Models in Finance - part 10

Master in Actuarial Science

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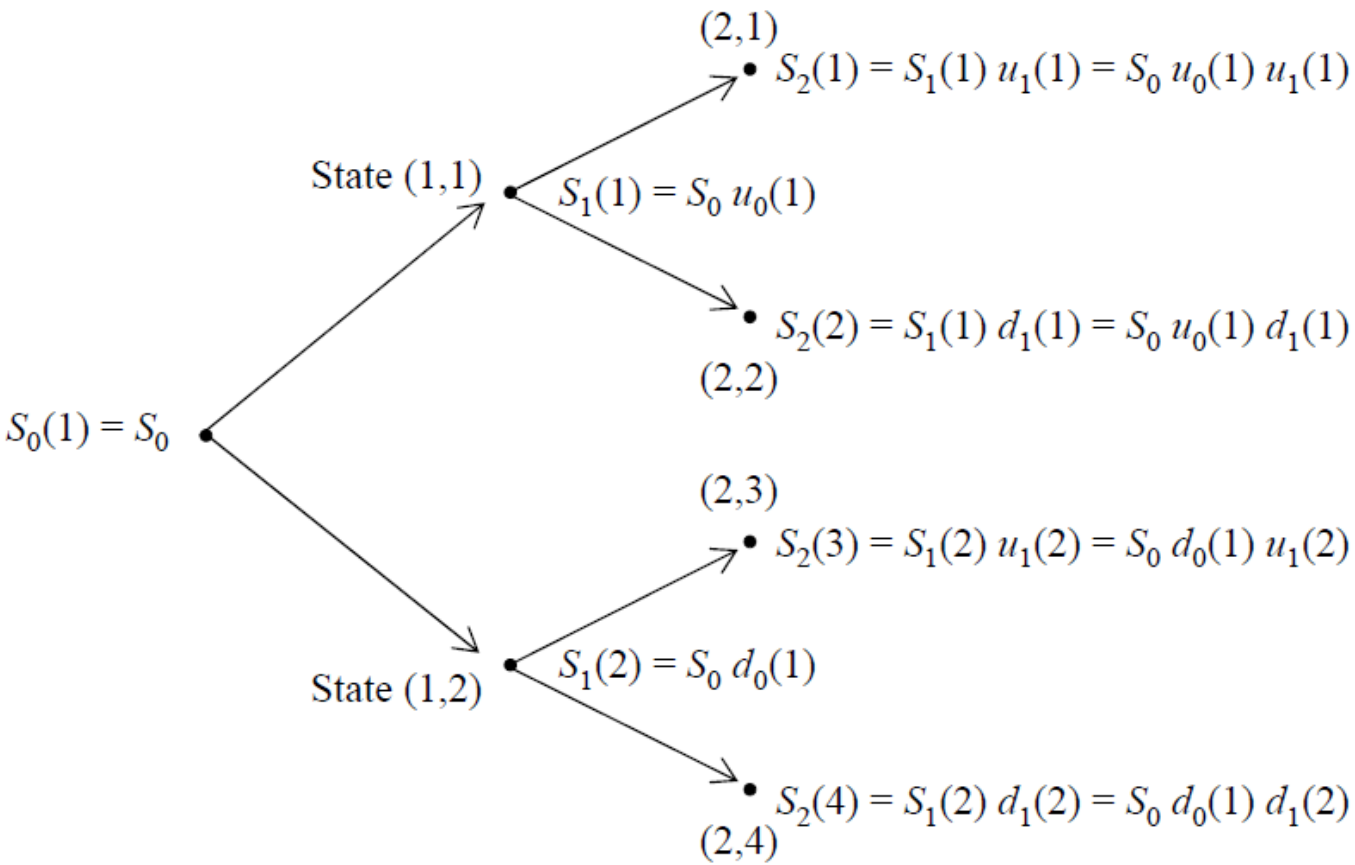
ISEG

Binomial two-period model



- a particular node is denoted $S_t(j)$: price in state (t, j).

Binomial two-period model

- Suppose that the derivative gives a payoff at time 2 of $c_2(j)$ if the price at time 2 of the stock is in state $(2, j)$.
- How do we calculate the price of the derivative at time 0?
- Working backwards from time 2.
- We calculate the value of the contract at time 1: Let $V_1(j)$ be the value of thecontract if we are in state j at time 1.

$$V_1(1) = e^{-r}\left[q_1(1)c_2(1) + (1 - q_1(1))c_2(2)\right],$$
$$V_1(2) = e^{-r}\left[q_1(2)c_2(3) + (1 - q_1(2))c_2(4)\right],$$

where:

$$q_1(1) = \frac{e^r - d_1(1)}{u_1(1) - d_1(1)},$$
$$q_1(2) = \frac{e^r - d_1(2)}{u_1(2) - d_1(2)}.$$

Binomial two-period model

- no-arbitrage conditions imply that $d_t(j) < e^r < u_t(j)$ (and hence $0 < q_t(j) < 1$) for all t and j.
- Exercise: Why do no-arbitrage conditions imply $d_t(j) < e^r < u_t(j)$?
- The price at time 0 is found by treating the values $V_1(1)$ and $V_1(2)$ in the same way as derivative payoffs at time 1. Thus:

$$V_0 = e^{-r} \left[q_0 V_1(1) + (1 - q_0) V_1(2) \right].$$

Binomial two-period model

- C_2: derivative payoff at time 2 (it takes one of the values $c_2(j)$ for $j = 1, 2, 3, 4$).
- V_t: random value of the contract at time t.
- $\mathcal{F}_t$: information generated by the process up to and including time t (sigma-algebra generated by the process up to and including time t).
- $\mathcal{F}$: sigma-algebra generated by the process up to the final time considered by the model.
- the risk-neutral probability measure Q: probability measure defined by the probabilities q_0, $q_1(1)$ and $q_1(2)$.
- Then

$$V_1(1) = e^{-r} E_Q\left[C_2 | \text{up in period 1}\right]$$
$$V_1(2) = e^{-r} E_Q\left[C_2 | \text{down in period 1}\right]$$

or

$$V_1 = e^{-r} E_Q\left[C_2 | \mathcal{F}_1\right]$$

Binomial two-period model

- With the same reasoning:

$$\begin{aligned} V_0 &= e^{-r} E_Q \left[V_1 | \mathcal{F}_0 \right] = \\ &= e^{-r} E_Q \left[e^{-r} E_Q \left[C_2 | \mathcal{F}_1 \right] | \mathcal{F}_0 \right] \\ &= e^{-2r} E_Q \left[C_2 | \mathcal{F}_0 \right], \end{aligned}$$

  by the "tower property" of the conditional expectation.

- Recall the "tower property": if $\mathcal{G} \subset \mathcal{H}$ then

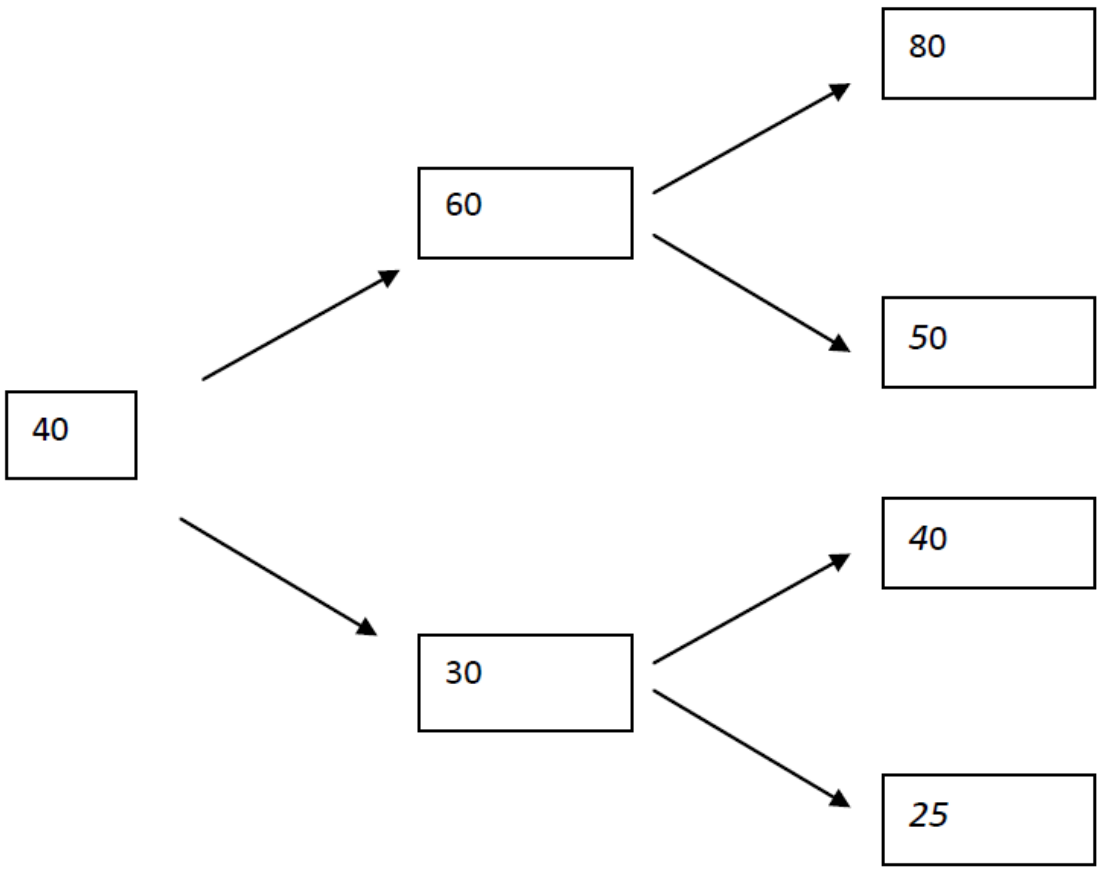
$$E\left[E\left[X|\mathcal{H}\right]|\mathcal{G}\right] = E\left[E\left[X|\mathcal{G}\right]|\mathcal{H}\right] = E\left[X|\mathcal{G}\right].$$

Numerical example

- $S_0 = 40$, over the first period the stock price can move up to 60 or down to 30.
- Following an up-movement in the first period, the price can go up to 80 or down to 50.
- Following a down-movement in the first period, the price can go up to 40 or down to 25.
- The actual (real world) probability of an up movement is $1/2$, cont. compounded risk free interest rate is 5% per time period.
- The exercise price of a call option is 45. What is the price of this call?

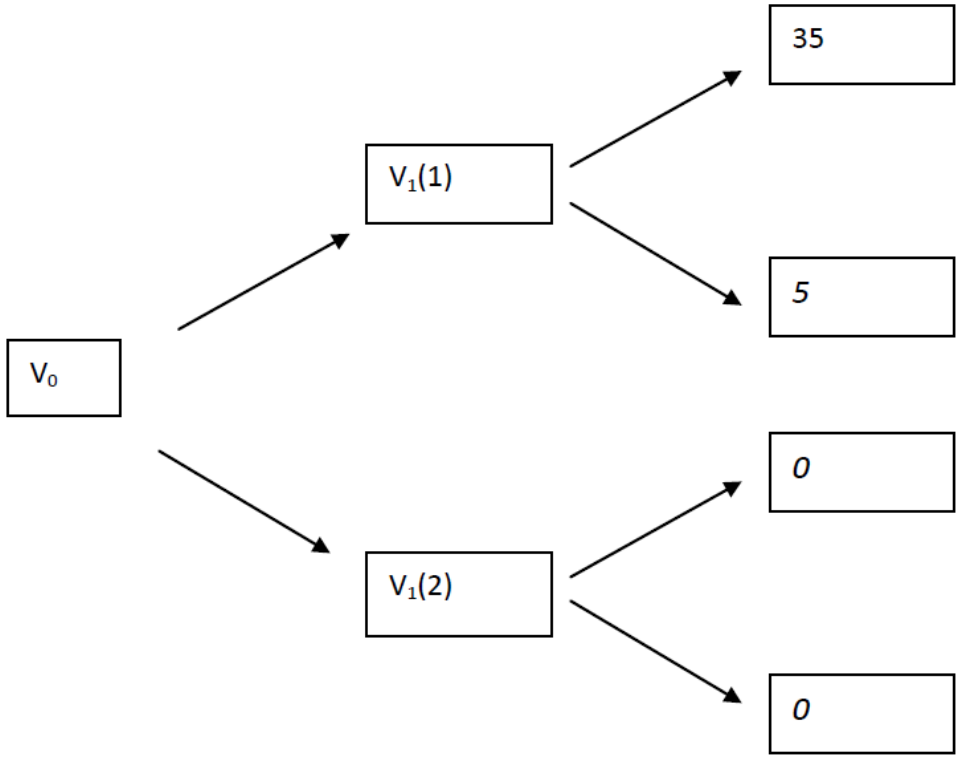
Numerical example

Tree:



Numerical example

Binomial tree for the derivative value:



Derivative value at final node = payoff at that node

Numerical example

- Factors: $u_0(1) = \frac{60}{40} = \frac{3}{2}$, $d_0(1) = \frac{30}{40} = \frac{3}{4}$, $u_1(1) = \frac{80}{60} = \frac{4}{3}$, $d_1(1) = \frac{50}{60} = \frac{5}{6}$, $u_1(2) = \frac{80}{60} = \frac{4}{3}$, $d_1(2) = \frac{25}{30} = \frac{5}{6}$.
- Risk-neutral probability:

$$q_1(1) = \frac{e^r - d_1(1)}{u_1(1) - d_1(1)} = \frac{e^{0.05} - 5/6}{4/3 - 5/6} = 0.43588,$$

$$q_1(2) = \frac{e^r - d_1(2)}{u_1(2) - d_1(2)} = \frac{e^{0.05} - 5/6}{4/3 - 5/6} = 0.43588.$$

- $q_1(1) = q_1(2)$ because the factors were the same (in general, they are not equal).
- We can also consider that the interest rate r is different from period 1 to period 2.

Numerical example

$$V_1(1) = e^{-r}\left[q_1(1)c_2(1) + (1-q_1(1))c_2(2)\right] =$$
$$= e^{-0.05}\left[0.43588 \times 35 + (1-0.43588) \times 5\right] = 17.195$$
$$V_1(2) = e^{-r}\left[q_1(2)c_2(3) + (1-q_1(2))c_2(4)\right] = \ldots = 0$$

- The computation of $q_1(2)$ was pointless in this concrete example!
- $$q_0 = \frac{e^r - d_0(1)}{u_0(1) - d_0(1)} = \frac{e^{0.05} - 3/4}{3/2 - 3/4} = 0.40169.$$
- The derivative price at time 0 is:

$$V_0 = e^{-r}\left[q_0 V_1(1) + (1-q_0)V_1(2)\right]$$
$$= e^{-0.05}\left(0.40169 \times 17.195 + (1-0.40169) \times 0\right)$$
$$= 6.570.$$

n-period binomial tree

- At time t there are 2^t possible states $(t,1),(t,2),...,(t,2^t)$.
- In state (t,j) the price of the underlying stock is $S_t(j)$. From this state the price can:
  1. go up to $S_{t+1}(2j-1) = S_t(j)u_t(j)$ and state $(t+1,2j-1)$;
  2. or down to $S_{t+1}(2j) = S_t(j)d_t(j)$ and state $(t+1,2j)$.
- The $2j$'s come in because of the ways the states are numbered in each column.
- The binomial model is recognised as an effective model (provided the time to maturity is broken up into a suitable number of sub-periods) for pricing and valuing derivative contracts. In this respect we might describe the binomial model as a good computational tool.

n-period binomial tree

- With constant r such that: $d_t(j) < e^r < u_t(j)$ then

$$q_t(j) = \frac{e^r - d_t(j)}{u_t(j) - d_t(j)}.$$

- The probabilities $q_t(j)$ define the risk-neutral probability Q and (as before):

$$E_Q[S_{t+1}|\mathcal{F}_t] = S_t e^r.$$

- Exercise: prove that $E_Q[S_{t+1}|\mathcal{F}_t] = S_t e^r$ for $t = 0, 1, 2, ...$

n-period binomial tree

- By the "tower property"

$$E_Q\left[S_{t+1}|\mathcal{F}_0\right] = E_Q\left[E_Q\left[S_{t+1}|\mathcal{F}_t\right]|\mathcal{F}_0\right]$$
$$= E_Q\left[S_t e^r|\mathcal{F}_0\right] = e^r E_Q\left[S_t|\mathcal{F}_0\right]$$

and by induction:

$$E_Q\left[S_{t+1}\right] = E_Q\left[S_{t+1}|\mathcal{F}_0\right] = S_0 e^{(t+1)r},$$

and the name "risk-neutral measure"still makes sense.

n-period binomial tree

- $C_n(j)$: payoff under a derivative maturing at time n in state (n, j).
- $V_t(j)$: price of the derivative at time t in state (t, j).
- The corresponding random variables are denoted by C_n and V_t respectively.
- we calculate prices by starting at the maturity date and working backwards.
- $V_n(j) = C_n(j)$ and for $t < n$:

$$V_t = e^{-r} E_Q[V_{t+1}|\mathcal{F}_t]$$

- Moving backwards from time n to time t:

$$V_t = e^{-r(n-t)} E_Q[C_n|\mathcal{F}_t]$$

n-period binomial tree

- By analogy with the 1-period model, we can construct a replicating strategy (or portfolio) (ϕ_{t+1}, ψ_{t+1}) where:

$$\begin{aligned}
\phi_{t+1}(j) &= \text{number of units of stock in state } (t,j) \\
&= \frac{V_{t+1}(2j-1) - V_{t+1}(2j)}{S_t(j)(u_t(j) - d_t(j))}, \\
\psi_{t+1}(j) &= \text{cash amount in state } (t,j) \\
&= e^{-r}\left(\frac{V_{t+1}(2j)\,u_j(t) - V_{t+1}(2j-1)\,d_j(t)}{u_t(j) - d_t(j)}\right).
\end{aligned}$$

- the holding of $\phi_{t+1}(j)$ shares means that the difference between the value of the holding in shares when prices go up and when prices go down precisely matches the difference between the two possible values of the option. This means that all risk has been removed:

$$\phi_{t+1}(j)\,S_t(j)\,(u_t(j) - d_t(j)) = V_{t+1}(2j-1) - V_{t+1}(2j).$$