{\partial t},$$
$$\lambda = \frac{\partial F}{\partial q}, \; \rho = \frac{\partial F}{\partial r}, \; Vega = \nu = \frac{\partial F}{\partial \sigma}.$$

where S is the price of the underlying security, t is the time, q is the continuous dividend yield on the security, r is the interest rate and σ is the volatility.

(b) Portfolio with zero delta means that $N \times (delta$ of one put$)$ + number of shares = 0. Therefore: $N \times (-0.16) + 10000 = 0$ and therefore $N = 62500$

(c) By the put-call parity relationship, the delta of a call is such that:

$\Delta_c = \Delta_p + 1$, where Δ_p is the Δ of the put. Moreover $\Gamma_c = \Gamma_p$. Therefore $\Delta_c = 0.84$ and $\Gamma_c = \Gamma_p = 0.25$.

The gamma of the portfolio with the put options and the shares is $\Gamma_p \times 10000 = 0.25 \times 10000 = 2500$.

In order to have a new portfolio with zero delta and zero gamma, we need that: $m \times \Delta_c + j \times \Delta_\Phi = 0$ and $N \times \Gamma_p + m \times \Gamma_c + j \times \Gamma_\Phi = 0$.

Therefore: $0.84m + 0.22j = 0$ and $62500 \times 0.25 + m \times 0.25 + j \times 0.18 = 0$. Then

$j = -3.8182m$ and $m = \frac{-62500 \times 0.25}{0.25 - 0.18 \times 3.8182} = 35733$ and $j = -3.8182 \times 35733 = -136440$.

6. (a) In the martingale approach it can be proved that exists a portfolio (φ_t, ψ_t) that replicates the derivative payoff.

We can calculate the price of the derivative by the general risk-neutral pricing formula:

$$V_t = e^{-r(T-t)} E_Q \left[X | F_t \right].$$

Moreover, using the martingale approach, we can prove that

$$\varphi_t = \Delta = \frac{\partial V\left(t, S_t\right)}{\partial s}$$

The martingale approach implies that if we start with an initial amount V_0 invested in cash and shares and we follow a self-financing strategy and continuously rebalance the portfolio in order to hold $\varphi_t = \Delta$ units of S_t with the rest in cash, then we can replicate the derivative payoff.

(b)The dynamics of the stock prices S_t under Q is given by the SDE

$$dS_t = r\, S_t\, dt + \sigma\, S_t\, d\overline{W}_t.$$

By Ito's lemma applied to $X_t = \ln\left(S_t\right)$, we can show that

$$X_t = \ln\left(S_0\right) + \left(r - \frac{\sigma^2}{2}\right) t + \sigma \overline{W}_t.$$

The price of the derivative is given by

$$F\left(t, S_t\right) = e^{-r(T-t)} E^Q_{t,s} \left[1_{\{K_1 < \ln(S_T) < K_2\}} \right]$$

where

$$\begin{aligned} dS_u &= r\, S_u\, du + \sigma\, S_u\, d\overline{W}_u, \\ S_t &= s \end{aligned}$$

Therefore

$$X_T = \ln(S_T) = \ln(s) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\overline{W}_T - \overline{W}_t\right).$$

Therefore

$$E^Q_{t,s}\left[1_{\{K_1<\ln(S_T)<K_2\}}\right] = P_Q\left[K_1 < \ln(S_T) < K_2\right]$$

and

$$F(t,S_t) = e^{-r(T-t)}P_Q\left[K_1 < \ln(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\overline{W}_T - \overline{W}_t\right) < K_2\right].$$

7. (a)We have that zero-coupon bond prices are related to the spot-rate and instantaneous forward-rate by:

$$R(t,T) = \frac{-1}{T-t}\log B(t,T) \quad \text{if } t<T$$

or

$$B(t,T) = \exp\left[-R(t,T)(T-t)\right].$$

and

$$f(t,T) = \lim_{S\to T} F(t,T,S) = -\frac{\partial}{\partial T}\log B(t,T).$$

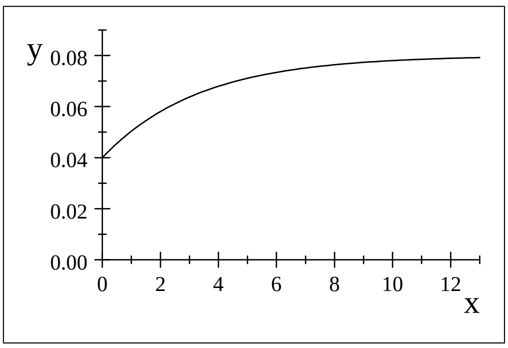
or (integrating):

$$B(t,T) = \exp\left[-\int_t^T f(t,u)\,du\right].$$

By $F(t,T,S)$ we represent the forward rate

$$F(t,T,S) = \frac{1}{S-T}\log\frac{B(t,T)}{B(t,S)} \quad \text{for } t<T<S.$$

(b)



$B(t,T) = \exp\left[-\int_t^T f(t,u)\,du\right] = \exp\left[-\int_t^T \left(0.04e^{-0.3(u-t)} + 0.08(1-e^{-0.3(u-t)})\right)du\right]$
$= \exp\left[-\int_t^T \left(0.08 - 0.04e^{-0.3(u-t)}\right)du\right] = \exp\left[-0.08\,(T-t) - 0.1333e^{-0.3(T-t)} + 0.1333\right]$

Moreover, $R(t,T) = \frac{-1}{T-t}\left[-0.08\,(T-t) - 0.1333e^{-0.3(T-t)} + 0.1333\right] = 0.08 - 0.1333\left[\frac{1-e^{-0.3(T-t)}}{T-t}\right]$.

2.7 Part 4

1. Consider that the share price of a non-dividend paying security is given by a stochastic process S_t which is the solution of the Stochastic Differential Equation (SDE)

$$dS_t = \alpha(t,S_t)\,dt + \sigma(t,S_t)\,dB_t,$$

where:

- B_t is a standard Brownian motion,
- $\alpha(t,x)$ and $\sigma(t,x)$ are differentiable functions with continuous and bounded partial derivatives.
- t is the time from now measured in years.

(a) Consider the process $Y_t = g(t,S_t)$, where $g : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a function of class $C^{1,2}\left(\mathbb{R}_0^+ \times \mathbb{R}\right)$ such that

$$\frac{\partial g}{\partial t}(t,x) + \frac{\partial g}{\partial x}(t,x)\alpha(t,x) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x)\left(\sigma(t,x)\right)^2 = 0,$$

for all $t \geq 0$ and $x \in \mathbb{R}$. Show that

$$dg(t,S_t) = \frac{\partial g}{\partial x}(t,S_t)\sigma(t,S_t)\,dB_t.$$

(b) Let $\alpha(t,S_t) = 0.05S_t$ and $\sigma(t,S_t) = 0.15S_t$. Calculate the probability that the 5-year return will be at least 20%.

2. Consider put and call options written on the same non-dividend paying share with price process S_t and with the same expiry date T and the same exercise price K.

(a) Show that for the European put option price, we have that (do not use the put-call parity relationship, just use an appropriate portfolio)

$$p_t \geq Ke^{-r(T-t)} - S_t,$$

$0 \leq t \leq T$.

(b) Assume that the current price of the share is 16.5€, the call option price is 1.2€, the put option price is 0.9€, the exercise price is 17€ and the continuously compounded risk-free interest rate is 4% p.a. Calculate the time to expiry of the options.

(c) What can you say to an investor that wants to exercise an American call option on a non-dividend paying share before the expiry date. Explain your reasons.

3. Consider a 3-period binomial model for the non-dividend paying share with price process S_t such that over each time period the stock price can either move up by 10% or move down by 8%. Assume that the (continuously compounded) risk-free interest rate is 5% per period and that $S_0 = 10$€.

   (a) Construct the binomial tree for the 3-period model and verify if the model is arbitrage free.

   (b) Calculate the price of a derivative with maturity date in 3 periods and with payoff $\max\{S_T^2 - K, 0\}$, where T is the maturity date, $K = 130$€. Assume that $S_0 = 5$€.

4. Consider a portfolio of 20000 European put options written on a share and N shares. Assume that the delta of an individual option is -0.20.

   (a) Explain what are the steps in the 5-step method which can be used to solve the problems of pricing and hedging of derivatives.

   (b) If the portfolio has a delta of zero, calculate the number N of shares in the portfolio.

   (c) Consider the Black-Scholes model and a put option written on a dividend paying share with expiry date 9 months from now, strike price 50€ and current price 45€. Assume that the (continuously compounded) free-risk interest rate is 6% p.a., the volatility is 0.15 and the dividends are payable continuously at the constant rate of 2% p.a. Calculate the price of this option.

5. Consider the zero-coupon bond market.

(a) List the desirable characteristics of a term structure model.

(b) Under the real-world probability measure $\mathbb{P}$, the price of a zero-coupon bond with maturity T is

$$B(t,T) = \exp\left\{-(T-t)r(t) + \frac{\sigma^2}{6}(T-t)^3\right\},$$

where $r(t)$ is the short rate of interest at time t.Derive formulas for the instantaneous forward rate $f(t,T)$, the spot rate $R(t,T)$ and the market price of risk $\gamma(t,T)$ in terms of $r(t)$. In order to derive the formula for $\gamma(t,T)$ assume that

$$dr(t) = \alpha r(t)\,dt + \sigma dZ_t,$$

where $\alpha > 0$ and Z_t is a standard Brownian motion under $\mathbb{P}$.

2.8 Part 4 - Solutions

1. (a) By Itô's lemma (or Itô's formula) applied to $g(t,x)$ (it is a $C^{1,2}$ function):

$$\begin{aligned} dg(t,S_t) &= \frac{\partial g}{\partial t}(t,S_t)dt + \frac{\partial g}{\partial x}(t,S_t)dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,S_t)(dS_t)^2 \\ &= \left[\frac{\partial g}{\partial t}(t,S_t) + \alpha(t,S_t)\frac{\partial g}{\partial x}(t,S_t) + \frac{1}{2}(\sigma(t,S_t))^2\frac{\partial^2 g}{\partial x^2}(t,S_t)\right]dt \\ &+ \sigma(t,S_t)\frac{\partial g}{\partial x}(t,S_t)dB_t \\ &= 0 + \sigma(t,S_t)\frac{\partial g}{\partial x}(t,S_t)dB_t \end{aligned}$$

where we have used $(dB_t)^2 = dt$.

(b) We have

$$dS_t = 0.05S_t dt + 0.15S_t dB_t,$$

which is the SDE of a geometric Brownian motion with $\alpha = 0.05$ and $\sigma = 0.15$. The solution is (it can be obtained by applying the Itô formula to $f(x) = \log(1/x)$)

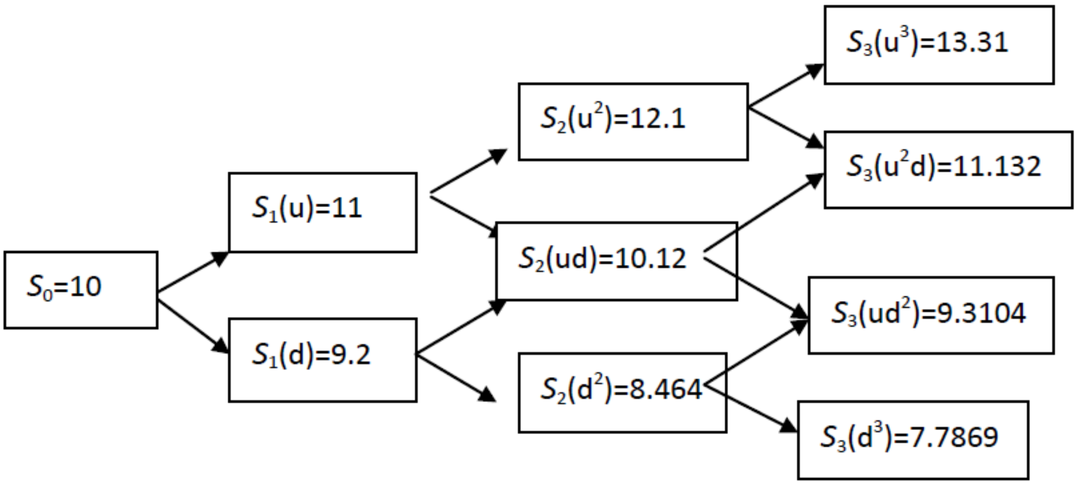
$$\begin{aligned} S_t &= S_0 \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right] \\ &= S_0 \exp\left[\left(0.05 - \frac{1}{2}(0.15)^2\right)t + 0.15B_t\right] \end{aligned}$$

Therefore
$$S_t = S_0 \exp\left[0.03875t + 0.15B_t\right].$$
Since $B_t \sim N(0;t)$, then $\log(S_t) \sim N(\log(S_0) + 0.03875t; 0.0225t)$.
$$\begin{aligned} P\left(\frac{S_5}{S_0} > 1.2\right) &= P\left(\exp\left[0.03875 \times 5 + 0.15B_5\right] > 1.2\right) \\ &= 1 - P\left(Z \leq \log(1.2)\right). \end{aligned}$$
where $Z = 0.19375 + 0.15B_5 \sim N(0.19375; 0.1125)$. Therefore: $P\left(\frac{S_5}{S_0} > 1.2\right) = 0.5136$.

3. (a) At time t, consider the portfolio: one European put + Share S_t and a cash account with value $Ke^{-r(T-t)}$. At time T, the portfolio value is $0 + S_T = S_T > K$ if $S_T > K$. If $S_T < K$ then the payoff from portfolio is $K - S_T + S_T = K$. The cash account at time T has a value of K. Therefore the portfolio payoff $\geq K \Longrightarrow p_t + S_t \geq Ke^{-r(T-t)}$ and we have the lower bound for the price of European put:
$$p_t \geq Ke^{-r(T-t)} - S_t.$$

(b) By the put-call parity:
$$c_t + Ke^{-r(T-t)} = p_t + S_t.$$
Therefore:
$$\begin{aligned} Ke^{-r(T-t)} &= p_t + S_t - c_t \\ (T-t) &= -\frac{1}{r}\log\left(\frac{p_t + S_t - c_t}{K}\right) \end{aligned}$$
and
$$\begin{aligned} (T-t) &= -\frac{1}{0.04}\log\left(\frac{0.9 + 16.5 - 1.2}{17}\right) \\ &= 1.2051 \end{aligned}$$
and the time to expiry is $T - t = 1.2051$ years.

(c) It is never optimal to exercise an american call on a non-dividend paying share early because if we exercise early, the payoff is $S_t - K$, but if we do not exercise, the value of the American call must be at least that



of the European call, i.e., by the lower bound for an European call option, $C_t \geq S_t - Ke^{-r(T-t)} > S_t - K$. So, we would receive more by selling the option than by exercising it.

4. (a) $\frac{S_{t+1}}{S_t} = 1.10$ or $\frac{S_{t+1}}{S_t} = 0.92$. Therefore $u = 1.10$ and $d = 0.92$. $e^r = e^{0.05} = 1.0513$ and we have $d < e^r < u$ and therefore the model is arbitrage free.

Binomial tree:

(b)The risk neutral probability of an up-movement is

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.05} - 0.92}{1.1 - 0.92} = 0.7293.$$

Payoff of the derivative: $C_3 = \max\left\{(S_3)^2 - K, 0\right\}$ with $K = 130$.

Using the usual backward procedure:

$C_3\left(u^3\right) = \max\left\{\left(S_0 u^3\right)^2 - 130, 0\right\} = 47.1561$, $C_3(u^2 d) = \max\left\{\left(S_0 u^2 d\right)^2 - 130, 0\right\} = 0$, $C_3\left(d^2 u\right) = \max\left\{\left(S_0 d^2 u\right)^2 - 130, 0\right\} = 0$ and

$C_3\left(d^3\right) = \max\left\{\left(S_0 d^3\right)^2 - 130, 0\right\} = 0$.

At time 2: $C_2\left(u^2\right) = \exp(-r)\left[qC_3\left(u^3\right) + (1-q)C_3\left(u^2 d\right)\right] = 32.7137$, $C_2\left(ud\right) = 0$, $C_2\left(d^2\right) = 0$.

At time 1: $C_1\left(u\right) = \exp(-r)\left[qC_2\left(u^2\right) + (1-q)C_2\left(ud\right)\right] = 22.6946$, $C_1\left(d\right) = 0$

At time 0, the price is $C_0 = \exp(-r)\left[qC_1\left(u\right) + (1-q)C_1\left(d\right)\right] = 15.7439$.

Or, we can calculate by $C_0 = e^{-3r} q^3 C_3\left(u^3\right) = 15.7439$.

5. (a) 1.Establish the equivalent martingale measure Q under which $D_t = e^{-rt}S_t$ is a martingale.

2. Propose $V_t = e^{-r(T-t)}E_Q\left[X|\mathcal{F}_t\right]$ as the "fair" price of the derivative.

3. Show that $E_t = e^{-rt}V_t = e^{-rT}E_Q\left[X|\mathcal{F}_t\right]$ is a martingale under Q.

4. Use the Martingale representation theorem to construct a hedging strategy (portfolio) (ϕ_t, ψ_t).

5. Show that the hedging strategy (ϕ_t, ψ_t) replicates the derivative payoff at maturity and therefore V_t is the fair price of the derivative at time t.

(b) A portfolio with zero delta means that: (number of put options)×(*delta* of one put) + number of shares (N) = 0, since the delta of a share is 1. Therefore: $20000 \times (-0.20) + N = 0$ and therefore $N = 4000$ shares.

(c)The option price is given by:

$$p_t = Ke^{-r(T-t)}\Phi\left(-d_2\right) - S_te^{-q(T-t)}\Phi\left(-d_1\right).$$

with:

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{45}{50}\right) + \left(0.04 + \frac{0.15^2}{2}\right) \times 0.75}{0.15\sqrt{0.75}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Therefore, $d_1 = -0.51517$, $d_2 = -0.64507$ and

$$\begin{aligned} p_t &= 50e^{-0.06\times0.75}\Phi\left(0.64507\right) - 45e^{-0.02\times0.75}\Phi\left(0.51517\right) \\ &= 4.5102. \end{aligned}$$

6. (a) Desirable characteristics of term structure models:

1. The model should be arbitrage free.

2. Interest rates should be positive.

3. $r(t)$ and other interest rates should be mean-reverting.

4. Computational efficiency: we aim for models which either give rise to simple formulae for bond and option prices or which make it straightforward to compute prices using numerical techniques.

5. The model should reproduce realistic dynamics for the interest rates and bond prices.

6. The model, with appropriate parameter estimates, should fit historical interest-rate data.

7. The model should be easily and accurately calibrated to current market data.

8. The model should be flexible enough to cope properly with a range of derivative contracts.

(b) Spot rate: $R(t,T) = \frac{-1}{T-t} \log B(t,T)$, instantaneous forward rate: $f(t,T) = -\frac{\partial}{\partial T} \log B(t,T)$.Therefore:

$$\begin{aligned} R(t,T) &= r(t) - \frac{\sigma^2}{6}(T-t)^3, \\ f(t,T) &= -\frac{\partial}{\partial T}\left[-(T-t)r(t) + \frac{\sigma^2}{6}(T-t)^3\right] \\ &= r(t) - \frac{\sigma^2}{2}(T-t)^2. \end{aligned}$$

For the market price of risk, $\gamma(t,T) = \frac{m(t,T)-r(t)}{S(t,T)}$, where $dB(t,T) = B(t,T)\,[m(t,T)dt + S(t,T)dZ_t$

By Itô's formula, we have that

$$\begin{aligned} dB(t,T) &= \frac{\partial B(t,T)}{\partial t}dt + \frac{\partial B(t,T)}{\partial r_t}dr(t) + \frac{1}{2}\frac{\partial^2 B(t,T)}{\partial r_t^2}(dr(t))^2 \\ &= B(t,T)\left[(r(t) - \alpha(T-t)r(t))\,dt - \sigma(T-t)dZ_t\right]. \end{aligned}$$

Therefore, $S(t,T) = -\sigma(T-t)$, $m(t,T) = r(t) - \alpha(T-t)r(t)$ and

$$\gamma(t,T) = \frac{r(t) - \alpha(T-t)r(t) - r(t)}{-\sigma(T-t)} = \frac{\alpha}{\sigma}r(t).$$

2.9 Part 5

1. Consider that the discounted share price of a non-dividend paying security is given by the stochastic process

$$\widetilde{S}_t = \exp\left\{-\sigma B_t - \left(\alpha^2 + \sigma^2\right)t\right\}$$

where:

- B_t is a standard Brownian motion under the real world measure P,
- α and σ are positive constants.

(a) Deduce the stochastic differential equation (SDE) satisfied by $\widetilde{S}_t$.

(b) The process $\widetilde{S}_t$ is a martingale under real world measure P? And under the equivalent martingale measure or risk neutral measure Q, what would be the SDE satisfied by $\widetilde{S}_t$?

2. Consider European put and call options written on a dividend paying share.

   (a) Explain how and why the time to expiry, the interest rates and the dividend income received on the underlying security affect the price of the European call and put options.

   (b) By constructing two portfolios with identical payoffs at the exercise date of the options, derive an expression for the put-call parity of European options on a dividend paying share, where the dividend H is known to be payable at some date t^* with $t < t^* < T$. More precisely, prove that

$$c_t + He^{-r(t^*-t)} + Ke^{-r(T-t)} = p_t + S_t.$$

   (c) Consider a call option with price (at time t) given by $c_t = 0.8$, a put option with price $p_t = 0.6$, written on the same underlying share, with time to maturity 15 months and the same strike price 25€. Assume that the current share price is 20€, the continuously compounded risk-free interest rate is 7% p.a. and the share pays a dividend of 1€ at a date 3 months before maturity. Is the put-call relationship satisfied? What can you say about the model used to calculate c_t and p_t?

3. Consider a binomial model for the non-dividend paying share with price process S_t such that the price at time $t+1$ is either $1.15S_t$ or $0.9S_t$ (assume that the time t is measured in years). Assume that the continuously compounded risk-free interest rate is 10% p.a. and that $S_0 = 10$. Consider a derivative D with payoff at time $t = 2$ given by

$$\begin{aligned} c_2(1) &= S_2 - 3.225 \quad \text{if } S_2 = S_0u^2, \\ c_2(2) &= S_2 - 5.35 \quad \text{if } S_2 = S_0ud, \\ c_2(3) &= 0 \quad \text{if } S_2 = S_0d^2, \end{aligned}$$

   where u and d are the sizes of the up-step and down-step in each period.

   (a) Calculate the risk-neutral probability measure and construct the binomial tree.

(b) Calculate the price of the derivative D at time $t = 0$ and describe how you could derive the hedging strategy (i.e., state a general formula for the portfolio composed of the underlying security and the risk free asset required to hedge the derivative security).

4. Consider the Black-Scholes model and a European call option written on a non-dividend paying share with expiry date 15 months from now, strike price 30€ and current price 25€. Assume that the (continuously compounded) free-risk interest rate is 8% p.a. and that the volatility is $\sigma = 0.2$.

   (a) Define the greeks Delta (Δ) and vega (ν) for a general derivative and calculate the delta for the call option (considering the Black-Scholes model).

   (b) Consider that an investor has 10000 call options as defined above. Calculate the corresponding hedging portfolio in shares and cash.

   (c) Consider a derivative Φ that has the following payoff at expiry date T depending on the price of the underlying non-dividend paying share at maturity T and at a previous time $T_0 < T$:

$$Payoff = \frac{S(T)}{S(T_0)}.$$

   Show that the price of the derivative at time t is given by (for $t < T_0 < T$)

$$e^{-r(T-t)}e^{\left(r-\frac{\sigma^2}{2}\right)(T-T_0)}E^Q_{t,s}\left[\exp\left(\sigma\left(Z_T - Z_{T_0}\right)\right)\right] = e^{-r(T_0-t)},$$

   where Z is a standard Brownian motion with respect to the measure Q.

5. Consider the zero-coupon bond market.

   (a) Present the stochastic differential equations (SDE), under the risk neutral measure Q, for the short rate in the Hull-White model and in the 2-factor Vacisek model, defining all the notation used.

   (b) Discuss the main differences and advantages/disadvantages between the Hull-White model and the one-factor Vacisek model

2.10 Part 5 - Solutions

1. (a) By Itô's lemma (or Itô's formula) applied to $f(t,x) = \exp(-\sigma x - (\alpha^2 + \sigma^2)t)$ (it is a $C^{1,2}$ function):

$$\begin{aligned} d\widetilde{S}_t &= \frac{\partial f}{\partial t}(t, B_t)dt + \frac{\partial f}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)\,(dB_t)^2 \\ &= -(\alpha^2 + \sigma^2)\widetilde{S}_t dt - \sigma \widetilde{S}_t dB_t + \frac{1}{2}\sigma^2 \widetilde{S}_t dt \\ &= -\left(\alpha^2 + \frac{\sigma^2}{2}\right)\widetilde{S}_t dt - \sigma \widetilde{S}_t dB_t. \end{aligned}$$

where we have used $(dB_t)^2 = dt$. Therefore

$$d\widetilde{S}_t = -\left(\alpha^2 + \frac{\sigma^2}{2}\right)\widetilde{S}_t dt - \sigma \widetilde{S}_t dB_t.$$

(b) In general, the discounted price process $\widetilde{S}_t$ is not a martingale under the real world probability $\mathbb{P}$. Indeed, since in the SDE above, the drift coefficient $-\left(\alpha^2 + \frac{\sigma^2}{2}\right)\widetilde{S}_t$ is not zero, the process $\widetilde{S}_t$ is not a martingale.

Under the equivalent martingale measure Q, the discounted price process $\widetilde{S}_t$ is a martingale, the drift coefficient is zero and the diffusion coefficient of the SDE remains the same, i.e.

$$d\widetilde{S}_t = -\sigma \widetilde{S}_t d\overline{B}_t,$$

where $\overline{B}_t$ is a standard Brownian motion under Q.

3. (a) The longer the time to expiry, the greater the chance that the underlying share price can move significantly in favour of the holder of the option before expiry. So the value of an option will increase with term to maturity.

Interest rates:

Call option: an increase in the risk-free rate of interest will result in a higher value for the option because the money saved by purchasing the option rather than the underlying share can be invested at this higher rate of interest, thus increasing the value of the option.

Put option: higher interest means a lower value (put options can be purchased as a way of deferring the sale of a share: the money is tied up for longer)

Dividend income:

Call option: the higher the level of dividend income received, the lower is the value of a call option, because by buying the option instead of the underlying share the investor loses this income.

Put option: the higher the level of dividend income received, the higher is the value of a put option, because buying the option is a way of deferring the sale of a share and the dividend income is received.

(b) Let us consider two portfolios. Portfolio A: one European call option + cash $He^{-r(t^*-t)} + Ke^{-r(T-t)}$

Portfolio B: one European put option + one dividend paying share.

At time T, the value of portfolio A is $S_T - K + He^{r(T-t^*)} + K = S_T + He^{r(T-t^*)}$ if $S_T > K$ and $He^{r(T-t^*)} + K$ if $S_T \leq K$.

At time T, the value of portfolio B is $0 + S_T + He^{r(T-t^*)}$ if $S_T > K$ and $K - S_T + S_T + He^{r(T-t^*)} = He^{r(T-t^*)} + K$ if $S_T \leq K$.

Therefore, the portfolios have the same value at maturity. Then, by the no-arbitrage principle, the porfolios have the same value for any time $t < T$, i.e.,

$$c_t + He^{-r(t^*-t)} + Ke^{-r(T-t)} = p_t + S_t.$$

(c) We have that

$$\begin{aligned} c_t + He^{-r(t^*-t)} + Ke^{-r(T-t)} &= 0.8 + e^{-0.07\times 1} + 25e^{-0.07\times\left(\frac{15}{12}\right)} \\ &= 24.638 \end{aligned}$$

and

$$p_t + S_t = 0.6 + 20 = 20.6$$

Therefore, the put-call relationship is not satisfied. This means that the model used to calculate the prices of the options is not arbitrage free.

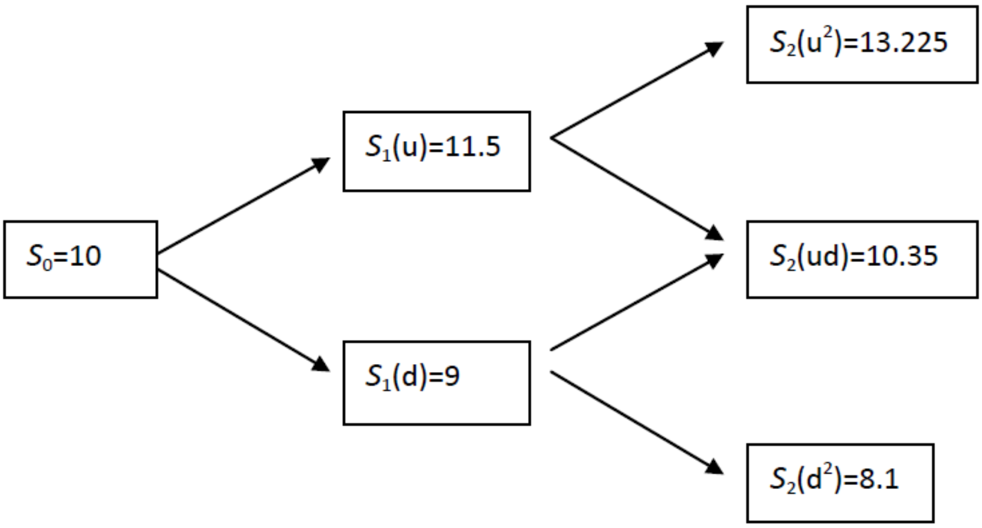
4. (a) $\frac{S_{t+1}}{S_t} = 1.15$ or $\frac{S_{t+1}}{S_t} = 0.9$. Therefore $u = 1.15$ and $d = 0.9$. $e^r = e^{0.10} = 1.1052$ and we have $d < e^r < u$ and therefore the model is arbitrage free.

The risk neutral probability of an up-movement is

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.10} - 0.9}{1.15 - 0.9} = 0.8207.$$

Binomial tree:

(b) Payoff of the derivative: $C_2\left(u^2\right) = S_2 - 3.225 = 10$, $C_2\left(ud\right) = S_2 - 5.35 = 5$, $C_2\left(d^2\right) = 0$.



Using the usual backward procedure:
At time 1: $C_1(u) = \exp(-r)\left[qC_2(u^2) + (1-q)C_2(ud)\right] = 8.2372$,
$C_1(d) = \exp(-r)\left[qC_2(ud^2) + (1-q)C_2(d^2)\right] = 3.713$
At time 0, the price is $C_0 = \exp(-r)\left[qC_1(u) + (1-q)C_1(d)\right] = 6.7193$.

In order to calculate the hedging strategy, we could use the formulas (generalization of the formulas for the one-period model): for time t and state j, we should apply the formulas

$$\phi_{t+1}(j) = \frac{C_{t+1}(ju) - C_{t+1}(jd)}{S_t(j)(u-d)},$$
$$\psi_{t+1}(j) = e^{-r}\left[\frac{C_{t+1}(jd)\,u - C_2(ju)\,d}{u-d}\right].$$

where ϕ represents the units of stock in the portfolio and ψ represents the units of cash.

5. (a) Let $f(t,s)$ be the value at time t of a derivative when the price of the the underlying asset at t is $S_t = s$.

Delta of the derivative and vega:

$$\Delta = \frac{\partial f}{\partial s},$$
$$\nu = \frac{\partial f}{\partial \sigma}.$$

Vega is the rate of change of the price of the derivative with respect to a change in the volatility of S_t. The delta of a call option can be derived from the Black-Scholes formula and is given by $\Delta = \frac{\partial c_t}{\partial S_t} = \Phi(d_1)$, where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = -0.2564$$

and $\Delta = \Phi(-0.2564) = 0.3988$.

(b) The option price is given by:

$$c_t = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2) = 1.4032$$

where $d_1 = -0.2564$ and $d_2 = d_1 - \sigma\sqrt{T-t} = -0.48$. Hence, the hedging portfolio is: $\Delta \times$number of options $= 0.3988 \times 10000 = 3988$ units of stock and $10000 \times 1.4032 - 3988 \times 25 = -85668$€ in cash.

(c) The dynamics of the stock prices S_t under Q is given by the SDE

$$\begin{aligned} dS_u &= r\, S_u\, du + \sigma\, S_u\, dZ_u, \\ S_t &= s \end{aligned}$$

This is a geometric Brownian motion and the solution is such that:

$$\begin{aligned} S_T &= s \exp\left[\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(Z_T - Z_t)\right], \\ S_{T_0} &= s \exp\left[\left(r - \frac{\sigma^2}{2}\right)(T_0 - t) + \sigma(Z_{T_0} - Z_t)\right] \end{aligned}$$

The price of the derivative is given by

$$\begin{aligned} F(t, S_t) &= e^{-r(T-t)} E_{t,s}^{Q}\left[\frac{S_T}{S_{T_0}}\right] \\ &= e^{-r(T-t)} e^{\left(r - \frac{\sigma^2}{2}\right)(T-T_0)} E_{t,s}^{Q}\left[\exp\left(\sigma\left(Z_T - Z_{T_0}\right)\right)\right] \\ &= e^{-r(T-t)} e^{\left(r - \frac{\sigma^2}{2}\right)(T-T_0)} e^{\frac{1}{2}\sigma^2(T-T_0)} = e^{-r(T_0-t)}. \end{aligned}$$

6. (a) The Hull-White model SDE for the short rate $r(t)$ under Q:

$$dr(t) = \alpha(\mu(t) - r(t))\, dt + \sigma d\widetilde{W}_t,$$

where $\widetilde{W}_t$ is a standard Bm under Q, the parameter α is positive and $\mu(t)$ is a deterministic function. In the 2-factor Vasicek model there are two processes: $r(t)$ and $m(t)$, the local mean reversion level:

$$\begin{aligned} dr(t) &= \alpha_r(m(t) - r(t))\, dt + \sigma_{r1} d\widetilde{W}_1(t) + \sigma_{r2} d\widetilde{W}_2(t), \\ dm(t) &= \alpha_m(\mu - m(t))\, dt + \sigma_{m1} d\widetilde{W}_1(t), \end{aligned}$$

where $\widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$ are independent, standard Brownian motions under the risk neutral measure Q.

(b) The SDEs for the Vasicek model gives us a time-homogeneous model. This implies lack of flexibility for pricing related contracts. A simple way to get theoretical prices to match observed market prices is to introduce some elements of time-inhomogeneity into the model. The Hull & White (HW) model does this. This model is similar to Vasicek model but now $\mu(t)$ is no longer a constant. The HW model can even be extended to include a time-varying deterministic $\sigma(t)$. This allows us to calibrate the model to traded option prices as well as zero-coupon bond prices. Moreover, since $\mu(t)$ is deterministic, the HW model is as tractable as the Vasicek model. The HW model suffers from the same drawback as the Vasicek model: interest rates might become negative.