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Advanced Econometrics
PhD in Economics
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Exercise Sheet 3 - Ordered Data and Count Data Models

1. Let Y be a Poisson Random variable with parameter $\mu > 0$. That is

$$\mathcal{P}\left(Y=k\right)=\frac{\mu^{k}e^{-\mu}}{k!}, k=0,1,2,...$$

   (a) Use the first order conditions to show that the maximum likelihood estimator of μ is given by $\bar{Y}=\sum_{i=1}^{n}Y_i/n$.

   (b) Show that $\bar{Y}$ satisfy the second order conditions for a local maximum of the log-likelihood objective function.

   (c) Show that

$$\sqrt{n}(\bar{Y}-\mu)\xrightarrow{d}\mathcal{N}(0,\mu).$$

   (d) Show that the estimator

$$S^2=\frac{1}{n-1}\sum\nolimits_{i=1}^{n}(Y_i-\bar{Y})^2.$$

   for μ is unbiased, that is show that $\mathrm{E}(S^2)=\mu$.

   (e) It is known that

$$\sqrt{n}[S^2-\mathrm{Var}(Y)]\xrightarrow{d}\mathcal{N}(0,m_4-\mathrm{Var}(Y)^2).$$

   where $m_4=\mathrm{E}[(Y-\mathrm{E}(Y))^4]$. Confirm the result stated by the theory that S^2 cannot be asymptotically more efficient than the maximum likelihood estimator for μ. [**Hint:** in the case of the Poisson random variable $m_4=\mathrm{E}[(Y-\mathrm{E}(Y))^4]=3\mu^2+\mu$]

2. Consider a count data model that satisfies the conditions

$$\begin{aligned}\mathrm{E}(Y|X) &= \exp\left(X\beta_0\right),\\ \mathrm{Var}(Y|X) &= f\left(X\right)\end{aligned}$$

where X is a univariate random variable, β_0 is a parameter and $f\left(X\right)$ is a positive function of X. Suppose additionally that a random sample $\{(Y_i,X_i)\}_{i=1}^{n}$ is available. Obtain the asymptotic variance of the pseudo-maximum likelihood estimator of β_0 based on a likelihood function obtained using:

   (a) the Poisson distribution.

   (b) the normal distribution with variance 1 (the non-linear least squares estimator).

(c) the negative binomial distribution with $\sigma^2 = 1$.

3. Define $Y_i = (Y_{i1}, \ldots, Y_{iT})'$ and $j_i = (j_{i1}, \ldots, j_{iT})'$, and let

$$\begin{aligned}
P(Y_{it} &= j_{it}|\mathbf{X}_i, \varepsilon_i) = \frac{\exp\left(-\lambda_{it}\right)\lambda_{it}^{j_{it}}}{j_{it}!} \\
\lambda_{it} &= \exp(\mathbf{X}_{it}'\beta + \varepsilon_i) \\
&= \exp(\mathbf{X}_{it}'\beta)\alpha_i, i = 1, ..., n,\ t = 1, ..., T
\end{aligned}$$

where ε_i is a random variable and $\alpha_i = \exp\left(\varepsilon_i\right)$. Assume strict-exogeneity and independence of the elements of $Y_i = (Y_{i1}, \ldots, Y_{iT})'$, conditional on ε_i and $\mathbf{X}_i = (\mathbf{X}_{i1}, ..., \mathbf{X}_{iT})'$. Prove that

$$\mathcal{P}\left(Y_i = j_i \middle| \mathbf{X}_i, \varepsilon_i, \sum_{t=1}^{T} Y_{it} = \sum_{t=1}^{T} j_{it}\right) = \frac{\left(\sum_{t=1}^{T} j_{it}\right)!}{\prod_{t=1}^{T} j_{it}!} \prod_{t=1}^{T} \left(\frac{\exp(\mathbf{X}_{it}'\beta_0)}{\sum_{t=1}^{T} \exp(\mathbf{X}_{it}'\beta_0)}\right)^{j_{it}}.$$