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2020/2021 Paulo M.D.C. Parente

Advanced Econometrics
PhD in Economics
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Exercise Sheet 1 - Maximum Likelihood

1. Let $Y \sim \mathcal{N}(\mu, \sigma^2)$, that is Y has density function

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\}, \text{ where } \sigma^2 > 0$$

Note that $\mathrm{E}(Y) = \mu$, $\mathrm{Var}(Y) = \sigma^2$, $\mathrm{E}[(Y-\mu)^3] = 0$ and $\mathrm{E}[(Y-\mu)^4] = 3\sigma^4$.

(a) Find the maximum likelihood estimator $(\hat{\mu}, \hat{\sigma}^2)$ of (μ, σ^2), of based on a random sample $(Y_1, ..., Y_n)$ using the first order conditions.

(b) Derive $E(\hat{\mu})$, and $var(\hat{\mu})$

(c) Derive $E(\hat{\sigma}^2)$ and $var(\hat{\sigma}^2)$. [**Hints:**1 $\mathcal{V} = \sum_{i=1}^{n}(Y_i - \bar{Y})^2/\sigma^2 \sim \chi^2(n-1)$ and if $X \sim \chi^2(k)$, $\mathrm{E}(X) = k$ and $\mathrm{Var}(X) = 2k$].

(d) Show that the expected value of the score vector is zero.

(e) Compute the Cramér–Rao lower bound for any regular consistent asymptotically normal estimators of (μ, σ^2).

(f) Show that the Information Identity holds.

(g) Let q_α be the quantile of order α of the standard normal random variable, that is $\Phi(q_\alpha) = \alpha$, where $\Phi(.)$.is the cumulative distribution of the standard normal random variable. Denote $\tilde{q}_\alpha$ be the quantile of order α of Y, that is the constant such that $\mathcal{P}(Y \leq \tilde{q}_\alpha) = \alpha$. What is the maximum likelihood estimator of $\tilde{q}_\alpha$?

2. Consider the simple linear model

$$y_i = \alpha + \beta x_i + u_i, \qquad i = 1, \ldots, n$$

where α and β are scalar parameters, and where the error terms u_i are independently identically normally distributed with mean zero and variance σ^2 and $\mathrm{Cov}(x, u) = 0$.

(a) Write the likelihood function for this model and derive $s_i(\theta) = \partial \log L_i(\theta)/\partial\theta$, where $\theta = (\alpha, \beta, \sigma^2)'$. Show that each element of this vector has expected value equal to zero when evaluated at the true values of the parameters.

(b) Derive the Maximum Likelihood estimators of α, β and σ^2, using the first order conditions and compare these with the corresponding Ordinary Least Squares estimators.

(c) Explain and discuss the general properties of Maximum Likelihood estimators

(d) Suppose you want to test the null hypothesis $H_0 : \beta = 0$ in the above model. Discuss how you would test this hypothesis by conducting a Likelihood-Ratio test. Derive the expression for the test statistic in terms of the ratio of the sum of squared residuals from the restricted and unrestricted model.

(e) Explain and contrast the general principles that underlie the Likelihood-Ratio, Wald and Lagrange Multiplier tests. Briefly discuss how you would construct a test for omitted variables in the linear model

$$y_i = \alpha + \beta x_i + \gamma z_i + u_i, \qquad i = 1, \ldots, n$$

where α, β and γ are scalar parameters, and where the error terms u_i are independently identically normally distributed with mean zero and variance σ^2.