PART II

Theory of Portfolio Management

1.

Portfolio Concepts

1. Portfolio Concepts

- Learning objectives
- Introduction
- Expected return
- Examples
- Risk
- Large Portfolios
- Questions

Learning objectives

- state the objective of modern portfolio theory,
- define the return of an asset,
- compute expected returns for assets and portfolios,
- compute variances of returns for assets and portfolios,
- derive formulas for the variances of portfolios,
- define positive definiteness and use it to identify covariance matrices,
- derive and compute the variance for very large portfolios,
- define and compute semi-variance,

Assumptions

- Our objective in mean variance theory (MVT) or modern portfolio theory (MPT) is to use mathematics to maximize the risk-return trade-off when investing in the markets
- We will generally work across a fixed time-frame.
- We should think of ourselves as a funds manager whose performance is assessed on a yearly basis.
- The funds manager will be given a statement by his/her client or the board stating the required risk-return trade-off and then it is his/her job to achieve it.

What we need to do

This will require us to do various things: at a minimum

1. Define return.
2. Define risk
3. Model asset price movements.
4. Model how investors make their choices.

Measuring return

The return on an asset over a time period is the percentage change in its value.

- A negative return is possible.
- Note that change can occur in multiple fashions.
  - First, the market price of a stock can vary both up and down due to company performance and general market conditions.
  - Second, the stock may pay dividends or any other cash-flows.
- Cash-flows – negative or positive – are always considered as part of the return.

Defining return

Definition

The return on a portfolio is the percentage change in its value taking into account all cash in flows and out flows.

- We are generally interested in the future rather than the past, so the return will normally be uncertain.
- It is therefore expected return that is important rather than actual return.

Expected return in discrete distributions

If we assume that the return, R, follows some probability distribution taking value R_i will probability p_i.

The *expected return* is

$$\mathbb{E}(R) = \bar{R} = \sum p_i R_i.$$

Example: If R has probabilities of taking values as follows

$\frac{1}{3}$	5%
$\frac{1}{6}$	6%
$\frac{1}{2}$	7%

then the expected percentage return is

$$\bar{R} = \frac{1}{3}5\% + \frac{1}{6}6\% + \frac{1}{2}7\% = 6.16\%$$

Expected return for portfolios

- The expectation operator is linear that is

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y) \ ,$$

  where a and b are constants and X and Y are random variables.
- So for a portfolio P consisting of assets A_i, with return R_i, in proportions x_i, we have the expected portfolio return

$$\mathbb{E}(R_P) = \bar{R}_P = \sum_{i=1}^{n} x_i \bar{R}_i.$$

- We can write this as

$$\bar{R}_P = \langle X, \bar{R} \rangle,$$

  with $X = (x_1, \ldots, x_n)'$ and $\bar{R} = (\bar{R}_1, \ldots, \bar{R}_n)'$
- Here $\langle X, Y \rangle$ denotes the inner product of two vectors: $\sum_{i=1}^{n} x_i y_i$.

The trivial solution

- Note that if our sole objective is to maximize expected return, the portfolio selection problem is easy to solve.
  $\Rightarrow$ We simply put as much money as possible into the asset with the highest expected return.
  I.e., the problem reduces to

$$\max_i \bar{R}_i.$$

- The reason that there is some work to the subject is that generally there is a requirement to control risk as well as maximize returns.
- So, we need to define risk.

Simple example with same mean

We look at some very simple examples which will help us to think about risk and return.

Example 1:

We have to choose between two assets:

- Asset A pays € 1 000 000 with 25% probability and pays 0 with 75% probability.
- Asset B however pays € 250 000 with 100% probability.

Which would an investor prefer?

Simple example with same mean

Example 1: solution

- Both assets have the same mean.
- However, B guarantees the mean whereas A involves a great deal of risk.

Generally B would be preferred as it involves no risk.

Simple example: higher mean

Example 2:

We have to choose between two assets.

- Asset A pays € 1 000 000 with 25% probability and pays 0 with 75% probability.
- Asset B however pays € 260 000 with 100% probability.

Which would an investor prefer?

Simple example: higher mean

Example 2: solution

- Asset B has higher mean and lower risk.
- You would have to be very risk loving to prefer A.

Almost all investors prefer B to A.

OBS: Note, however, that if you play roulette or a lottery, then A is the sort of investment you are making. Of course, owning a casino or running a lottery is a different matter and is highly recommended.

Simple example: lower mean

Example 3: We have to choose between two assets.

- Asset A pays € 1 000 000 with 25% probability and pays 0 with 75% probability.
- Asset B however pays €240 000 with 100% probability.

Which would an investor prefer?

Simple example: lower mean

Example 3: solution

- Asset B has lower mean and lower risk.
- Most investors would go for B on the grounds that the extra risk is not worth the money to be gained on average.
- However, some might go for A.
- If one had the opportunity to do many such independent investments then the risk could average out and A would be preferable.

Two risky assets

Example 4: Now suppose we have to risky assets A and B.

- A coin is tossed and A pays 1 on heads and zero otherwise.
- B pays 1 on tails and zero otherwise.
- The two assets are based on the same coin toss.

How much are A and B worth?

Two risky assets

Example 4: solution

- The mean pay-off for each asset is 0.5
- We would expect the value to be lower because of risk-aversion.
- We would also expect the two assets to trade at the same price.
- If consider the portfolio of A and B together then it will be worth 1 always.
- We therefore conclude that the individual assets are worth 0.5 despite risk aversion.

OBS: This illustrates the fact that a risk premium is generally not available for risk that is diversifiable or hedgeable. Whilst we will generally not be able to remove all risk, we will be able to remove some via portfolio diversification.

Defining risk with variance

- There are many ways to define and control risk.
- The first and simplest way is to use variance.
  The variance of a random variable is defined via
  $$\mathrm{Var}(R) = \mathbb{E}((R - \bar{R})^2) = \mathbb{E}(R^2) - \mathbb{E}(R)^2.$$
  The standard deviation is a related measure of risk. It is defined by
  $$\sigma_R = \sqrt{\mathrm{Var}(R)} = (\mathrm{Var}(R))^{\frac{1}{2}}.$$
  It therefore contains the same information as the variance.
- In financial markets, σ_R is called volatility.

Scaling

- The volatility is harder to work with because of the square root, but has the virtue that it has the same scale as the expectation.
- That is we have

$$\begin{aligned}\mathbb{E}(\lambda R) =& \lambda\mathbb{E}(R),\\ \mathrm{Var}(\lambda R) =& \lambda^2\,\mathrm{Var}(R),\\ \sigma_{\lambda R} =& |\lambda|\sigma_R.\end{aligned}$$

for some λ constant.

OBS: Note the important modulus sign $|\cdot|$ in the final equation: standard deviation is always positive.

Portfolio variance

- We will be interested in the variance of portfolios' returns given the variances of individual assets' returns.
- If we have assets with returns $R_1, \ldots, R_n$, held in amounts $x_1, \ldots, x_n$ then we can compute the variance of the portfolio.
- We proceed by direct computation. We want the value of

$$\sigma_P = \mathrm{Var}(R_P) = \mathrm{Var}(\sum_{i=1}^{n} x_i R_i).$$

Portfolio variance

$$\begin{aligned}\sigma_P^2 =& \mathrm{Var}\left(\sum_{i=1}^{n} x_i R_i\right) = \mathbb{E}\left(\left(\sum_{i=1}^{n} x_i R_i - \mathbb{E}\left(\sum_{i=1}^{n} x_i R_i\right)\right)^2\right)\\ =& \mathbb{E}\left(\left(\sum_{i=1}^{n} x_i (R_i - E(R_i))\right)^2\right),\\ =& \mathbb{E}\left(\sum_{i=1}^{n} x_i (R_i - \mathbb{E}(R_i)). \sum_{j=1}^{n} x_j (R_j - \mathbb{E}(R_j))\right),\\ =& \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \mathbb{E}\left[(R_i - \mathbb{E}(R_i))(R_j - \mathbb{E}(R_j))\right]\\ =& \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij}\end{aligned}$$

Variance and covariance

- We define the covariance of R_i and R_j via

$$\sigma_{ij} = \mathrm{Cov}(R_i, R_j) = \mathbb{E}((R_i - \bar{R}_i)(R_j - \bar{R}_j)).$$

Notation: recall $\bar{R}_i = \mathbb{E}(R_i)$.

- So

$$\mathrm{Var}\left(\sum_i x_i R_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i \sigma_{ij} x_j.$$

- If we let V be the variance-covariance matrix of returns

$$V_{ij} = \sigma_{ij} = \mathrm{Cov}(R_i, R_j),$$

we can rewrite the variance of a portfolio as

$$\sigma_P^2 = \mathrm{Var}\left(\sum_i x_i R_i\right) = X'VX.$$

with $X = (x_1, \ldots, x_n)'$ and $\bar{R} = (\bar{R}_1, \ldots, \bar{R}_n)'$.

Positive definiteness

Definition

If V is a symmetric matrix and

$$X'VX \geq 0,$$

for all X V is said to be positive semi-definite. It is said to be positive definite if $X'VX > 0$, for $X \neq 0$.

- So all covariance matrices are positive semi-definite.
- It can be shown that any positive semi-definite matrix is the covariance of some collection of random variables.

Matrix equations

- Recall, we regard X as a vector $(n \times 1)$, and V is a matrix $(n \times n)$ rows.

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad V = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix}$$

  The transpose of X is written X' and has one row and n columns. So,

$$X' = (x_1, x_2, \ldots, x_n) .$$

- The matrix V is size $n \times n$.
- So, in

$$X'VX$$

  we are multiplying a $(1 \times n)$ matrix by a $(n \times n)$ matrix, and then by a $(n \times 1)$ matrix to get a (1×1) matrix, i.e. a scalar (a number).

Simple use of covariance

- Note that the variance of an asset is the covariance of an asset with itself.

$$\sigma_i^2 = \text{Var}(R_i) = \text{Cov}(R_i, R_i) = V_{ii} = \sigma_{ii}$$

- Uncorrelated returns:
  It follows from the portfolio variance formula, that the variance of a portfolio will be the sum of the individual assets *if and only if* the covariance between assets are zero. That is if and only if all returns are uncorrelated.
  In that case, we have

$$\text{Var}\left(\sum x_i R_i\right) = \sum x_i^2 \,\text{Var}\,(R_i) .$$

Independence and correlation

- We can always write the covariance as

$$\sigma_{ij} = \text{Cov}(R_i, R_j) = \sigma_i \sigma_j \rho_{ij},$$

  where ρ_{ij} is the correlation coefficient and is defined in such a way as to make this true.
- Assets' returns will have zero correlation if and only if they have zero covariance.
- One condition that will lead to zero correlation, is the much stronger condition of independence.
- If two returns R_i, R_J, are independent then

$$\mathbb{E}(R_i R_j) = \mathbb{E}(R_i)\mathbb{E}(R_j),$$

  so

$$\sigma_{ij} = \text{Cov}(R_i, R_j) = 0.$$

Variances of large homogeneous portfolios

What happens if we take a large number of independent assets and put the same fraction in each?

- In homogeneous portfolios of n assets, we invest $1/n$ in each asset.
- If they are all independent

$$\text{Var}\left(\sum_{i=1}^{n}\frac{1}{n}R_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(R_i).$$

Q:What happens to the portfolio variance and risk as $n \to \infty$?

Variances of large homogeneous portfolios

- If we assume that $\text{Var}(R_i) \leq C$ for some constant C for all i (i.e. finite return variances) then we have

$$\text{Var}\left(\sum_{i=1}^{n}\frac{1}{n}R_i\right) \leq \frac{C}{n},$$

  as n goes to infinity the variance will go to zero.
  OBS: This says that given enough independent *assets, we can achieve an arbitrarily small amount of risk.*

- The expected return on our almost riskless portfolio will be the average of the returns on the individual assets. *Q:Why?*

Diversification with residual variance

- What if we allow covariances to be non-zero?
- Then, we get

$$\begin{aligned}\text{Var}\left(\frac{1}{n}\sum R_i\right) &= \frac{1}{n^2}\sum \text{Var}(R_i) + \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j<i}\text{Cov}(R_i,R_j)\\ &= \frac{1}{n}\overline{\text{Var}(R_i)} + \frac{n-1}{n}\overline{\text{Cov}(R_i,R_j)}\\ &= \frac{1}{n}\overline{\sigma_i} + \frac{n-1}{n}\overline{\sigma_{ij}}\end{aligned}$$

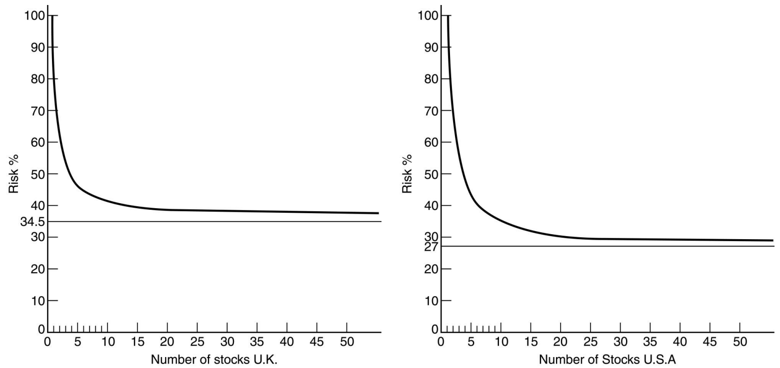
- Letting n tend to infinity this will converge to

$$\overline{\sigma_{ij}} = \overline{\text{Cov}(R_i,R_j)}.$$

- Thus by taking equal proportions of a large number of assets, we obtain a portfolio whose variance is the average covariance of the assets in the pool.

Variances of large homogeneous portfolios

Illustration:



The background covariance in a pool of assets affects how much risk we can diversify away.

Semi-variance

- Variance can be criticized for penalizing upside volatility as well as down-side volatility.
- We generally only care about our possibility of loss, not our possibility of gaining a lot extra.
- We can define the semi-variance of a variable R via

$$\mathbb{E}\left[(R - \mathbb{E}(R))^2 I_{R<\mu}\right].$$

Note the indicator function $I_{R<\mu}$ equals 1 for $R < \mu$ and 0 otherwise.

OBS: Here, we will stick to cases where X is reasonably symmetric and then the semi-variance will not give much beyond the variance and so we will not study it further.

Theory questions

1. What is the objective of modern portfolio theory?
2. How is return defined in MPT?
3. How is expected return defined?
4. How do we maximize return if there is no risk constraints?
5. Derive the formula for the variance of returns of a portfolio.
6. What is a covariance matrix?
7. What special properties does a covariance matrix have?
8. Derive the formula for the variance of return of a large pool of correlated assets.
9. Define semi-variance.