2.

Mean-Variance Theory (MVT)

2. Mean-Variance Theory (MVT)

- Mean-variance Approach
- The two-assets case
- Including the riskless asset
- Two risky assets with the riskless asset
- The general case
- Safety Criteria
- International Diversification

2.1 Mean-Variance Approach

- Learning Objectives
- Defining Mean-Variance Analysis
- Efficient Portfolios
- Investment Opportunity Set

Learning objectives

- define mean-variance efficiency,
- distinguish between the opportunity set and efficient frontier,

Mean variance analysis

MVT has its flaws, but it provides a good starting point for portfolio theory. It is still the most commonly used tool for portfolio construction.

MVT assumptions

1. Investors only care about the mean and variance of future returns.
   - Investors prefer higher means to lower means.
   - Investors prefer lower variances to higher variances.
2. We know the means, variances and covariances of future returns for the assets we can invest.

Investor Preferences

On Assumption 1:

- Clearly, in general, investors do not care only about mean and variance of future returns.
- Most investor worry about bad outcomes, i.e. also care about the left tail of return distributions. Investors may want to impose, for instance, safety restrictions.
- Most investor preferences cannot be represented just in terms of mean and variance of returns.
- However assumption 1 gets to be automatically satisfied if:
  - If investor with quadratic utility function (or approximately); OR
  - The mean and variance are sufficient statistics to the future return distribution.

MVT estimation of inputs

On Assumption 2:

- In a world with n risky assets. The MVT inputs are a vector of future expected returns.

$$\bar{R} = \begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \vdots \\ \bar{R}_n \end{pmatrix}$$

and the future variance-covariance matrix (recall $V_{ij} = \sigma_{ij}$)

$$V = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix}$$

- They are not things we can observe in the market.
- Historical data only tell us about past realised returns, not future returns .

Efficient portfolios

Definition (Efficiency)

A portfolio is *efficient* provided

- No other portfolio has at least as much expected return and lower standard deviation, and
- No other portfolio has higher expected return and standard deviation which is smaller or equal.

- So no other portfolio does at least as well on both risk and return, and better on one of them.
- We get the same set of portfolios if we replace standard deviation by variance in the definition.
- MVT Assumption 1 implies investors only want efficient portfolios.

Investment Opportunity set

Definition (Investment Opportunity Set)

The set of all possible pairs of standard deviations and returns attainable from investing in a collection of assets is called the *investment opportunity set*.

OBS: An efficient asset is therefore in the edge of the opportunity set. If it would not be in the edge, there would be a direction that gives better return or risk.

Definition (Efficient Frontier)

The subset of the investment opportunity set which is efficient is called the *efficient frontier*.

Efficiency and opportunities

- It is important to realize that efficiency is only defined relative to a set of investment opportunities.
- If we change the set of assets which the investor can put his money into then the set of efficient portfolios changes too.
- In general, if we allow an extra asset then portfolios that were previously efficient are no longer efficient.
- Similarly, if we throw away an asset both from the set of investment opportunities and from an efficient portfolio, then the portfolio containing the remaining assets may not be efficient.

2.2 The two-assets case

- Learning Objectives
- Graphs
- Geometry
- Questions

Learning objectives

- Derive and sketch the opportunity set and efficient frontier for two assets for a variety of correlations with and without short selling,
- find the minimum variance combination of two assets,
- state what sort of curve the opportunity set takes,
- discuss convexity in the context of efficiency.

The two assets case

- Let us consider there are only two assets C and S, both are risky.
- All possible combinations of C and S consist of investing fractions x_S and x_C such that
$$x_S + x_C = 1.$$
- Because we can always write $x_C = 1 - x_S$, the return of all portfolios can be described in terms of only the variable which is the fraction of investments put into C
$$R_P = x_c R_c + \underbrace{(1 - x_c)}_{x_s} R_s$$
- This means that the opportunity set will be one dimensional.

The two assets case: mean and variance

- The expected return of any combination of C and S can be written as
$$\begin{aligned}\bar{R}_P = \mathbb{E}(R_P) =& x_C \mathbb{E}(R_C) + (1 - x_C)\mathbb{E}(R_S), \\ =& x_C(\mathbb{E}(R_C) - \mathbb{E}(R_S)) + \mathbb{E}(R_S) \\ =& x_C(\bar{R}_C - \bar{R}_S) + \bar{R}_S \ .\end{aligned}$$
- The variance of any combination of C and S can be written as
$$\sigma_P^2 = \mathbb{E}\left[(R_p - \bar{R}_p)^2\right] = x_C^2\sigma_C^2 + (1 - x_C)^2\sigma_S^2 + 2x_C(1 - x_C)\sigma_{CS}$$

OBS: The expected return is linear in x_C whilst the variance is quadratic.

Shape

- To represent all combinations of C and S in the space $(\sigma, \bar{R})$.
- We need to solve
$$\begin{cases}\bar{R}_P = x_C(\bar{R}_C - \bar{R}_S) + \bar{R}_S \\ \sigma_P^2 = x_C^2\sigma_C^2 + (1 - x_C)^2\sigma_S^2 + 2x_C(1 - x_C)\sigma_{CS}\end{cases}$$
- We can solve for x_C in terms of the expected return, provided the two assets have different expected returns, and we get x_C is a linear function on the expected return $\bar{R}_p$:
$$x_C = \frac{\bar{R}_P - \bar{R}_S}{\bar{R}_C - \bar{R}_S}.$$
- If one substitutes this back into the expression for variance, one gets the investment opportunity curve
$$\sigma_P^2 = \alpha\bar{R}_P^2 + \beta\bar{R}_P + \gamma$$
for some constants α, β and γ that depend only on the MVT inputs.

Minimal variance portfolio

- The variance of all combinations of C and S is given by
$$\sigma_P^2 = x_C^2\sigma_C^2 + (1 - x_C)^2\sigma_S^2 + 2x_C(1 - x_C)\sigma_{CS}$$
- To find the minimum variance portfolio (MV) we can solve
$$\frac{\partial \sigma_P^2}{\partial x_c} = 0$$
to find the value of x_C^* that gives least variance,
$$x_C^{MV} = \frac{\sigma_S^2 - \sigma_{CS}}{\sigma_C^2 + \sigma_S^2 - 2\sigma_{CS}} = \frac{\sigma_S^2 - \sigma_C\sigma_S\rho_{CS}}{\sigma_C^2 + \sigma_S^2 - 2\sigma_C\sigma_S\rho_{CS}}.$$
- In the two risky assets case, the portfolio of minimal variance will always be efficient.

Special cases

- Having done a little work on the general case for two assets, we now study some special cases in order to develop some intuition.
- For illustration purposes we consider the following parameters

$$\begin{aligned}\sigma_S &= 15\%,\\ \sigma_C &= 10\%,\\ \bar{R}_S &= 6\%,\\ \bar{R}_C &= 5\%.\end{aligned}$$

- What will change across cases is the correlation parameter ρ_{CS}.

$\rho = 1$: two perfectly correlated assets

- If the assets are perfectly correlated, i.e. $\rho_{CS} = 1$
- The variance of any combination is

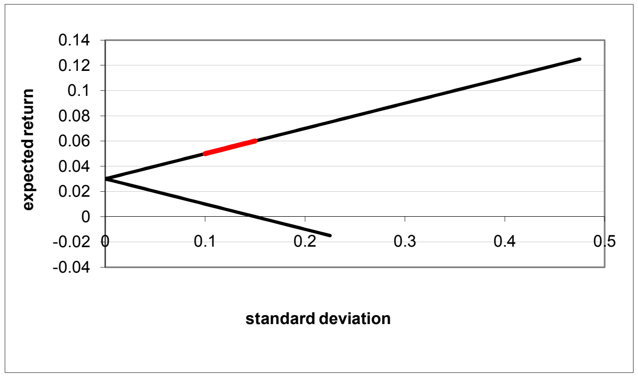
$$\begin{aligned}\sigma_P^2 &= x_C^2\sigma_C^2 + (1-x_C)^2\sigma_S^2 + 2x_C(1-x_C)\sigma_C\sigma_S,\\ &= (x_C\sigma_C + (1-x_C)\sigma_S)^2,\end{aligned}$$

 which implies

$$\sigma_P = |x_C\sigma_C + (1-x_C)\sigma_S|$$

 The opportunity set is describe by two a straight lines in $(\sigma, \bar{R})$ space, reflecting at the zero volatility axis, since standard deviation is always positive.
- If shortselling is not allowed, there is no risk reduction arising from diversification, because in this case both the volatility and expected returns are linear functions of x_C.

$\rho = 1$: investment opportunity curve in $(\sigma, \bar{R})$



- The red portion is the opportunity curve without shortselling.

$\rho = -1$: perfect negative correlation

- Suppose we have $\rho_{CS} = -1$.
- We have

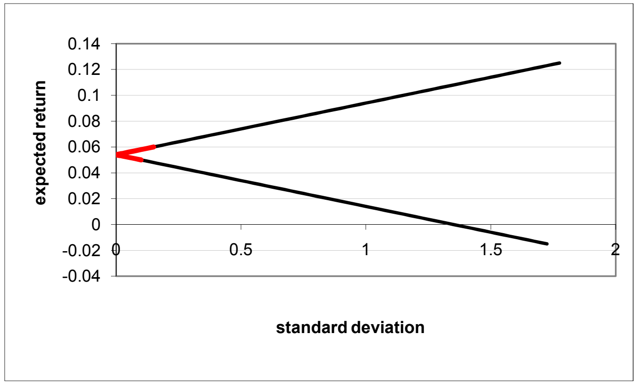
$$\begin{aligned}\sigma_P^2 &= x_C^2\sigma_C^2 + (1-x_C)^2\sigma_S^2 - 2x_C(1-x_C)\sigma_C\sigma_S,\\ &= (x_C\sigma_C - (1-x_C)\sigma_S)^2.\end{aligned}$$

 This implies

$$\sigma_P = |x_C\sigma_C - (1-x_C)\sigma_S|\,.$$

- In this case, even without assuming shortselling positions, there will be a point where the two pieces of risk cancel each other out and we obtain zero risk.

$\rho = -1$: investment opportunity curve in $(\sigma, \bar{R})$



- The red portion is the opportunity curve without shortselling.

$\rho = 0$: uncorrelated assets

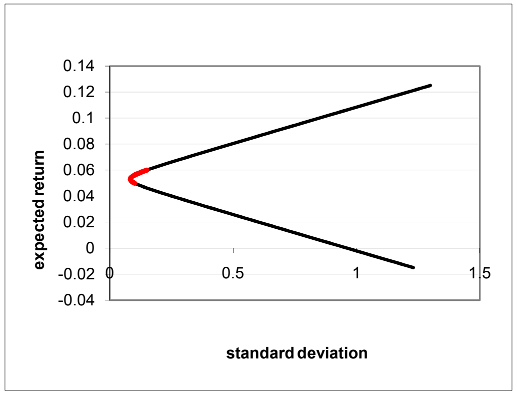
- We now consider the case where $\rho_{CS} = 0$.
- This corresponds to taking the length of the sum of two orthogonal vectors.

$$\sigma_P^2 = x_C^2\sigma_C^2 + (1 - x_C)^2\sigma_S^2.$$

- This time we get a curve – the hyperbola.
- The minimum variance portfolio is given by

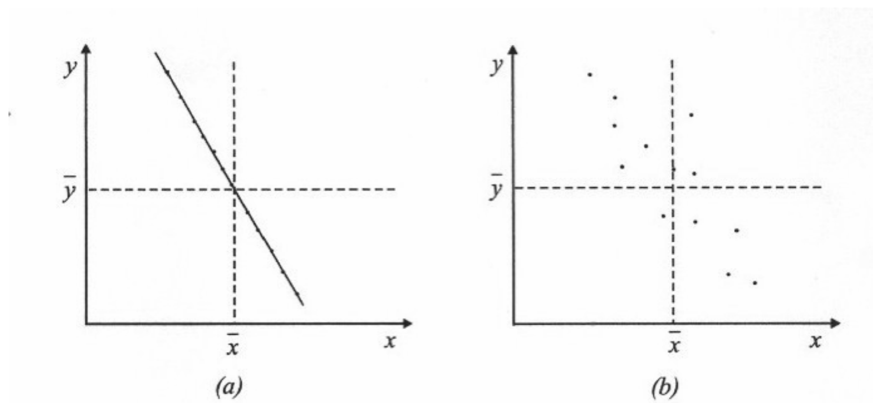
$$x_C^{MV} = \frac{\sigma_S^2}{\sigma_C^2 + \sigma_S^2}.$$

$\rho = 0$: investment opportunity curve in $(\sigma, \bar{R})$

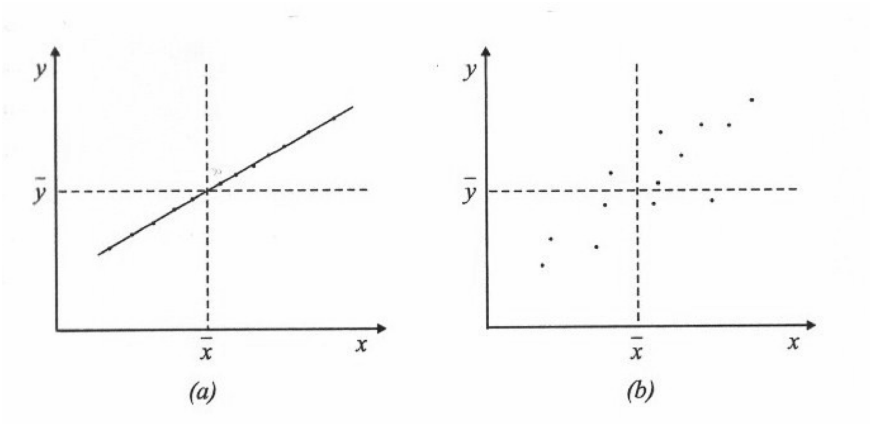


- The red portion is the opportunity curve without shortselling.

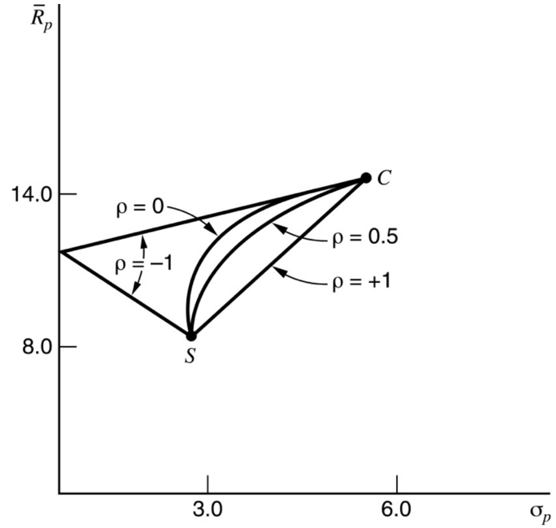
Real-life negative correlations



Real-life positive correlations



Real-life two-assets investment curves (no shortselling)



Understanding the curve

Consider two risky assets ($n = 2$).

- We saw that, in general, the investment opportunity set is a *hyperbola*.

$$\sigma_P^2 = \alpha \bar{R}_P^2 + \beta \bar{R}_P + \gamma$$

  for some constants α, β and γ that depend only on the MVT inputs.
- Recall

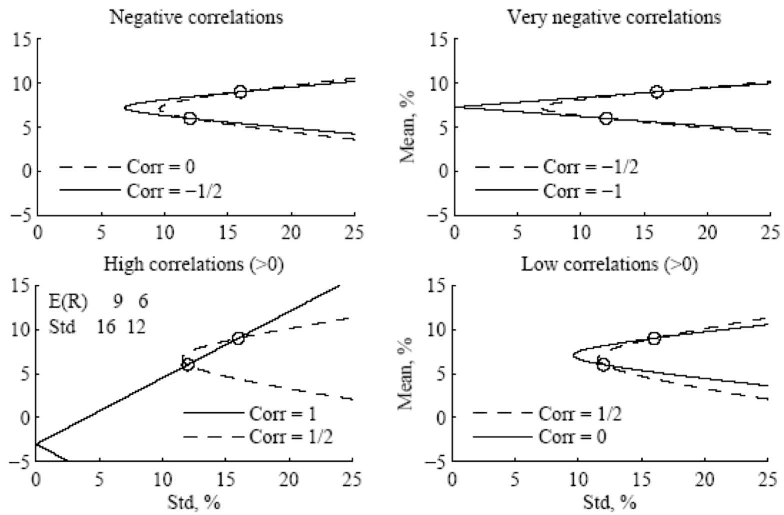
$$\begin{cases} \bar{R}_P = x_C(\bar{R}_C - \bar{R}_S) + \bar{R}_S \\ \sigma_P^2 = x_C^2\sigma_C^2 + (1 - x_C)^2\sigma_S^2 + 2x_C(1 - x_C)\sigma_{CS} \end{cases}$$

HW: Find out the mathematical expressions for α, β and γ.

Convexity

- A function is said to be convex if the chord between any two points lies above the graph.
- A function is said to be concave if the chord between two points lies below the graph.
- Hyperbolas are concave above the turning point.
- Straight lines are trivially both convex and concave.
- Hence, the efficient frontier of two assets is concave.
- Note the area below the turning point is not concave and not efficient.

Real-life two-assets investment curves



Theory questions

1. What are the assumptions of mean-variance portfolio theory?
2. What does it mean for an asset to be mean-variance efficient?
3. Define the opportunity set and efficient frontier in mean-variance analysis? How do they relate to each other?
4. If a portfolio is efficient and we add in a new asset to invest in will the original portfolio remain efficient?
5. If a portfolio is efficient and we discard one of its elements from the set of possible assets, will the part of the portfolio excluding this element always be efficient?
6. Derive expressions for the variance and expected returns of a portfolio of two assets in terms of the amount invested in the first, their expected returns, their variances and covariance.

Theory questions

7. Describe the shape of the graph of expected return of two asset portfolios as a function of the investment fraction.
8. Describe the shape of the graph of standard deviation of two asset portfolios as a function of the expected return.
9. Derive an expression for the composition of the minimal variance portfolio for possible investment assets.
10. Will the minimal variance portfolio always be efficient?
11. What does it mean for a graph to be concave? Convex? Is the efficient frontier convex and/or concave in general?

2.3 Including the riskless asset

- Recap
- Learning objectives
- The risk-free asset
- The investment opportunity set
- The efficient frontier

Recap

Up to now, we studied efficiency for two risky assets:

- The investment opportunity set is a hyperbola in $(\sigma, \bar{R})$ space.
- The efficient frontier is the upper part of the hyperbola in $(\sigma, \bar{R})$. space.
- Expected return of any combinations is linear in the portfolio weight of one of the assets.
- Variance of any combination is quadratic in the portfolio weight of one of the assets
- Variance of any combination of the two assets is quadratic in its own expected return.

Learning objectives

- define a riskless asset,
- identify the opportunity set with a riskless asset and one risky asset,
- define and compute the market price of risk,
- prove that an efficient portfolio containing a riskless asset remains efficient after the riskless asset has been discarded,
- sketch the efficient set when there is a riskless asset,
- state and prove the Tobin separation theorem,
- discuss why tangency is required for optimality,
- show how the investment line with three assets meets the opportunity set with two assets.

Risk-free asset

Definition

An asset whose return is known in advance is said to be risk-free. An asset, F, is *risk-free* if and only if

- The variance of its returns is zero ($\sigma_f^2 = 0$)
- $\Rightarrow$ The standard deviation of returns is zero ($\sigma_f = 0$).

Result:

All risk-free assets have the same return.

If there would be two risk-free assets with different returns, then everyone would sell the risk-free asset with lower return and buy the one with higher return until the returns agreed.

The opportunity set with a risk-free asset

Consider one risk-free asset F with return R_f and one risky asset (or portfolio) A with expected return $\bar{R}_A$ and volatility σ_A.

- If our portfolio P is $1-x$ units of the risk-free asset, F and x units of some risky asset A,
- its expected return is

$$\bar{R}_P = (1-x)R_f + x\bar{R}_A \ ,$$

- its variance is given by

$$\begin{aligned}\mathrm{Var} R_p =& \mathrm{Var}(xR_A),\\ \sigma_P^2 =& x^2\sigma_A^2.\end{aligned}$$

- So, taking square roots.

$$\sigma_P = |x|\sigma_A.$$

The investment line

- If we restrict to $x \geq 0$ (positive investment risky asset), we have from the previous slide
$$x = \frac{\sigma_P}{\sigma_A}.$$
i.e., the investment fraction in A is the ratio of the portfolio's standard deviation to the risky asset's standard deviation.
- This implies
$$\bar{R}_P = \left(1 - \frac{\sigma_P}{\sigma_A}\right) R_f + \frac{\sigma_P}{\sigma_A}\bar{R}_A,$$
- Hence all combinations of the risk-free asset F with the risky asset A are represented by a straight line
$$\bar{R}_p = R_f + \frac{\bar{R}_A - R_f}{\sigma_A}\sigma_P.$$

Q: What would mean an $x < 0$?
How can that be represented graphically? Is that ever efficient?

Interpreting the line

- All the efficient combinations of a riskless asset and a risky asset are
$$\bar{R}_P = R_f + \frac{\bar{R}_A - R_f}{\sigma_A}\sigma_P.$$
which is a straight line in the space $(\sigma, \bar{R})$.
- The line goes trough F and A.
- Its y-cross is R_f.
- Its slope
$$\theta_A = \frac{\bar{R}_A - R_f}{\sigma_A},$$
is the market price of risk of asset A, and it represents the excess expected return per unit of risk of the risky asset A.
- θ_A is also know as the Sharpe ratio of asset A.

Example of market price of risk

If we have
$$R_f = 3\%,$$
$$\sigma_A = 12\%,$$
$$\bar{R}_A = 12\%$$

The market price of risk for A is
$$\theta_A = \frac{12\% - 3\%}{12\%} = \frac{3}{4} = 0.75$$

Q: Consider another risky asset B has $\bar{R}_B = 15\%$, and $\sigma_B = 6\%$.
What is the market price of risk for B?
In which asset would you rather invest? Why?

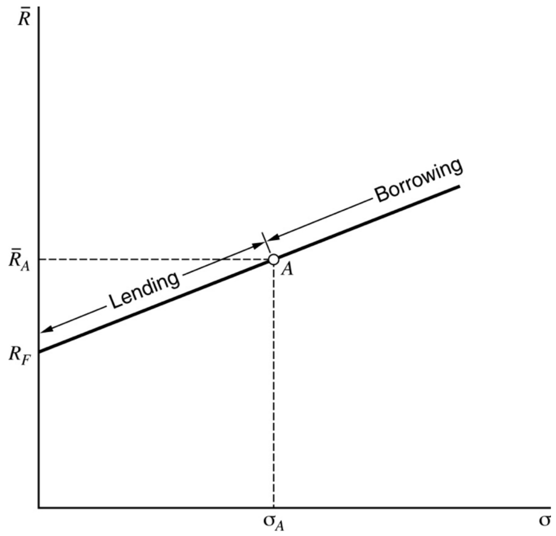
Interpreting the line

We can interpret the line as follows:

- Between the two points F and A, i.e. desired risk levels between $[0, \sigma_A]$, we are dividing our portfolio into the risk-free asset and the risky-asset.
- Above the risky point A, we are short-selling the risk-free asset, and putting more money into the risky asset, to be able to reach volatilities above σ_A.

Q: What exactly does it mean to shortsell the risk-free asset?

Interpreting the line



Different active and passive riskless rates

In real life lending rates (passive rates) and borrowing rates (active rates) are not the same:

- Riskless borrowing is only possible if you are a government with an impeccable credit rating. $\Rightarrow$ Only then can one talk about buying or shorting the a risk-less asset at a constant rate R_f.
- In general, since shorting the risk-less asset is the same as borrowing, the borrower is taking the risk that you will not pay back the money (credit risk). He may
  - not give a loan for investment in risky assets.
  - give a loan, but demand a risk-premium, making the active rate higher than the passive rate $R_f^a > R_f^p$.

HW: How would this reality change the previously derived investment opportunity line?

2.4 Two risky assets with the riskless asset

- Recap
- Learning Objectives
- Efficiency Results
- Tangent Portfolios
- Examples
- Borrowing restrictions
- Questions

Recap

Up to now we have considered ...

- Combinations of two risky assets (with and without shortselling allowed)
- Combinations of the riskless asset with one risky asset. (even when the lending and borrowing rates differ)

Now it is time to ...

- Consider two risky-assets and one riskless asset (already 3 assets).
- For reasons that will become clear later, this is a reference situation .
- Even the general case (with n risky assets), in the end, reduce to this one.

Learning objectives

- Be able to use graphs and formulas to find the minimal variance portfolio, tangent portfolio and efficient frontiers.
- Sketch and derive the efficient frontier with two risky assets and one riskless asset with:
  - equal lending and borrowing rates => Scenario 1
  - equal lending and borrowing rates, but no shortselling => Scenario 2
  - one riskless asset, but no borrowing => Scenario 3
  - different different borrowing and lending rates => Scenario 4
- Be able to find efficient portfolios for a specified level of expected returns, under different market scenarios.
- Understand how different shortselling restrictions impact the investment opportunity sets and efficient frontiers.

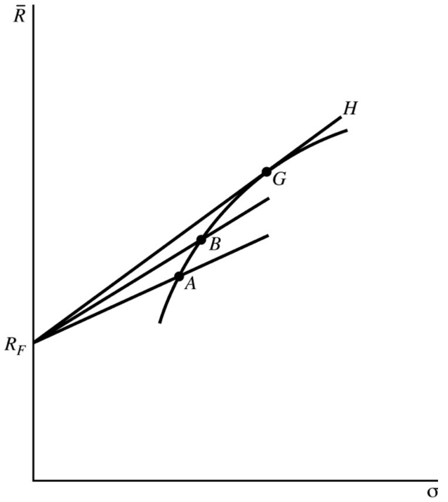
Scenario 1: two risky assets and one R_f

Scenario1

Consider two risky assets A and B with some $\bar{R}_A$, $\bar{R}_B$, σ_A, σ_B and σ_{AB}, and a riskless asset F with return R_f, that can be used to both lending and borrowing.

- Combinations of two risky assets give us an hyperbola.
- Combining a riskless asset with any risky portfolio, give us straight lines passing through F and the risky portfolio.

Scenario 1: Illustration



Scenario 1: Illustration

- From the previous picture, we see the efficient frontier must be the straight line going through the riskless asset F and the risky portfolio G.
- Combining F with any other combination of just risky assets is not efficient.
- G can be characterised as the combination of risky assets with the highest slope. It is also the portfolio tangent to the hyperbola – so it is common to denote it by T (instead of G) to highlight that fact.

Tasks:

1. What can you say about the investment opportunity set?
2. Find the so-called tangent portfolio T. Compute its expected return, $\bar{R}_T$ and volatility σ_T.
3. Write down the efficient frontier equation and interpret it.

Scenario 1: the efficient set is a straight line

- We showed that if we took a portfolio of risky assets T, then by combining with the risk-free asset F we got a straight line.
- So, if we have an efficient portfolio P

$$P = x_f F + (1 - x_f) T,$$

then the entire line through the points $(0, R_f)$ and $(\sigma_T, \bar{R}_T)$ including P can be interpreted as combinations of T and F.

- In fact, this entire line is efficient.

Theorem (Investment line)

If there is a risk-free asset, all efficient portfolios lie on a straight line in $(\sigma, \bar{R})$ space.

Proof.

- If we had an efficient portfolio not on this line $\Rightarrow$ the entire line through it and $(0, R_f)$ would also be in the investment opportunity set.
- Two straight lines through R_f will:
  - either be the same, or
  - one will be below the other for all $\sigma_P > 0$.
- In the segund case, this means none of the portfolios on the longer line is efficient.
- So, the two lines must be the same. ∎

Scenario 1: Investors

- All efficient portfolios can be obtained as a mixture of a single portfolio of risky assets T and the risk-free asset F.
- So all mean-variance investors with the same investment situation, will hold the same portfolio of risky assets T, but may hold differing proportions of T.
- The crucial point here is that the investors have to have the same views (expectations) in terms of the MVT inputs (expected returns, variances and covariances).
- But, they don't have to have the same risk preferences.

Scenario 1: Tobin separation theorem

Theorem (Tobin separation theorem)

Two mean-variance investors facing the same investment situation will hold the same portfolio of risky assets (uniqueness).

- Note that up to now we have not proven that the tangent portfolio T is unique – i.e. that it is the only combination of just risky assets that is efficient.
- However, T will be generally unique (think graphically).

HW: One exception occurs when it is possible to make a risk-free asset from a combination of risky assets. Think about this situation. When can it happen? What can you conclude about the efficient frontier in that case?

Scenario 1: investment opportunity set

1.
- It is the cone with vertice at the riskless asset F and with limiting lines tangent to the hyperbola that is the investment opportunity set of just risky investments.
- Because you can both lend and borrow at the same rate R_f it is an open set.
- Sketch the investment opportunity set:

Scenario 1: efficient frontier shape and portfolio T

- So, if we have a risk-free asset, F, with return R_f, and two risky assets.
- And additionally, we know portfolio E is efficient.
- Then, from before, we know we can write E in the form

$$E = x_f F + (1 - x_f)T,$$

for some portfolio of risky assets, T.
- Thus, we just need to know how to find the tangent portfolio T.

Q: Why? What can you say about T?

Scenario 1: finding T

2. We know T is the portfolio with the highest Sharpe ratio.
So, it must be the portfolio P that solves the following optimisation problem

$$\max_{x_A, x_B} \quad \theta = \frac{\bar{R}_p - R_f}{\sigma_p}$$

s.t.

$$\bar{R}_p = x_A \bar{R}_A + x_B \bar{R}_B$$
$$\sigma_p = \left(x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}\right)^{\frac{1}{2}}$$
$$x_A + x_B = 1$$

Scenario 1: finding T

2. Including the restrictions:
   - We can substitute the expressions for $\bar{R}_P$ and σ_P in the objective function.
   - Also, we have $R_f = (x_A + x_B) R_f$, because $x_A + x_B = 1$.

So, the original problem is equivalent to the following unrestricted problem

$$\max_{x_A, x_B} \quad \theta = \frac{x_A \bar{R}_A + x_B \bar{R}_B - R_f(x_A + x_B)}{\left(x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}\right)^{\frac{1}{2}}}$$

And to get the optimal, we need to solve the FOC:

$$\begin{cases} \frac{\partial \theta}{\partial x_A} = 0 \\ \frac{\partial \theta}{\partial x_B} = 0 \end{cases}$$

Scenario 1: finding T

2

- Note the objective function is symmetric in A and B. So we can solve for A (and it will be similar for B)

$$\frac{\partial\theta}{\partial x_A} = \frac{(\bar{R}_A - R_f)\,\sigma_p - \frac{1}{2}\sigma_p^{-1}\left(2x_A\sigma_A^2 + 2x_B\sigma_{AB}\right)\left(\bar{R}_p - R_f(x_A + x_B)\right)}{\sigma_p^2}$$

where the blue terms depend on x_A and x_B.

- By setting $\frac{\partial\theta}{\partial x_A} = 0$, and recalling $x_A + x_B = 1$, we get

$$\frac{(\bar{R}_A - R_f)\,\sigma_p - \frac{1}{2}\sigma_p^{-1}\left(2x_A\sigma_A^2 + 2x_B\sigma_{AB}\right)\left(\bar{R}_p - R_f\right)}{\sigma_p^2} = 0$$

$$(\bar{R}_A - R_f)\,\sigma_p - \left(x_A\sigma_A^2 + x_B\sigma_{AB}\right)\frac{\bar{R}_p - R_f}{\sigma_p} = 0$$

$$(\bar{R}_A - R_f) - \frac{\bar{R}_p - R_f}{\sigma_p^2}\left(x_A\sigma_A^2 + x_B\sigma_{AB}\right) = 0$$

Scenario 1: finding T

2

- For any concrete portfolio P (including the optimal/tangent portfolio) the ratio $(\bar{R}_p - R_f)/\sigma_p^2$ is a constant.
- So we can define $\lambda = (\bar{R}_p - R_f)/\sigma_p^2$ and do a change of variable

$$z_i = \lambda x_i\ , \qquad \text{for } i = A, B\ ,$$

where the z values are proportional to x, but do not add up to 1.

- Using the variable z, $\frac{\partial\theta}{\partial x_A} = 0$ becomes

$$\begin{aligned}(\bar{R}_A - R_f) - \left(z_A\sigma_A^2 + z_B\sigma_{AB}\right) &= 0\\ \bar{R}_A - R_f &= z_A\sigma_A^2 + z_B\sigma_{AB}\end{aligned}$$

- By symmetry, solving the FOC is equivalent to

$$\begin{cases}\bar{R}_A - R_f = z_A\sigma_A^2 + z_B\sigma_{AB}\\ \bar{R}_B - R_f = z_A\sigma_{AB}^2 + z_B\sigma_B^2\end{cases}$$

Scenario 1: finding T

2

- In vector notation is

$$\begin{pmatrix}\bar{R}_A - R_f\\ \bar{R}_B - R_f\end{pmatrix} = \begin{pmatrix}\sigma_A^2 & \sigma_{AB}\\ \sigma_{AB} & \sigma_B^2\end{pmatrix}\begin{pmatrix}z_A\\ z_B\end{pmatrix}$$

- Denoting $\tilde{R} = \bar{R} - R_f$, we can write the solution as

$$\tilde{R} = VZ \quad \Leftrightarrow \quad Z = V^{-1}\tilde{R}$$

- Since $Z = \lambda X$, the tangent portfolio weights follow from

$$x_A^T = \frac{z_A}{z_A + z_B} \qquad \text{and} \qquad x_B^T = \frac{z_B}{z_A + z_B}\ ,$$

and we finally reached the composition of the tangent portfolio T,

$$X_T = \begin{pmatrix}x_A^T\\ x_B^T\end{pmatrix}$$

Scenario 1: efficient frontier

3

- The tangent portfolio T, expected return and variance are give by

$$\begin{aligned}\bar{R}_T &= x_A^T\bar{R}_A + x_B^T\bar{R}_B\\ \bar{R}_T &= X_T'\bar{R}\end{aligned}$$

$$\begin{aligned}\sigma_T^2 &= (x_A^T)^2\sigma_A^2 + (x_B^T)^2\sigma_B^2 + 2x_A^T x_B^T\sigma_{AB}\\ \sigma_T^2 &= X_T' V X_T\end{aligned}$$

- The efficient frontier contains only combinations of the riskless asset F with the tangent portfolio T.
- Its equation in the $(\sigma, \bar{R})$ space is

$$\bar{R}_p = R_f + \frac{\bar{R}_T - R_f}{\sigma_T}\sigma_p$$

Scenario 2: two risky assets and one R_f, no shortselling

What if shortselling is not allowed?

Scenario2

Consider two risky assets A and B with some $\bar{R}_A$, $\bar{R}_B$, σ_A, σ_B and σ_{AB}, and a riskless asset F with return R_f, that can be used to both lending and borrowing. Suppose you are not allowed no shortselling the risky assets A and B.

1. What can you say about the investment opportunity set.
2. What can you say about the tangent portfolio T.
3. What can you say about the efficient frontier.

Scenario 2: investment opportunity set

1. Opportunity set:
   - Recall, that the possible combinations of the two risky assets are just a small portion of the hyperbola.
   - Including the riskless asset with a fixed R_f to lending and borrowing allows to consider all combinations of the riskless asset with all feasible portfolios.
   - The investment opportunity set will be a cone tangent from above and from bellow to portion of the hyperbola that do not require shortselling.
   - It is still and open set.

HW: Sketch this in the $(\sigma, \bar{R})$ space.
Make sure you know how to determine the lines limiting the cone.

Scenario 2: tangent portfolio T

2. To find the tangent portfolio T we have the usual problem

$$\max_{x_A, x_B} \quad \theta = \frac{\bar{R}_p - R_f}{\sigma_p}$$

s.t.

$$\bar{R}_p = x_A \bar{R}_A + x_B \bar{R}_B$$
$$\sigma_p = \left(x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}\right)^{\frac{1}{2}}$$
$$x_A + x_B = 1$$
$$x_A \geq 0$$
$$x_B \geq 0$$

with two inequality (because we just have two risky assets) additional restrictions.

Scenario 2: tangent portfolio T

The unrestricted solution

- If none of the two "no shortselling restrictions" is binding $\Rightarrow$ the solution to the restricted problem is the solution to the unrestricted problem.

Or, the trivial solution

- If the optimal solution of the unrestricted problem requires shortselling of the risky asset A, then the restrict solution implies zero investment in that asset $x_A = 0$. This holds in general, i.e. also for more than two risky assets, i.e. for $n \geq 2$.
- And the tangent portfolio implies full investment in the other asset B, i.e. trivially $x_B = 1$. This part of the solution only hold for $n = 2$.

OBS: In the two risky assets case we can deal with shortselling restrictions.

Scenario 2: efficient frontier

3. Given the tangent portfolio T, the efficient frontier contains only combinations of the riskless asset F with the tangent portfolio T. Its equation in the $(\sigma, \bar{R})$ space is

$$\bar{R}_p = R_f + \frac{\bar{R}_T - R_f}{\sigma_T}\sigma_p$$

Scenario 1+2: up to now ...

We have shown that

- If there is a risk-free asset with a unique rate R_f for both lending and borrowing, the efficient set is a straight line.
- The investment opportunity set is a open cone, with vertice at the riskless asset F and tangent, from above and below, to the hyperbola (or part of the hyperbola) which is the efficient set for the two risky assets.
- This efficient frontier is the straight line passing through the risk-free asset and tangent from above to the hyperbola (or part of the hyperbola).
- The efficient set of portfolios is linear combinations of the risk-free asset and the portfolio where the straight line is tangent from above to the hyperbola.
- That portfolio is called the tangent portfolio. and is the portfolio that maximizes the Sharpe ratio.

Theorem: Efficiency of T after discarding the riskless asset

If we throw away the risk-free asset, it turns out that T is efficient amongst the set of risky assets:

Theorem (Efficiency of T even w/o the riskless asset)

If E is efficient then the portfolio T consisting of the risky assets in E is efficient relative to investing solely in risky assets.

- The intuitive reason this holds is that R_f has no variance or covariance.
- So no diversification effects can come into play.

OBS: If we added or subtracted a RISKY asset, a similar result would NOT hold.

Proof by contra-positive

- We use proof by contra-positive.
- We show that if T is not efficient then E is not efficient.
- This is logically equivalent to the statement that if E is efficient then T is efficient.
- Note, in the proof F is riskless so $\mathbb{E}(R_f) = R_f$.

Logic of proof: If T is not efficient, then either there exists

1. a portfolio of risky assets A with higher return and the same or lower variance,
2. or a portfolio B with the same return and lower variance.

Proof by contra-positive (cont.)

1. If T is not efficient, then either there exists a portfolio of risky assets A with higher return and the same or lower variance,

$$\mathbb{E}(x_f R_f + (1-x_f)R_A) = x_f R_f + (1-x_f)\bar{R}_A, \\ > x_f R_f + (1-x_f)\bar{R}_T,$$

$$\mathrm{Var}(x_f R_f + (1-x_f)R_A) = \mathrm{Var}((1-x_f)R_A), \\ = (1-x_f)^2\sigma_A^2, \\ \leq (1-x_f)^2\sigma_T^2.$$

$\Rightarrow \quad x_f F + (1-x_f)T$ is not efficient.

Proof by contra-positive (cont.)

2. If T is not efficient, then either there exists or a portfolio B with the same expected return and lower variance,

$$\mathbb{E}(x_f R_f + (1-x_f)R_B) = x_f R_f + (1-x_f)\bar{R}_B, \\ = x_f R_f + (1-x_f)\bar{R}_T,$$

$$\mathrm{Var}(x_f R_f + (1-x_f)R_B) = \mathrm{Var}((1-x_f)R_B), \\ = (1-x_f)^2\sigma_B^2), \\ < (1-x_f)^2\sigma_T^2.$$

$\Rightarrow \quad x_f F + (1-x_f)T$ is not efficient.

- So if T is not efficient then neither is $x_f F + (1-x_f)T$.
- Turning this round, $x_f F + (1-x_f)T$ efficient $\Rightarrow$ T is efficient. ■

Scenario 3: two risky assets and lending at R_f, but no borrowing

What if borrowing for investment in risky assets is not allowed?

Scenario 3

Consider two risky assets A and B with some $\bar{R}_A$, $\bar{R}_B$, σ_A, σ_B and σ_{AB}, and a riskless asset F that can be used only for lending/deposit at rate R_f.

1. What can you say about the investment opportunity set.
2. What can you say about the tangent portfolio T.
3. What can you say about the efficient frontier.

Scenario 3: investment opportunity set

1. Opportunity set:

- We saw the tangency portfolio T corresponds to having all money in the risky assets.
- If we say no borrowing then the investment line must terminate at T.
- What we just said about T is actually true for any feasible combination of the two-risky assets.
- So the investment opportunity set is the cone with vertice in the riskless asset up to the hyperbola (to portion of the hyperbola is shortselling is forbidden) and the hyperbola itself (or portion of the hyperbola).

HW: Sketch this in the $(\sigma, \bar{R})$ space.
Consider both the case when shortselling is allowed and when it is not.

Scenario 3: tangent portfolio T

2. The tangent portfolio T can be determined using the methods describe in Scenario 1 or Scenario 2, depending on whether shortselling is allowed or not.

HW: Explain why this is so.

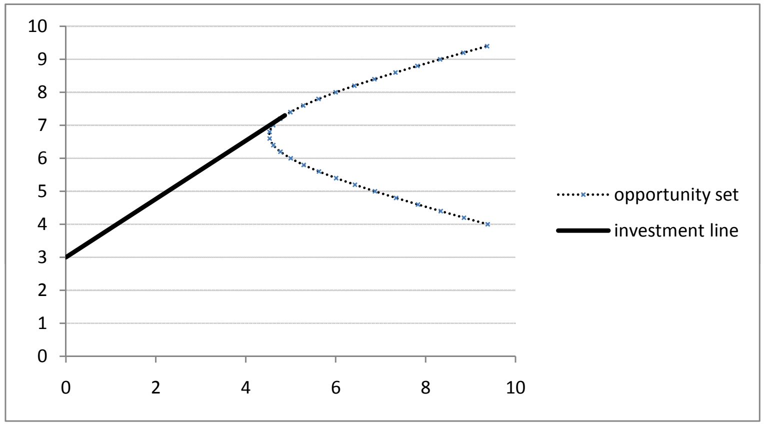
Scenario 3: efficient frontier

3. The efficient frontier when borrowing is not allowed comes in two pieces:
   - The investment line between the riskless asset F and the tangent portfolio T.
   - For volatility levels higher than σ_T, it is described by the upper part of the hyperbola that results from combining the two risky assets.

$$\begin{cases} \text{investment line} & \text{for } \sigma_p < \sigma_T \\ \text{hyperbola} & \text{for } \sigma_p \geq \sigma_T \end{cases}$$

OBS: Note that we already know how to determine both branches of the efficient frontier.

Scenario 3: efficient frontier



Scenario 4: Different borrowing and lending rates

What if borrowing is possible but only at a rate higher than the lending rate?

Scenario 4

Consider two risky assets A and B with some $\bar{R}_A$, $\bar{R}_B$, σ_A, σ_B and σ_{AB}, and a riskless asset F that can be used for lending at rate R_f^p or borrowing at rate R_f^a, with $R_f^a > R_f^p$.

1. What can you say about the investment opportunity set.
2. What can you say about the tangent portfolios.
3. What can you say about the efficient frontier.

Scenario 4: $R_f^p < R_f^a \Rightarrow$ two tangent portfolios T, T'

- When we buy the risk-less bond we are lending at the risk-less rate to an essentially risk-less counterparty: the government.
- When we borrow, we are charged a risk premium.
- So the borrowing rate R_f^a (active rate) should be higher than the lending rate R_f^p (passive rate).
- To obtain the efficient frontier we have to compute two tangent portfolios – one using the passive rate (T) and another using the active rate (T').

1. *HW: Sketch the full investment opportunity set.*
2. *OBS: Note that we already know hold to find both T and T'.*

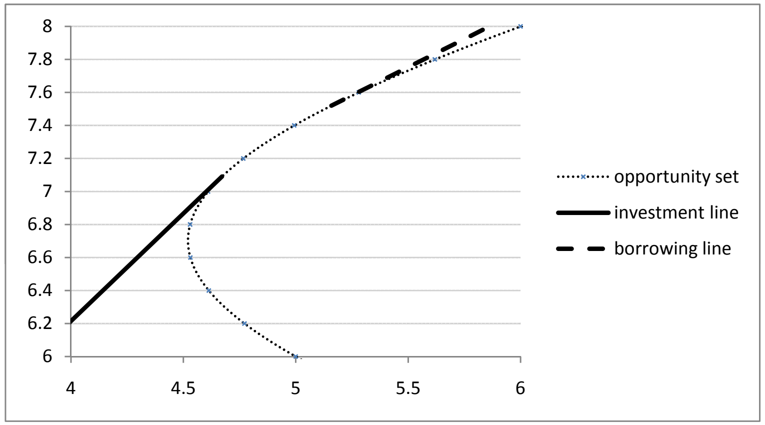
Scenario 4: efficient frontier

3. The efficient frontier will be in three pieces:

- the straight line to the tangent portfolio with riskless asset with the low lending late $\Rightarrow$ investment line
- the hyperbola between tangency portfolio (T) with the low lending rate and the tangent portfolio with high borrowing rate (T'),
- the investment line beyond the tangent portfolio for the high borrowing rate $\Rightarrow$ borrowing line

$$\begin{cases} \bar{R}_p = R_f^p + \frac{\bar{R}_T - R_f^p}{\sigma_T}\sigma_p & \sigma_p < \sigma_T \\ \textit{hyperbola} & \sigma_T \leq \sigma_p \leq \sigma_{T'} \ , \ \bar{R}_p \geq \bar{R}_T \\ \bar{R}_p = R_f^a + \frac{\bar{R}_{T'} - R_f^a}{\sigma_{T'}}\sigma_p & \sigma_p > \sigma_{T'} \end{cases}$$

Efficient frontier with different borrowing and lending rates



Example: bond and stock funds

Suppose we can invest in a bond fund, B, and an index tracker on the stock market, S We want to find the efficient frontier.

Bonds offer lower returns but also lower volatility than stocks. The two funds are correlated since both are affected by the overall economy.

$$\begin{aligned} \bar{R}_S &= 10.3\%, \\ \sigma_S &= 12.2\%, \\ \rho_{S,B} &= 0.34, \\ \bar{R}_B &= 6.2\%, \\ \sigma_B &= 5.5\% \end{aligned}$$

Example: finding the minimal variance portfolio

Using the formula for the minimal variance portfolio, we compute

$$x_B = \frac{0.122^2 - 0.122 \times 0.055 \times 0.34}{0.12^2 + 0.055^2 - 2 \times 0.122 \times 0.055 \times 0.34} = 0.944.$$

- To get MV on invest 94.4% in bonds and only 5.6% in stocks.
- Substituting, we get that the minimal volatility of 5.46% which is slightly lower than the volatility of just bonds.

Example: adding the riskless investment

Suppose we add in a risk-less investment.

We let $R_f = 5\%$.

To find the tangent portfolio T we must solve:

$$\begin{pmatrix} 0.103 - 0.05 \\ 0.062 - 0.05 \end{pmatrix} = \begin{pmatrix} 0.122^2 & 0.122 \times 0.055 \times 0.34 \\ 0.122 \times 0.055 \times 0.34 & 0.055^2 \end{pmatrix} \begin{pmatrix} z_S \\ z_B \end{pmatrix}$$

$$(\cdots)$$

$$X_T = \begin{pmatrix} x_S^T \\ x_B^T \end{pmatrix} = \begin{pmatrix} 0.697 \\ 0.303 \end{pmatrix}$$

The tangent portfolio has

$$\bar{R}_T = 9.06\%,$$
$$\sigma_T = 9.21\%$$

Example: tangent line and portfolio

The tangent line has slope:

$$\frac{9.06\% - 5\%}{9.21\%} = 0.44,$$

and so its equation is

$$\bar{R}_p = 0.05 + 0.44\,\sigma_p\ .$$

- if instead we start from the above efficient frontier, we can recover the portfolio T composition. We know

$$\bar{R}_T = x_S \bar{R}_S + (1 - x_S)\bar{R}_B,$$

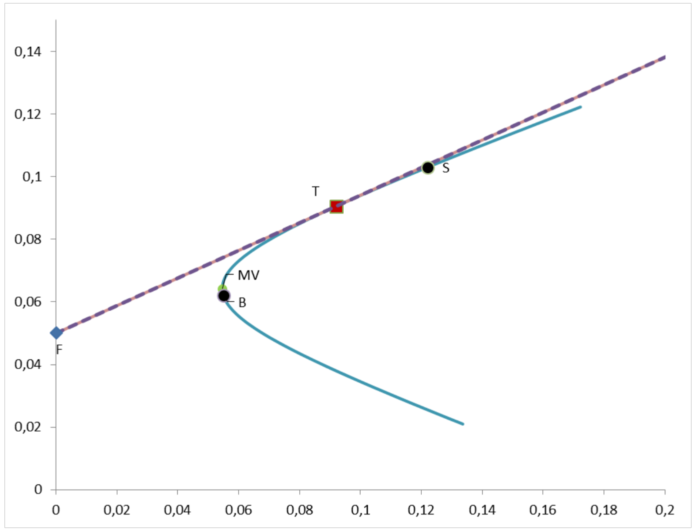
if $\bar{R}_T = 9.06\%$,

$$9.06\% = x_S(10.3\%) + (1 - x_S)6.2\%,$$

which implies

$$x_S = 0.697\%, \text{ and } x_B = 0.303.$$

Example: opportunity set and efficient frontier



Theory questions

1. What does the Tobin separation theorem say?
2. Show that if a portfolio consisting of x units of A and $(1-x)$ units of the riskless assets is efficient then A is efficient if we cannot invest in the riskless asset.
3. Show that if a portfolio consisting of x units of A and $(1-x)$ units of the riskless assets is efficient then A by itself is efficient if we can invest in the riskless asset.
4. We have two risky assets and one riskless asset, sketch the efficient frontier with and without the riskless asset on the same graph.
5. Describe the shape of the efficient frontier in the space of weights when we have a riskless asset.

Theory questions

6. How would borrowing and lending rates vary for most investors?
7. What is the shape of the efficient frontier with two risky assets and one riskless asset, if no borrowing is possible?
8. What is the shape of the efficient frontier with two risky assets and one riskless asset, with different borrowing and lending rates?
9. Describe how to find the weights in all efficient portfolios with two risky assets and different borrowing and lending rates.

2.5 The General Case

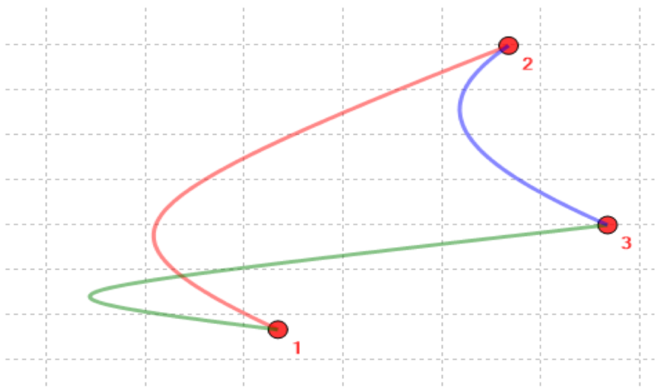
- Learning objectives
- Three or more risky assets
- Solving for the efficient frontier:
  - Tangent portfolios
  - Minimal variance portfolio
  - The envelop hyperbola
- Multi-asset Example
- Questions

Learning objectives

- understand the impact of three or more risky assets in the investment opportunity set and the efficient frontier
- find the efficient frontier under various market conditions
- find the tangent portfolio(s) in the multi-asset case,
- find the minimal variance portfolio in the multi-asset case,
- describe the geometry of the efficient frontier in weight space with and without a risk-free asset,
- solve problems which involve finding a portfolio prescribed expected return or standard deviation in the multi-asset case.

3 risky assets

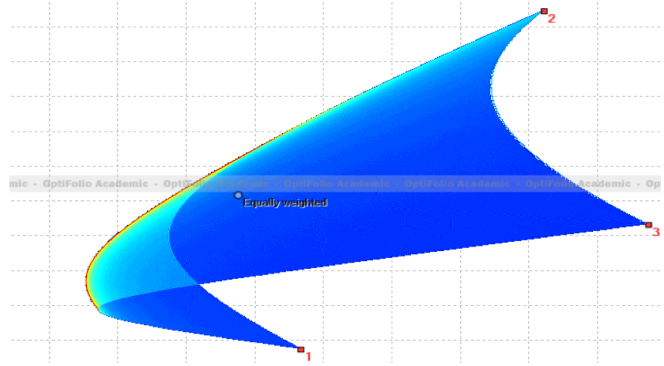
Just pairwise combinations ...



... and assuming shortselling is not allowed.

3 risky assets

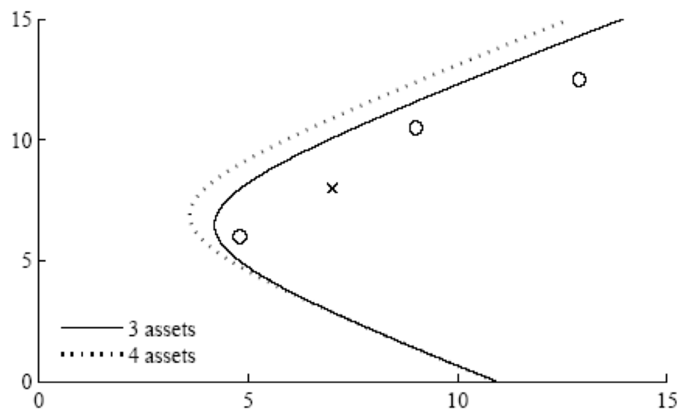
A more realistic picture...



... still assuming shortselling is not allowed.

From 3 to 4 assets ...

Let us just focus on the frontier of the investment opportunity set



Solving for the efficient frontier

We have the consider the same various scenarios:

- Scenario 1: Shortselling allowed + same R_f for lending and borrowing
- Scenario 2: Shortselling not allowed + same R_f
- Scenario 3: Shortselling allowed + no borrowing
- Scenario 4: different deposit and lending rate
- Scenario 5: no riskless asset

$\Downarrow$

From before we know it all reduces to be able :

1. determine tangent portfolios.
2. determine the minimum variance portfolio of risky assets.
3. derive the equation of the efficient frontier without the riskless asset.

Tangent portfolios

- Shortselling allowed

If X is a vector of portfolio weights, $\bar{R}$ is the vector of the assets' expected returns and V is the variance-covariance matrix, we have for all risky portfolios P

$$\bar{R}_p = \langle X, \bar{R} \rangle, \text{ and } \sigma_p = (X'VX)^{\frac{1}{2}}.$$

So we must maximize

$$\theta(X) = \frac{\langle X, \bar{R} \rangle - R_f}{(X'VX)^{\frac{1}{2}}}$$

subject to $\sum x_i = 1$

Tangent portfolios

OBS: Note that this is not harder than the $n = 2$ case we studied before.

From before we actually know:

- how to include the restriction in the objective function
- how all parcial derivatives $\frac{\partial \theta}{\partial x_i}$ look like,
- that solving the the FOC is equivalent to solving the system

  $$\tilde{R} = VZ \qquad \Leftrightarrow Z = V^{-1}\tilde{R}$$

  where $\tilde{R}$ is a column vector with entries $\tilde{R}_i = \bar{R}_i - R_f$, and $Z = \lambda X$ with λ constant.
- So, from Z we can obtain the individual weights as

  $$x_i = \frac{z_i}{\sum_{j=1}^{n} z_j}.$$

Tangent portfolios

Note that if we set

$$\mathbb{1}' = (1, 1, \dots, 1),$$

we can rewrite the algorithm as

- Set

  $$\tilde{R} = \bar{R} - R_f \mathbb{1},$$

- Put

  $$Z = V^{-1}\tilde{R} = V^{-1}\bar{R} - R_f V^{-1}\mathbb{1} \ .$$

- Let

  $$X = Z \langle Z, \mathbb{1} \rangle^{-1}.$$

So every value is equal to the fixed vector $V^{-1}\bar{R}$ plus a varying multiple of $V^{-1}\mathbb{1}$. So all values of Z lies on a straight line. We only need to compute $V^{-1}\bar{R}$ and $V^{-1}\mathbb{1}$ once to get all values of Z.

Tangent portfolios

- Shortselling not allowed

As before, we must maximize

$$\theta(X) = \frac{\bar{R}_p - R_f}{\sigma_p} = \frac{\langle X, \bar{R} \rangle - R_f}{(X'VX)^{\frac{1}{2}}}$$

such that

$$\sum x_i = \langle X, \mathbb{1} \rangle = 1,$$

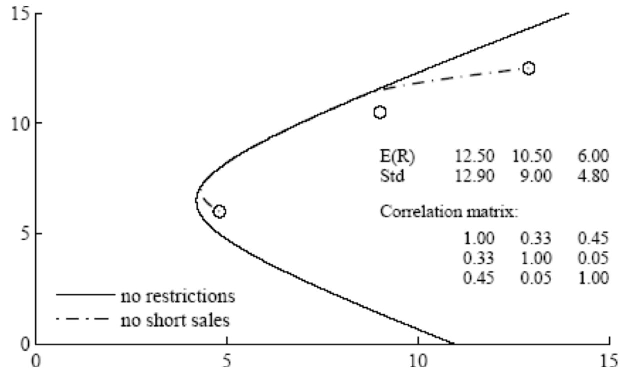
$$x_i \geq 0 \quad \text{for all} \quad i = 1, 2, \cdots, n \ .$$

$$\Downarrow$$

Additional n inequality restrictions.
We have to rely on numerical solutions.

No shortselling illustration

Impact of no shortselling on the frontier of the investment opportunity set



Tangent portfolios

If we know the solution to the unrestricted problem
(when shorselling is allowed)
$\Downarrow$
we already know some results about the solution to the restricted problem
(when shortselling is forbidden).

- if the unrestricted solution requires no shortselling positions that is also the solution to the restricted problem.
- a short position in the unrestricted tangent portfolio implies no investment in the restricted tangent portfolio.
- a long position in the unrestricted tangent portfolio does NOT imply long position in the restricted tangent portfolio.

Tangent portfolios

- Shortselling allowed, but restricted *a la* Lintner
- Lintner definition of portfolio:

$$\sum_{i=1}^{n} |x_i| = 1$$

Q: How can this be connected to shortselling restrictions?

- For this portfolio definition the problem becomes

$$\theta(X) = \frac{\langle X, \bar{R} \rangle - R_f}{(X'VX)^{\frac{1}{2}}}$$

subject to $\sum |x_i| = 1$.

Tangent portfolios

- Lintner solution:
  1. Convince yourself that from the FOC we get the same vector Z as in the unrestricted problem (*why?*):

  $$Z = V^{-1}\tilde{R}$$

  for $\tilde{R} = \bar{R} - R_f \mathbb{1}$.
  2. Lintner weights for the risky assets can, thus, easily be obtained by

  $$x_i = \frac{z_i}{\sum_{i=1}^{n} |z_i|}$$

  3. What is not invested in risky assets is assumed to be invested in the risk-free asset

  $$x_f = 1 - \sum_{i=1}^{n} x_i$$

Tangent portfolios

Q: How to represent this Lintner solution in the $(\sigma, \bar{R})$ space? Explain.

Tangent portfolios

- Real-life Shortselling limits

We must maximize

$$\theta(X) = \frac{\langle X, \bar{R}\rangle - R_f}{(X'VX)^{\frac{1}{2}}}$$

subject to

$$\sum x_i = 1,$$

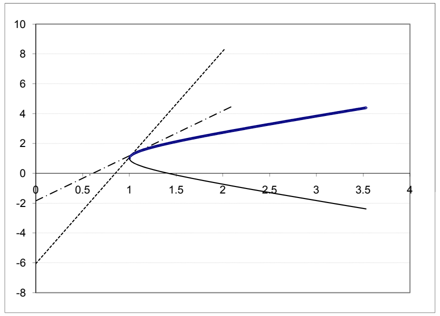
$$x_i \geq -c_i\ , \quad \text{for all} \quad i = 1, 2, \cdots, n$$

$$\sum_{x_i<0} x_i \leq -C$$

$$\left(\vdots\right)$$

for $c_1, c_2, \cdots, c_n$ and C positive constants.⇒ Numerical Solutions

Minimum variance portfolio



- as the risk-free rate gets lower and lower, the slope of the investment line gets steeper and steeper, and the tangent portfolio gets closer and closer to the tip.
- So we can find the weights in the minimal variance portfolio by letting the risk-free rate tend to minus infinity.

Minimum variance portfolio

- When here are no shortselling restrictions, we can write the tangent portfolio weights , as

$$X_T = \frac{Z}{\langle Z, \mathbb{1}\rangle} = \frac{V^{-1}\left[\bar{R} - R_f\right]}{\langle V^{-1}\left[\bar{R} - R_f\right], \mathbb{1}\rangle}$$

$$= \frac{V^{-1}\bar{R} - R_f V^{-1}\mathbb{1}}{\langle V^{-1}\bar{R}, \mathbb{1}\rangle - R_f\langle V^{-1}\mathbb{1}, \mathbb{1}\rangle},$$

$$= \frac{-R_f^{-1}V^{-1}\bar{R} + V^{-1}\mathbb{1}}{-R_f^{-1}\langle V^{-1}\bar{R}, \mathbb{1}\rangle + \langle V^{-1}\mathbb{1}, \mathbb{1}\rangle}$$

Letting $R_f \to -\infty$ this converges to

$$X_{MV} = \frac{V^{-1}\mathbb{1}}{\langle V^{-1}\mathbb{1}, \mathbb{1}\rangle} ,$$

so, we can find the minimal variance portfolio with n assets easily.

Minimum variance portfolio

- Or, one could explicitly solve the optimization problem:

$$\min_X \quad \sigma_P^2 = X'VX$$
$$\text{s.t.} \quad \sum x_i = 1 \; ,$$

using the Lagrangean to get the same solution.

- In the case of no shortselling or real-life shortselling restrictions we would need to include additional short selling conditions and solve the problem numerically.

HW: How to interpret the MV in the case of Lintner portfolios ? How to determine its composition?

Efficient risky portfolio for fixed $\bar{R}_P$

Consider only the n risky assets.

Often we are given a predetermined level of expected return $\bar{R}_P$ and our task is to find, among all risky portfolios with that specific expected return, the only efficient one.

I.e, we need to solve the optimization problem:

$$\min_X \quad \sigma_p^2 = X'VX$$
$$\text{s.t.} \quad \sum x_i = 1$$
$$\langle X, \bar{R} \rangle = \bar{R}_p^* \; ,$$

OBS: There is only two equality restrictions. So, the problem is not hard to solve.

The envelop Hyperbola

Theorem (Envelop)

When there are n risky assets. The investment opportunity set is limited:

- *From the outside by an hyperbola that is the envelop to all hyperbolas combining any two points in the set.*
- *If shotselling is allowed there is re no inner limits*
- *If shortselling is not allowed it the limited from bellow by a set of hyperbolas.*

⇓

The efficient frontier is the upper-part of some enveloping Hyperbola.

OBS: To get the exact expression of an hyperbola it is enough to know two portfolios on that hyperbola and their return covariance.

The envelop Hyperbola

"Two tangents strategy" to find the outer hyperbola:

1. Choose two fictitious values for the return of the riskless asset, R_h and R_g
2. Find the two tangent portfolios, H and G associated with each of the fictitious riskless returns.
3. Determine $\bar{R}_H, \bar{R}_G, \sigma_H, \sigma_G, \sigma_{HG}$
4. Derive the expression for the hyperbola that represents all combinations of H and G. This is the envelop hyperbola!

⇓

That hyperbola is nothing but our Envelop Hyperbola!

The envelop Hyperbola

For the case of unlimited shortselling be get:

$$\sigma_P^2 = \frac{A\bar{R}_P^2 - 2B\bar{R}_P + C}{AC - B^2}$$

where A, B, C are the scalars

$$A = \mathbb{1}'V^{-1}\mathbb{1} \qquad B = \mathbb{1}'V^{-1}\bar{R} \qquad C = \bar{R}'V^{-1}\bar{R}\ .$$

Using this simpler notation the minimum variance portfolio is

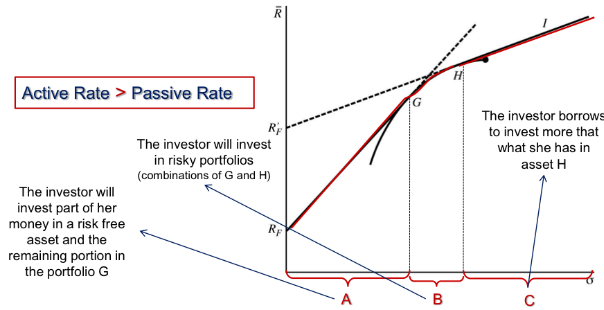
$$X_{MV} = \frac{1}{A}V^{-1}\mathbb{1}$$

OBS: For a particular instance with $n \geq 3$ check that the hyperbola you get from the above expression is the same as the hyperbola you get from the previous slide "two tangents strategy".

The two-fund theorem

Theorem

Two efficient funds (portfolios) can be established so that any efficient portfolio can be duplicated, in terms of mean and variance, as a combination of these two. In other words, all investors seeking efficient portfolios need only invest in combinations of these funds.



Example

We are given

Asset	$\bar{R}$	σ
A	15%	10%
B	10%	6%
C	20%	15%
R_f	3%	

and pairwise correlations $\rho_{AB} = 0.4$, $\rho_{BC} = 0.3$, and $\rho_{AC} = 0.5$.

Setup A

- there is a single risk-free rate R_f for both lending and borrowing,
- shortselling is allowed.

Example

To get the covariance matrix, we multiply each element of the correlation matrix by the standard deviations for each of the corresponding assets:

$$\begin{pmatrix} 1 \times (10\%)^2 & 0.4 \times 10\% \times 6\% & 0.5 \times 10\% \times 15\% \\ 0.4 \times 10\% \times 6\% & 1 \times (6\%)^2 & 0.3 \times 6\% \times 15\% \\ 0.5 \times 10\% \times 15\% & 0.3 \times 6\% \times 15\% & 1 \times (15\%)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.01 & 0.024 & 0.075 \\ 0.024 & 0.036 & 0.027 \\ 0.075 & 0.027 & 0.0225 \end{pmatrix}$$

Example

We have

$$\bar{R} = \begin{pmatrix} 15\% \\ 10\% \\ 20\% \end{pmatrix},$$

this implies that

$$\tilde{R} = \begin{pmatrix} 15\% - 3\% \\ 10\% - 3\% \\ 20\% - 3\% \end{pmatrix} = \begin{pmatrix} 12\% \\ 7\% \\ 17\% \end{pmatrix}.$$

The equation to solve is therefore

$$\begin{pmatrix} 0.01 & 0.024 & 0.075 \\ 0.024 & 0.036 & 0.027 \\ 0.075 & 0.027 & 0.0225 \end{pmatrix} \begin{pmatrix} z_A \\ z_B \\ z_C \end{pmatrix} = \begin{pmatrix} 0.12 \\ 0.07 \\ 0.17 \end{pmatrix}.$$

Example

The solution is

$$\begin{pmatrix} z_A \\ z_B \\ z_C \end{pmatrix} = \begin{pmatrix} 5.962 \\ 12.410 \\ 4.079 \end{pmatrix}$$

We need the weights x_i to add up to one.

Since we know the weights x_i are proportional to z_i, and $\sum z_i = 22.45$, we just need to compute

$$X_T = \begin{pmatrix} x_A^T \\ x_B^T \\ x_C^T \end{pmatrix} = \begin{pmatrix} \frac{z_A}{\sum z_i} \\ \frac{z_B}{\sum z_i} \\ \frac{z_C}{\sum z_i} \end{pmatrix} = \begin{pmatrix} 0.2656 \\ 0.5528 \\ 0.1817 \end{pmatrix}$$

Example

Substitute to get that the standard deviation is $\sigma_T = \sqrt{X_T' V X_T} = 6.722\%$ and the expected return is $\bar{R}_T = X_T' \bar{R} = 13.145\%$.

The efficient line goes through F and T, i.e in the space $(\sigma, \bar{R})$ passes the points

$$(0, 3\%) \text{ and } (6.722\%, 13.145\%).$$

The slope of the line is $\dfrac{0.13145 - 0.03}{0.06722} = 1.509$.

The efficient frontier has equation

$$\bar{R}_p = 0.03 + 1.509\ \sigma_p,$$

and we are done!

All efficient portfolios can be seen as combinations of F and T.

Example

What if shortselling is not allowed?

$\Downarrow$

Setup B

- there is a single risk-free rate R_f for both deposit and lending,
- shortselling not allowed.

OBS: Given the data in our Example, this is trivial! Why?

Example

What if lending is possible at R_f but not borrowing ?

$\Downarrow$

Setup C

- riskless rate R_f only available for lending.
- shortselling allowed.

- The tangent portfolio is the same, but for volatility levels higher than $\sigma_T = 6.722\%$ it is not efficient to invest in the riskless asset.
- The efficient portfolios for higher volatilities lie on the hyperbola (just risky assets).

Example

To get the hyperbola equation we can use

$$\sigma_P^2 = \frac{A\bar{R}^2 - 2B\bar{R} + C}{AC - B^2}\sigma_P$$

and for our case we have

$$A = \mathbb{1}'V^{-1}\mathbb{1} = 291.039$$
$$B = \mathbb{1}'V^{-1}\bar{R} = 31.1828$$
$$C = \bar{R}'V^{-1}\bar{R} = 3.8866$$

And we can conclude our efficient frontier is

$$\begin{cases} \bar{R}_p = 0.03 + 1.509\ \sigma_p & \sigma_p < 6.722\% \\ \sigma_p^2 = 1.8327\bar{R}_p^2 - 0.3927\bar{R}_p + 0.0245 & \sigma_p \geq 6.722\%\ ,\ \bar{R}_p \geq 13.245\% \end{cases}$$

Example

What if the active riskless rate differs from the passive riskless rate?

$\Downarrow$

Setup D

- active riskless rate R_f^a differs from the passive riskless rate R_f^p,
- shortselling allowed.

Let us keep $R_f^p = 3\%$ and set $R_f^a = 7\%$.

- The tangent portfolio T was found maximizing the slope $\frac{\bar{R}_P - 3\%}{\sigma_P}$;
- We now need to find the second tangent portfolio T' and, thus, maximize $\frac{\bar{R}_P - 7\%}{\sigma_P}$

Example

Solving for T'

$$Z = V^{-1}\left[\bar{R} - R_f\mathbb{1}\right] = \begin{pmatrix} 4.4140 \\ 2.3746 \\ 4.0215 \end{pmatrix}$$

Since we know the weights x_i are proportional to z_i, and $\sum z_i = 10.81$, we just need to compute

$$X_{T'} = \begin{pmatrix} 0.4083 \\ 0.2197 \\ 0.3720 \end{pmatrix}$$

and for our second tangent portfolio we have $\sigma_{T'} = \sqrt{X_{T'}'VX_{T'}} = 9\%$, $\bar{R}_{T'} = X_{T'}'\bar{R} = 15.76\%$.

The straight line passing through $(0, R_f^a)$ and $(\sigma_{T'}, \bar{R}_{T'})$ is:

$$\bar{R}_p = 0.07 + 0.9732\ \sigma_p$$

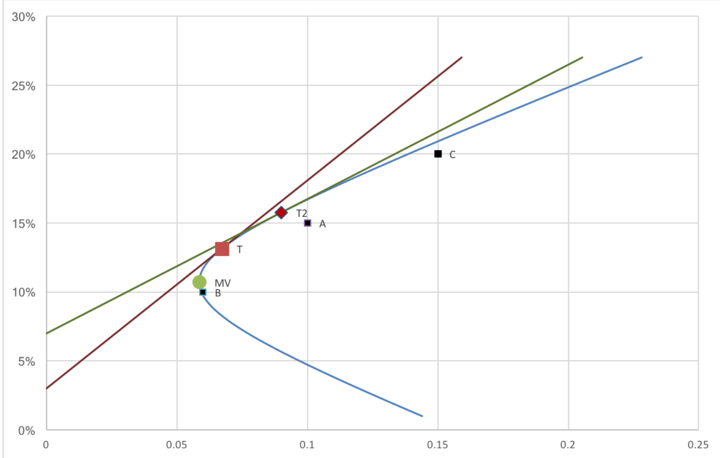
Example

The efficient frontier comes in three pieces

$$\begin{cases} \bar{R}_p = 0.03 + 1.509\ \sigma_p & \sigma_p < 6.722\% \\ \sigma_p^2 = 1.8327\bar{R}_p^2 - 0.3927\bar{R}_p + 0.0245 & 6.722\% \leq \sigma_p \leq 9\%\ , \\ & \bar{R}_p \geq 13.245\% \\ \bar{R}_p = 0.07 + 0.9732\ \sigma_p & \sigma_p > 9\% \end{cases}$$

OBS: Make sure you understand how to explain and compute the composition of all possible efficient portfolios.
HW: Determine the efficient portfolios with $\bar{R}_p = 10\%$, 15% or 20%?

Example



HW: Check that although close, asset B does not belong to the hyperbola. Even it would belong, it would not be efficient. Why?

HW Challengue

Consider the following data:

	S&P	Bonds	Canadian	Japan	Emerging Market	Pacific	Europe	Small Stock
Expected return	14.00	6.50	11.00	14.00	16.00	18.00	12.00	17.00
Standard deviation	18.50	5.00	16.00	23.00	30.00	26.00	20.00	24.00
Correlation Coefficients								
S&P	1.00	0.45	0.70	0.20	0.64	0.30	0.61	0.79
Bonds		1.00	0.27	−0.01	0.41	0.01	0.13	0.28
Canadian			1.00	0.14	0.51	0.29	0.48	0.59
Japan				1.00	0.25	0.73	0.56	0.13
Emerging Market					1.00	0.28	0.61	0.75
Pacific						1.00	0.54	0.16
Europe							1.00	0.44
Small stock								1.00

HW Challengue

1. Find the combination of risky assets with the lowest possible risk, *MV*.
2. Take $R_f^p = 5\%$, $R_f^a = 8\%$.
   - Determine the two tangent portfolios T and T'.
   - Find the efficient frontier. Interpret.
   - Show the minimum variance portfolio MV is no longer efficient.
   - Consider an investor who wants the risk level $\sigma^* = 15\%$. How should he invest.
3. Find the efficient portfolio with $\bar{R}_p^* = 15\%$.
4. For which risk levels is it efficient to lend at least part of the initial wealth?
5. For which risk levels is it efficient to borrow to invest more than the initial wealth in risky assets?

OBS: It is recommended the usage of matrix notation and Excel (or a matrix calculator)

Theory questions

1. What data is required to compute tangent portfolios?
2. Give the algorithm for finding the tangent portfolio.
3. Give the algorithm for finding the minimal variance portfolio.
4. How do the risky assets investment opportunities set looks like in $(\sigma, \bar{R})$ space for $n \geq 3$?
5. What shape does the efficient frontier take if there are $n \geq 3$ risky assets and no-risk-free asset in weight space and in $(\sigma, \bar{R})$ space?
6. What shape does the efficient frontier take if there are $n \geq 3$ risky assets and a risk-free asset in weight space and in $(\sigma, \bar{R})$ space
7. How does shortselling constraints affect the risky assets investment opportunity set?
8. What is the connection of Lintner definition of a portfolio with shortselling restrictions?

2.6 Portfolio Protection

- Learning objectives
- Safety criteria
- Roy criteria
- Kataoka criteria
- Telser criteria
- Mean-variance representation
- Questions

Learning objectives

- Understand the role of portfolio protection in portfolio management
- Identify and interpret the safety criteria of Roy, Kataoka and Telser
- For normally distributed returns and pre-defined market conditions :
  - represent safety criteria in the plane $(\sigma, \bar{R})$
  - determine and compare the optimal portfolios of Roy, Kataoka and Telser.

Safety criteria

- To evaluate portfolio risk we may be interested in knowing more than just its volatility.
- Many times criteria of some sort of portfolio protection are imposed by managers and/or investors.
- In typical situations one may wish to exclude from the analysis portfolios that do not satisfy some safety criteria.
- Our notion of "safety" may differ. We may want to,
  - minimize the likelihood of returns below a give threshold R_L;
  - establish a limit to what happens in the worst $\alpha\%$ worst scenarios;
  - exclude from the analysis all portfolios that have a probability higher than $\alpha\%$ of returns below a given threshold R_L.

Roy criterion

- An investor may wish to minimize the risk of returns below a pre-defined threshold R_L.
- According to this criterion the best portfolio is the one that solves:

$$\min_P \Pr(R_P < R_L)$$

- The threshold is pre-determined, it can take all sort of values:

$$R_L = \cdots, -10\%, \cdots, 0, \cdots, R_f, \cdots, 5\%, \cdots$$

- In general this criterion cannot be represented on the plane $(\sigma, \bar{R})$.

Roy criterion: Gaussian returns

- If, however we assume that all risky asset returns are normally distributed, then also the returns of any portfolio P are normally distributed:

$$\begin{aligned}\Pr(R_P < R_L) &= \Pr\left(\frac{R_P - \bar{R}_P}{\sigma_P} < \frac{R_L - \bar{R}_P}{\sigma_P}\right)\\ &= \Pr\left(z < \frac{R_L - \bar{R}_P}{\sigma_P}\right)\\ &= \Phi\left(\frac{R_L - \bar{R}_P}{\sigma_P}\right)\end{aligned}$$

where $\Phi(\cdot)$ is the standard Gaussian distribution function.

Roy criterion: Gaussian returns

- Thus, for Gaussian returns we have

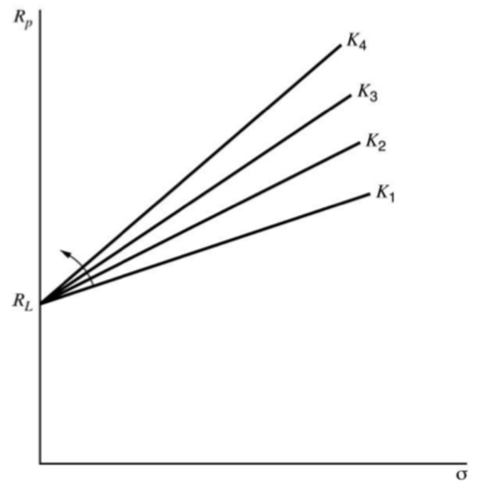
$$\min_P \Pr(R_P < R_L) \Leftrightarrow \min_P \Phi\left(\frac{R_L - \bar{R}_P}{\sigma_P}\right) \Leftrightarrow$$
$$\Leftrightarrow \min_P \frac{R_L - \bar{R}_P}{\sigma_P} \Leftrightarrow \max_P \frac{\bar{R}_P - R_L}{\sigma_P}$$

- Finding the safest portfolio according to Roy is, thus finding P that maximizes the ratio $\frac{\bar{R}_P - R_L}{\sigma_P}$.

OBS: Luckily we already know how to solve this, right?

Roy criterion: Gaussian returns MV representation

- The safest Roy portfolio is the one with highest slope.



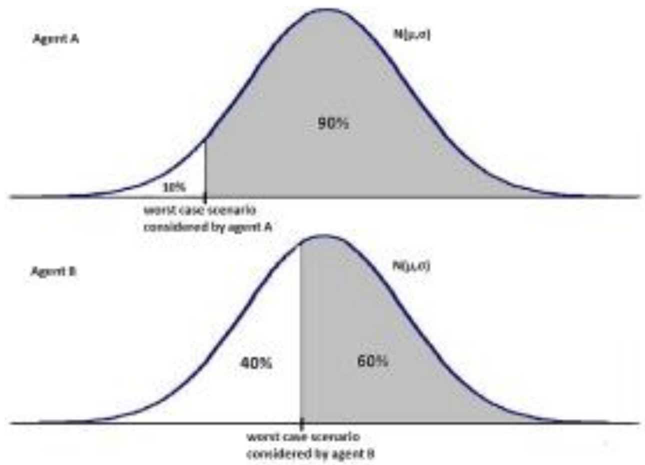
Kataoka criterion

- Alternatively, one can define bad outcomes in terms of the likelihood of their occurrence.
- One may be worried about what happens in the $\alpha\%$ worst scenarios

$$\max_{P} R_L$$
$$\text{s.t.} \quad \Pr(R_P < R_L) \leq \alpha\%$$

- The focus this time is on what, unlikely bad scenarios, may mean.
- Note that the higher the R_L of a given portfolio the safer it is, in the senses losses are no as severe as in portfolios with a lower R_L.

Kataoka criterion



Kataoka criterion: Gaussian returns

- For Gaussian returns we get

$$\Pr(R_P < R_L) \leq \alpha\% \quad \Leftrightarrow \quad \Phi\left(\frac{R_L - \bar{R}_P}{\sigma_P}\right) \leq \alpha\%$$
$$\Leftrightarrow \quad \frac{R_L - \bar{R}_P}{\sigma_P} \leq \Phi^{-1}(\alpha\%) \quad \Leftrightarrow \quad R_L \leq \Phi^{-1}(\alpha\%)\sigma_P + \bar{R}_P$$

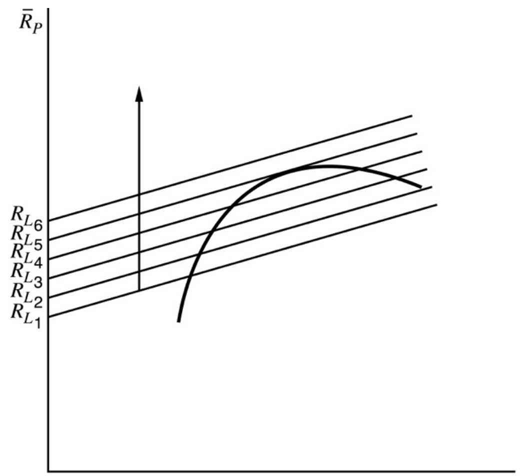
- I.e., for each portfolio P the best we can do is to choose

$$R_L = \Phi^{-1}(\alpha\%)\sigma_P + \bar{R}_P$$

- Remember that in the plane $(\sigma, \bar{R})$, these are represented by straight lines, where R_L are the $y-$crosses.

Kataoka criterion: Gaussian returns MV representation

- The safest Kataoka portfolio is the one with highest $y-$cross.



Telser criterion

- If safety is defined *a la* Telser than one pre-defines both:
  - what are bad outcomes, fixing R_L
  - what is highest likelihood acceptable for those bad outcomes $\alpha\%$
- For given R_L and $\alpha\%$, acceptable portfolios are only those that verify

$$\Pr(R_P \leq R_L) \leq \alpha\%$$

- From all portfolios that satisfy the above condition and since risk has already been taken into account, Telser recommends to choose the one with the highest expected return.
- Telser criterion is thus

$$\max_P \bar{R}_P$$
$$\text{s.t.} \quad \Pr(R_P \leq R_L) \leq \alpha\%$$

Telser criterion: Gaussian returns

- For Gaussian returns, we already know

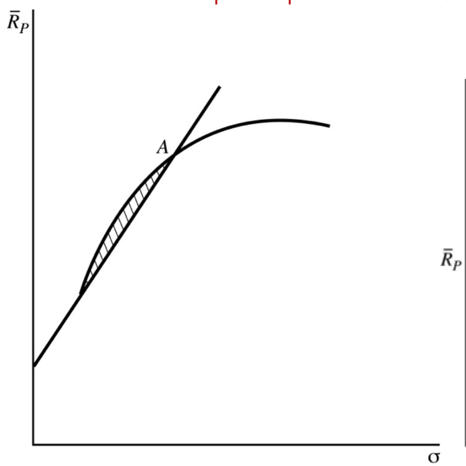
$$\Pr(R_P \leq R_L) \leq \alpha\%$$
$$\Leftrightarrow R_L \leq \Phi^{-1}(\alpha\%)\sigma_P + \bar{R}_P$$
$$\Leftrightarrow \bar{R}_P \geq \underbrace{R_L - \Phi^{-1}(\alpha\%)\sigma_P}_{\text{straight-line equation}}$$

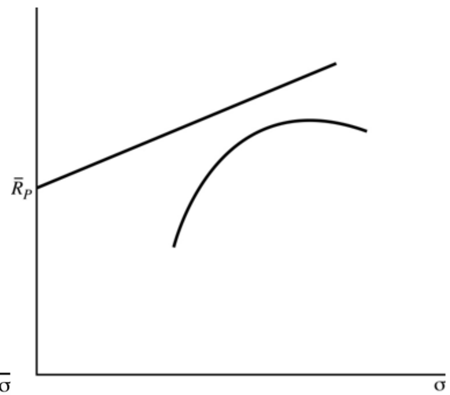
- In the $(\sigma, \bar{R})$ plane, Telsser safe portfolios are those above a pre-determined straight-line since we fix both $y-$cross and slope.

Telser criterion: Gaussian returns MV representation

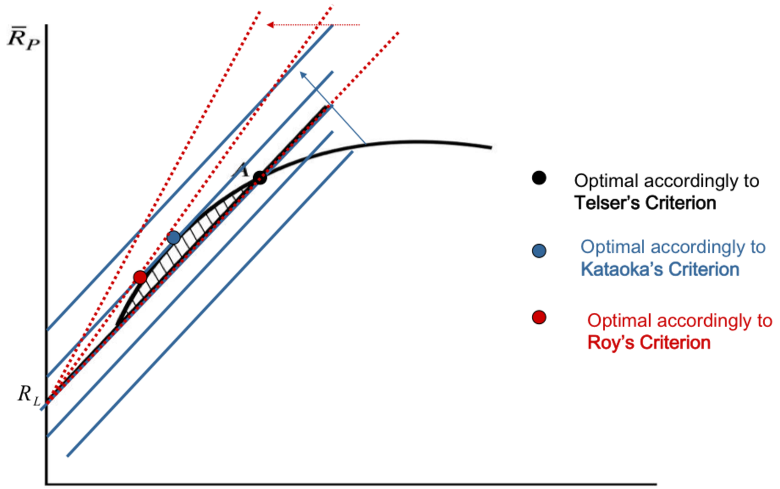
- Either we get:

A set of acceptable portfolios OR No acceptable portfolios





Safety-first criteria: MV comparison



Safety-first criteria: MV comparison

- The criteria definitions are independent of our market setup.
- I.e., for any investment opportunity set and associated efficient frontier, one can always determine the safest portfolios according to Roy, Kataoka and Telser.
- In the slides above the criteria were explained considering as investments opportunity sets of just risky assets.
- Whenever the riskless asset exists, some of the solutions to safety first criteria may be trivial.

HW: Sketch the Roy, Kataoka and Telser solutions for the safest portfolios considering the various possible scenarios of two-risky assets and the riskless asset.

Questions

- Why is portfolio protection important?
- What are the similarities and differences between the safety criteria of Roy, Kataoka and Telser?
- In general are safety criteria mean-variance efficient? Why or why not?
- For Gaussian returns, how to represent the Roy criterion in the $(\sigma, \bar{R})$ plane? What gets to be pre-determined?
- For Gaussian returns, how to represent the Kataoka criterion in the $(\sigma, \bar{R})$ plane? What gets to be pre-determined?
- For Gaussian returns, how to represent the Telser criterion in the $(\sigma, \bar{R})$ plane?
- For Gaussian returns, how to compare the safest portfolios of Roy, Kataoka and Telser?

2.7 International Diversification

- Learning objectives
- International correlations
- Exchange rate risk
- The world portfolio
- Questions

Learning Objectives

- Discuss the advantages and disadvantages of including foreign assets in portfolios.
- Compute domestic returns of a foreign asset.
- Understand how exchange risk affect the expected returns, variances and covariances of returns.
- Explain how international diversification may change the investment opportunity set and the associated efficient frontier.
- Define and determine the world portfolio.

International investments

Most portfolio managers have for decades routinely invested a large fraction of their portfolio in securities that were issued in other countries or in foreign currency.

Hence it is important to know how a world market will affect

- The allocation decision
- The investment opportunity set
- The efficient frontier
- The optimal portfolio decision

The allocation decision: international correlations

On the one hand, inclusion of foreign assets is good because

- It augments the investment opportunity set.
- Correlations across returns from different countries tend to be lower than domestic correlations. => from a diversification point of view, we want a portfolio with the lowest possible average correlation.

	Australia	Austria	Belgium	Canada	France	Germany	Hong Kong	Italy	Japan	Netherlands	Spain	Sweden	Switzerland	United Kingdom	United States
Australia															
Austria	0.279														
Belgium	0.304	0.459													
Canada	0.608	0.316	0.299												
France	0.400	0.505	0.677	0.465											
Germany	0.393	0.671	0.612	0.454	0.749										
Hong Kong	0.501	0.350	0.225	0.572	0.387	0.395									
Italy	0.248	0.358	0.396	0.361	0.487	0.495	0.231								
Japan	0.430	0.245	0.317	0.355	0.415	0.307	0.289	0.330							
Netherlands	0.480	0.578	0.738	0.514	0.758	0.740	0.424	0.429	0.432						
Spain	0.460	0.422	0.523	0.455	0.681	0.606	0.415	0.575	0.482	0.599					
Sweden	0.490	0.364	0.348	0.486	0.600	0.639	0.393	0.480	0.461	0.577	0.693				
Switzerland	0.363	0.530	0.610	0.410	0.598	0.537	0.327	0.304	0.465	0.697	0.567	0.494			
United Kingdom	0.543	0.519	0.577	0.460	0.642	0.594	0.437	0.313	0.474	0.722	0.602	0.523	0.494		
United States	0.505	0.281	0.504	0.709	0.534	0.489	0.491	0.301	0.348	0.592	0.530	0.466	0.523	0.646	
Average Correlation Coefficient				0.475											

The allocation decision: exchange rate risk

On the other hand, inclusion of foreign assets is bad because

- Foreign assets bear exchange rate risk.
- Exchange rates affect: expected returns, volatilities and even correlations.
- The same set of basic assets A, B, C, D may have very different representations in the planes:

$$(\sigma, \bar{R})^{€} \qquad (\sigma, \bar{R})^{\$} \qquad (\sigma, \bar{R})^{¥} \qquad \cdots$$

Investing in a foreign asset

Foreign assets can be understood as portfolios of

- The foreign currency
- The asset its self (as it would be seen by a domestic investor)

Take the case of an European investor, going long on a US stock:

$$W_0^{€} \rightarrow W_0^{\$} = W_0^{€} \times E_0^{\$/€} \rightarrow W^{\$} = (1+R^{\$})W_0^{\$} \rightarrow W^{€} = \frac{W^{\$}}{E^{\$/€}}$$

$$\begin{aligned}(1+R^{€})W_0^{€} &= W^{€} \\ &= \frac{W^{\$}}{E^{\$/€}} \\ &= \frac{(1+R^{\$})W_0^{\$}}{E^{\$/€}} \\ &= \frac{(1+R^{\$})W_0^{€} \times E_0^{\$/€}}{E^{\$/€}}\end{aligned}$$

Investing in a foreign asset

Using $E^{\$/€} = 1/E^{€/\$}$ and $E_0^{€/\$}(1 + R^{€/\$})$:

$$1 + R^{€} = (1 + R^{\$})(1 + R^{€/\$})$$

The expected return in euros is thus

$$1 + \bar{R}^{€} = \underbrace{\mathbb{E}\left[(1 + R^{\$})(1 + R^{€/\$})\right]}_{\text{product } \rightarrow \text{ covariance dependent}}$$

- Even the €– expected return ($\bar{R}^{€}$) of investing in a \$ – denominated asset, depends on the covariance between returns of exchange rates and returns in the foreign stock market.
- The same is true for variances any any covariances.

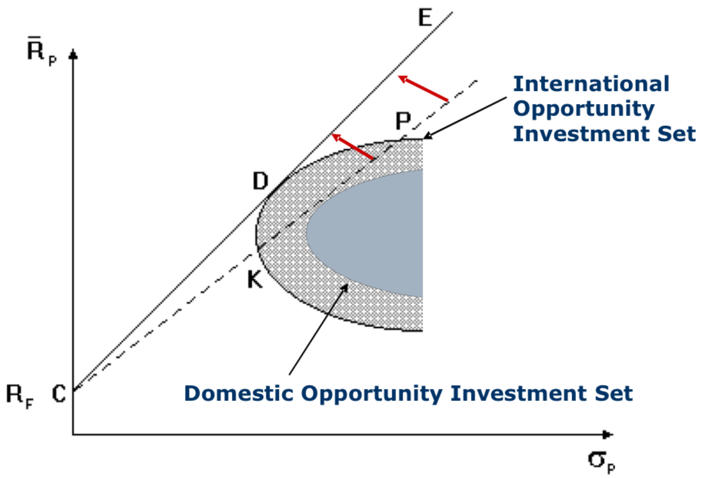
Investing in a foreign asset: Example

Taking the perspective of a US investor:

Stocks	Domestic Risk	Exchange Risk	Total Risk
Australia	13.94	8.66	17.92
Austria	24.80	10.59	24.50
Belgium	16.15	10.21	15.86
Canada	15.02	4.40	17.13
France	18.87	10.61	17.76
Germany	20.41	10.55	20.13
Hong Kong	29.75	0.43	29.79
Italy	24.55	11.13	25.29
Japan	22.04	12.46	25.70
Netherlands	16.04	10.59	15.50
Spain	22.99	11.18	23.27
Sweden	24.87	11.18	24.21
Switzerland	17.99	11.61	17.65
U.K.	14.45	10.10	15.59
United States	13.59	0.00	13.59
Equally Weighted Index (Non-U.S.)	21.57	10.03	23.43
Value-Weighted Index (Non-U.S.)			16.70

OBS: Notice that risk must be computed from the investor point of view, including exchange risk and its possible covariance with market risk.

The Investment Opportunity Set



The World Portfolio: Example

Again from the perspective of a US investor:

Area or Country	Percent of Total[a]
Austria	0.1%
Belgium	0.4%
Denmark	0.4%
Finland	1.6%
France	5.5%
Germany	4.3%
Ireland	0.2%
Italy	2.1%
Netherlands	2.5%
Norway	0.2%
Portugal	0.2%
Spain	1.3%
Sweden	1.6%
Switzerland	2.8%
U.K.	9.7%
Europe	**32.8%**
Australia	1.1%
Hong Kong	1.0%
Japan	12.6%
Malaysia	0.5%
New Zealand	0.1%
Singapore	0.4%
Pacific	**15.5%**
Canada	2.1%
United States	49.5%
North America	**51.6%**
Total	**100.0%**

Questions

- Explain how lower average correlations between assets denominated in different currencies may affect the allocation decision?
- How does the inclusion of foreign assets influence:
  - the determination of mean-variance inputs?
  - the investment opportunity set?
  - the efficient frontier?
- Will two investors facing the same set of assets denominated in a variety of currencies always choose the same world portfolio? Why or why not?