# Models in Finance - part 8

## Master in Actuarial Science

João Guerra

ISEG

# Bounds for option prices - Lower bounds

- European calls: At time $t$, consider portfolio A: one European call + cash $Ke^{-r(T-t)}$.
- At time $T$, value of A is equal to $S_T - K + K = S_T$ if $S_T > K$. If $S_T < K$ then the payoff from portfolio A is $0 + K > S_T$.
- Therefore the portfolio payoff $\geq S_T \implies c_t + Ke^{-r(T-t)} \geq S_t$.
- Lower bound for the price of European call:

$$c_t \geq S_t - Ke^{-r(T-t)}. \tag{1}$$

# Bounds for option prices - Lower bounds

- European puts: At time $t$, consider portfolio B: one European put + Share $S_t$.
- At time $T$, value of B is equal to $0 + S_T = S_T > K$ if $S_T > K$. If $S_T < K$ then the payoff from portfolio B is $K - S_T + S_T = K$.
- Therefore the portfolio payoff $\geq K \Longrightarrow p_t + S_t \geq Ke^{-r(T-t)}$.
- Lower bound for the price of European put:

$$p_t \geq Ke^{-r(T-t)} - S_t. \tag{2}$$

- Exercise: What is the lower bound for a 3-month European put option on a share $X$ if the share price is 95 EUR, the exercise price is 100 EUR and the (continuously compounded) risk-free rate is 12% p.a.

# Bounds for option prices - Lower bounds

- American calls: It is never optimal to exercise an american call on a non-dividend paying share early (surprising result?)
and therefore, the previous relation holds for american calls:

$$C_t \geq S_t - Ke^{-r(T-t)}. \tag{3}$$

- Why?
if we exercise early, the payoff is $S_t - K$, but if we do not exercise, the value of the American call must be at least that of the European call, i.e. $C_t \geq S_t - Ke^{-r(T-t)} > S_t - K$. So, we would receive more by selling the option than by exercising it.

# Bounds for option prices - Lower bounds

- The lower bound for an American put is

$$P_t \geq K - S_t. \tag{4}$$

- For an American put, the early exercise can be optimal.

# Upper bounds

- European call: the payoff $\max\{S_T - K, 0\} < S_T$. Therefore
$$c_t \leq S_t. \quad (5)$$
- European put: the maximum payoff at maturity is $K$ (if $S_T = 0$). Therefore
$$p_t \leq Ke^{-r(T-t)}. \quad (6)$$
- American call:
$$C_t \leq S_t. \quad (7)$$
- Possibility of early exercise of an American put $\Longrightarrow$ complex case (there is no explicit formula for the price of an American put). However, we have:
$$P_t \leq K. \quad (8)$$

# Put-call parity

- Two "natural" portfolios at time $t$:
- A: one call + cash $Ke^{-r(T-t)}$
- B: one put + one share $S_t$
- Note: as always in this chapter, we consider only non-dividend paying shares.

# Put-call parity

- Portfolio A: payoff at $T$:

$$\begin{cases} S_T - K + K = S_T & \text{if } S_T > K \text{ (call option exercised)} \\ 0 + K = K & \text{if } S_T \leq K \text{ (call expires wothless)} \end{cases} \tag{9}$$

- Portfolio B: payoff at $T$:

$$\begin{cases} 0 + S_T = S_T & \text{if } S_T > K \text{ (put expires worthless)} \\ K - S_T + S_T = K & \text{if } S_T \leq K \text{ (put option exercised)} \end{cases} \tag{10}$$

- At expiry $T$, both portfolios have a payoff $\max\{K, S_T\}$.

# Put-call parity

- Now, since the portfolios have the same value at $T$, and the options cannot be exercised before, the portfolios have the same value at any time $t < T$, i.e.

$$c_t + Ke^{-r(T-t)} = p_t + S_t. \tag{11}$$

- This relationship - eq. (11) is known as the put-call parity.
- If the result was not true then this would $\Longrightarrow$ arbitrage opportunity: the failure of put-call parity would allow an investor to trade on calls, cash, puts and shares with a net cost of zero at time 0 and certain profit at time $T$.
- Consequence of put-call parity: having found the value of a European call we can use (11) to find the value of the correponding put (or vice-versa).

# Put-call parity

- Unlike the forward pricing formula, put-call parity does not tell us what $c_t$ and $p_t$ are individually: only the relationship between the two. To calculate values for $c_t$ and $p_t$ we require a model.
- In this chapter, the pricing of derivatives is based upon the principle of no arbitrage.
- No model has been assumed for stock prices. All we have assumed is that we will make use of buy-and-hold investment strategies.
- Any model we propose for pricing derivatives must, therefore, satisfy both put-call parity and the forward-pricing formula. If a model fails one of these simple tests then it is not arbitrage free.

# Put-call parity for dividend-paying securities

- If we consider dividend-paying shares, then the put-call parity relationship is:

$$c_t + Ke^{-r(T-t)} = p_t + S_t e^{-q(T-t)}, \tag{12}$$

where $q$ is the continuously compounded dividend rate (we are assuming that all dividends are reinvested immediately in the same share).