Models in Finance - Part 9

Master in Actuarial Science

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ISEG

The Binomial model

- Model in discrete time. The model will seem very simple. However:
- (a) it introduces the key concepts of financial economic pricing and
- (b) it leads us to the celebrated Black-Scholes model as a limiting case.

The Binomial model

- Basic assumptions of the binomial model:
  (1) no trading costs or taxes
  (2) no minimum or maximum units of trading
  (3) stocks and bonds can only be bought and sold at discrete times $t = 1, 2, ...$
  (4) The principle of no arbitrage applies.

The Binomial model

- We will use S_t to represent the price of a non-dividend-paying stock at discrete time intervals t $(t = 0, 1, 2, ...)$. For $t > 0$, S_t is random.
- Besides the stock we can also invest in a bond or a cash account which has value B_t at time t per unit invested at time 0. This account is assumed to be risk free and we will assume that it earns interest at the constant risk-free continuously compounding rate of r per annum (p.a.). Thus

$$B_t = e^{rt} \tag{1}$$

  (we assume $r > 0$).
- there are no constraints (positive or negative) on how much we hold in stock or cash.

One period model

- At time $t = 0$ the stock price is S_0.
- At time $t = 1$, we have two possibilities:

$$S_1 = \begin{cases} S_0 u & \text{if the price goes up} \\ S_0 d & \text{if the price goes down} \end{cases}, \tag{2}$$

where $u > 1$ (fixed) and $d < 1$ (fixed).

One period model

- Binomial tree:

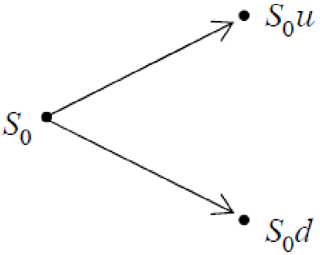


Figure 1: One-period binomial model for stock prices

One period model

- Proposition: In order to avoid arbitrage we must have
$$d < e^r < u. \tag{3}$$
- Proof: Suppose this is not the case:
- (1) Assume first $e^r < d < u$.
Then we could borrow the amount S_0 in Euros and buy a share by S_0.
- At time 0 this would have a net cost of 0€.
- At time 1 our portfolio would be worth $S_0 d - e^r S_0$ or $S_0 u - e^r S_0$.
In both cases, the value is > 0. So, we have an arbitrage opportunity.
- (2) If $d < u < e^r$ the cash investment would outperform the share investment in all circumstances.
An investor could (at time 0) sell the share and invest S_0 in a cash account.
At time 1 he could buy again the share and have a certain positive profit of $S_0 e^r - S_0 u > 0$ or $S_0 e^r - S_0 d > 0$ (arbitrage opportunity).

One period model

- Suppose that we have a derivative which pays c_u if the price of the underlying stock goes up and c_d if the price of the underlying stock goes down.
- At what price should this derivative trade at time 0?
- At time 0 suppose we hold ϕ units of stock and ψ units of cash (portfolio (ϕ, ψ)).
  The value of this portfolio at time 0 is V_0.
- At time 1 the same portfolio has the value:

$$V_1 = \begin{cases} \phi S_0 u + \psi e^r & \text{if } S_1 = S_0 u \text{ (price up)} \\ \phi S_0 d + \psi e^r & \text{if } S_1 = S_0 d \text{ (price down)} \end{cases} \tag{4}$$

- We will choose (ϕ, ψ) such that the portfolio replicates the payoff of the derivative, no matter what the outcome of the share price process.

One period model

- Let us choose ϕ and ψ so that $V_1 = c_u$ if the stock price goes up and $V_1 = c_d$ if the stock price goes down.
  Then:
  $$\begin{cases} \phi S_0 u + \psi e^r = c_u \\ \phi S_0 d + \psi e^r = c_d \end{cases} . \tag{5}$$
- This choice of ϕ and ψ ensures that the value of the portfolio (ϕ, ψ) is equal to the derivative payoff at time $t = 1$ (expiry date).
  Then, by the no arbitrage principle, the value of the derivative at time 0 is equal to the value of the portfolio V_0.
- Solving eqs. (5), we obtain:
  $$\phi = \frac{c_u - c_d}{S_0 (u - d)},$$
  $$\psi = e^{-r} \left[\frac{c_d u - c_u d}{u - d} \right].$$

One period model

- Then (Exercise: calculate this):

$$\begin{aligned} V_0 &= \phi S_0 + \psi = ... \\ &= e^{-r}\left[\frac{e^r - d}{u-d}c_u + \frac{u-e^r}{u-d}c_d\right] \end{aligned}$$

- That is:

$$V_0 = e^{-r}\left[qc_u + (1-q)c_d\right], \tag{6}$$

where:

$$q = \frac{e^r - d}{u-d}. \tag{7}$$

- Note that the no arbitrage condition ($d < e^r < u$) ensures that $0 < q < 1$.

One period model

- If we "pretend" that q is a probability, we can write (6) in the form

$$V_0 = e^{-r} E_Q [C_1], \tag{8}$$

  i.e., the price of the derivative at $t = 0$ is the discounted value of the expected derivative payoff at time $t = 1$, if we assume the probability measure Q is such that q is the "new" probability of going "up" and $1 - q$ the new probability of "going down".
- But Q is not the "real world" probability.
- If we denote the payoff of the derivative at $t = 1$ by the random variable C_1, we can write:

$$V_0 = e^{-r} E_Q [C_1], \tag{9}$$

  where Q is an artificial probability measure which gives probability q to an upward move in prices and $1 - q$ to a downward move.

Probability measures

- We can see that q depends only upon u, d and r and not upon the potential derivative payoffs c_u and c_d.
- Any function that assigns a value $q \in [0, 1]$ to the up-branch of the tree and a value $1 - q$ to the down-branch is a probability measure.
- Q is the particular probability measure, amongst many possible measures, that assigns the probability $q = \frac{e^r - d}{u - d}$ to the up-branch of the tree between times 0 and 1 and the probability $1 - q$ to the down-branch.
- $E_Q [C_1]$ represents the expected payoff of the derivative with respect to the probability measure Q.
- Q is called the risk neutral probability measure.

Replicating portfolio

- (ϕ, ψ) is called a replicating portfolio because it replicates, precisely, the payoff at time 1 on the derivative without any risk. Therefore, any payoff can be replicated by the replicating portfolio and the market is complete.
- This portfolio is also a simple example of a hedging strategy: that is, an investment strategy which reduces the amount of risk carried by the issuer of the contract.
- A hedging strategy is one that reduces the extent of, or in this case eliminates, any variation in the market value of a portfolio.
- In this case: if we hold portfolio (ϕ, ψ) and sell derivative C, then the total value of the portfolio at time 1 will be zero, however the stock price moves from time 0 to time 1.
- Therefore, by the principle of no arbitrage, the value of the combined portfolio is also zero at time 0: perfectly hedged position.

Real world probabilty

- The previous hedging analysis would also be true if we sell the replicating portfolio (ϕ, ψ) and hold the derivative C.
- The real world probability measure P: Let the "real-world" probability of an up-movement be p and a down movement be $1-p$: this defines a probability measure P.
- In general, p will not be equal to q. Note that q depends only upon u, d and r and not upon p.
- The real-world probability p is irrelevant to our calculation of the derivative price.
- Expected stock price at time 1, under P:

$$E_P[S_1] = S_0[pu + (1-p)d].$$

Risk neutral probability

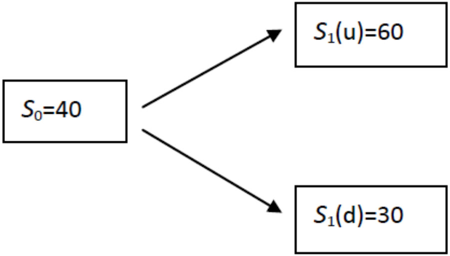
- Expected stock price at time 1, under Q:

$$E_Q[S_1] = S_0[qu + (1-q)d] = \\ = ... = S_0 e^r.$$

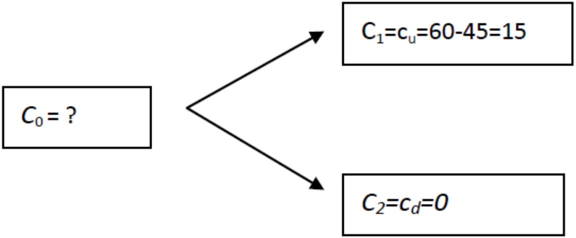
- Under Q we see that the expected return on the risky stock is the same as that on a risk-free investment in cash. In other words under the probability measure Q investors are neutral with regard to risk: they require no additional return for taking on more risk.
- This is why Q is sometimes referred to as a risk-neutral probability measure.
- Under the real-world measure P the expected return on the stock will not normally be equal to the return on risk-free cash. Under normal circumstances investors demand higher expected returns in return for accepting the risk in the stock price. Thus we would normally find that $p > q$.

Numerical example

- $S_0 = 40$, over the single period the stock price can move up to 60 or down to 30. The actual probability of an up movement is $1/2$, cont. compounded risk free interest rate is 5% per time period. Current value of European call option, V_0, with exercise price 45.
- Binomial tree:



Numerical example



Numerical example

- $d = \frac{30}{40} = 0.75$; $u = \frac{60}{40} = 1.5$.
- Risk neutral probability q:

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.05} - 0.75}{1.5 - 0.75} = 0.40169,$$
$$1 - q = 0.59831.$$

Therefore:

$$\begin{aligned} V_0 &= e^{-r} E_Q [C_1] = e^{-r} [q c_u + (1-q) c_d] \\ &= e^{-0.05} [0.40169 \times 15 + 0.59831 \times 0] \\ &= 5.732. \end{aligned}$$

and $C_0 = 5.732$.

Numerical example

- Exercise: Find the constituents of the replicating portfolio (ϕ, ψ) and show that it costs 5.732 to set up this portfolio.