Models in Finance - part 11

Master in Actuarial Science

João Guerra

ISEG

Recombining binomial trees

- the previous binomial model allows for different values of volatility when in different states (it allows different up and down factors for different states): $u_t(j)$ and $d_t(j)$ vary with t and j.
- However, the previous model has a drawback: the number of states at time n is 2^n states: if n is large, it is a big number (for computational purposes), since computation times even for simple derivative securities are at best proportional to the number of states.
- With 20 periods, at time $t = 20$ we have $2^{20} = 1048600$ states.

Recombining binomial trees

- One solution to this problem is assuming that the volatility is the same at all states (the up and down factors are the same irrespective of wether they appear in the binomial tree).
- Assume: $u_t(j) = u$; $d_t(j) = d$ $\Rightarrow$then: $q_t(j) = q \quad$ for all t, j with $d < e^r < u$, and $0 < q < 1$.

Recombining binomial trees

- Let N_t be the number of up-steps between time 0 and time t. Then:
$$S_t = S_0 u^{N_t} d^{t-N_t}.$$
- At time n we have $n+1$ possible states instead of 2^n.
- So, in a 20-period model, we have 21 states at time $t = 20$, instead of 1048600 states.

Recombining binomial trees

- Computing times are substantially reduced if the payoff of the derivative is not path-dependent: that is, it depends upon the number of up-steps and down-steps but not of their order.
- For non-path-dependent derivatives, we have the payoff $C_n = f(S_n)$ for some function f.
- For example, for the European put option: $f(x) = \max\{K - x, 0\}$ and $f(S_n) = \max\{K - S_n, 0\}$.

Recombining binomial trees

- This form of the n period model is called a "recombining binomial tree" or a "binomial lattice".
- Under this model, the q-probabilities are equal, and all steps are made independent of one another.
- The number of up-steps up to time t, N_t, has a binomial distribution with parameters t and q.

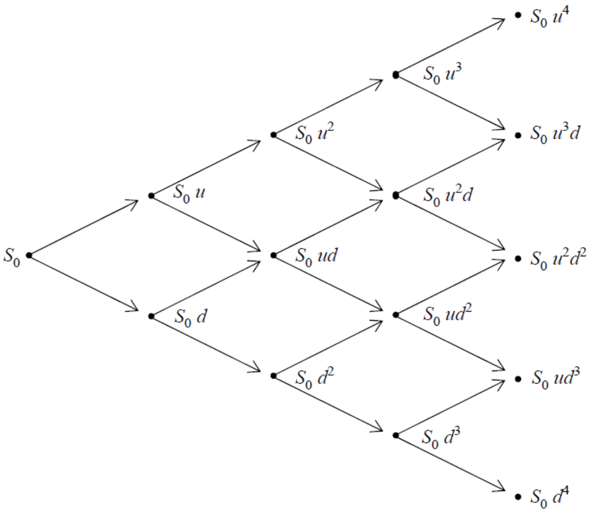
Recombining binomial trees

- For $0 < t < n$, N_t is independent of $N_n - N_t$ (number of up-steps (and down-steps) in non-overlapping time intervals is independent) and $N_n - N_t$ has a binomial distribution with parameters $n - t$ and q.
- The price at time t of the financial derivative is:

$$V_t = e^{-r(n-t)} \sum_{k=0}^{n-t} f\left(S_t u^k d^{n-t-k}\right) \frac{(n-t)!}{k!\,(n-t-k)!} q^k (1-q)^{n-t-k}.$$

- Unlike the non-recombining model, there will usually be more than one route from the initial node to any particular final node.

Recombining binomial trees



Calibrating binomial models

- It is often convenient when calibrating the binomial model to have the mean and variance implied by the binomial model corresponding to the mean and variance of a log-normal distribution.
- For recombining binomial models an additional condition that leads to a unique solution is:
$$u = \frac{1}{d}.$$
- Recall that the solution of the lognormal (or geometric Brownian motion) model with SDE $dS_t = \alpha S_t dt + \sigma S_t dB_t$ is such that $\left(\frac{S_t}{S_0}\right)$ has a lognormal distribution with parameters $\left(\alpha - \frac{1}{2}\sigma^2\right) t$ and $\sigma^2 t$.

Calibrating binomial models

- If we parametrize the lognormal distribution under the risk-neutral probability measure Q, so that:

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left[\left(r - \frac{1}{2}\sigma^2\right)(t - t_0), \sigma^2 (t - t_0)\right],$$

then the conditions that must be met are (where δt is the time interval of each step in the binomial model):

$$E_Q\left[\frac{S_{t+\delta t}}{S_t}\right] = \exp(r\delta t), \qquad (1)$$

$$\mathrm{var}_Q\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right] = \sigma^2 \delta t \qquad (2)$$

Calibrating binomial models

- Note also that in the binomial model:

$$E_Q\left[\frac{S_{t+\delta t}}{S_t}\right] = qu + (1-q)\,d.$$

And from Eq. (1), we get

$$q = \frac{e^{r\delta t} - d}{u-d}. \tag{3}$$

- If we use Eq. (2) and the assumption $u = 1/d$, we obtain:

$$\mathrm{var}_Q\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right] = q\,(\ln u)^2 + (1-q)\,(-\ln u)^2$$
$$-\left\{E\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right]\right\}^2 = (\ln u)^2 - \left\{E\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right]\right\}^2$$

Calibrating binomial models

- The last term involves terms of higher order than δt, i.e. $\left\{E\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right]\right\}^2 = f\left((\delta t)^2\right)$ which tends to zero as $\delta t \to 0$.
- So, ignoring the terms of order higher than δt, we obtain:

$$(\ln u)^2 = \sigma^2 \delta t.$$

Solving, we obtain (σ is the volatility):

$$u = \exp\left(\sigma\sqrt{\delta t}\right), \tag{4}$$

$$d = \exp\left(-\sigma\sqrt{\delta t}\right). \tag{5}$$

Calibrating binomial models

- When a continuously payable dividend rate ν is paid on the underlying asset, it is convenient to adjust the steps to be:

$$u = \exp\left(\sigma\sqrt{\delta t} + \nu\delta t\right),$$
$$d = \exp\left(-\sigma\sqrt{\delta t} + \nu\delta t\right).$$

The state price deflator approach: 1 period

- Recall the 1-period binomial model where:

$$V_1 = \begin{cases} c_u \text{ if } S_1 = S_0 u \\ c_d \text{ if } S_1 = S_0 d \end{cases},$$

$$V_0 = e^{-r} E_Q [V_1] = e^{-r} [q c_u + (1-q) c_d].$$

- We can re-express V_0 in terms of the real world probability p:

$$V_0 = e^{-r} \left[p \frac{q}{p} c_u + (1-p) \frac{(1-q)}{(1-p)} c_d \right]$$
$$= E_P [A_1 V_1],$$

where A_1 is the random variable:

$$A_1 = \begin{cases} e^{-r} \frac{q}{p} \text{ if } S_1 = S_0 u \\ e^{-r} \frac{(1-q)}{(1-p)} \text{ if } S_1 = S_0 d \end{cases}.$$

The state price deflator approach: 1 period

- A_1 is called a state-price deflator (or deflator, or state-price density, or pricing kernel or stochastic discount factor).
- Note that the discount factor A_1 depends wheter the share price goes up or down (it is a stochastic discount factor).
- Note that:
  1. if $V_1 = 1$ then: $V_0 = E_p \left[A_1 \times 1 \right] = e^{-r}$.
  2. if $V_1 = S_1$ then: $V_0 = E_p \left[A_1 \times S_1 \right] = S_0$.

The state price deflator approach: n periods

- In the n-period recombining binomial model (risk-neutral aproach) we have:

$$
\begin{aligned}
V_n &= f(S_n), \\
S_n &= S_0 u^i d^{n-i} \text{ if } i \text{ is the number of up-steps.}
\end{aligned}
$$

- Therefore, define $V_n(i) = f\left(S_0 u^i d^{n-i}\right)$.
  Then we have:

$$
\begin{aligned}
V_0 &= e^{-rn} E_Q\left[V_n\right] \\
&= e^{-rn} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} q^k (1-q)^{n-k} f\left(S_0 u^k d^{n-k}\right).
\end{aligned}
$$

The state price deflator approach: n periods

- We can re-express this in terms of the real world probability p by:

$$\begin{aligned} V_0 &= e^{-rn} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} \left(\frac{q}{p}\right)^k \left(\frac{1-q}{1-p}\right)^{n-k} V_n(k) \\ &= \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} A_n(k)\, V_n(k) \\ &= E_P[A_n V_n], \end{aligned}$$

where $A_n = e^{-rn} \left(\frac{q}{p}\right)^{N_n} \left(\frac{1-q}{1-p}\right)^{n-N_n}$ and N_n is the number of up-steps up to time n.

- The discount factor A_n is again random and we call it the state-price deflator or stochastic discount factor.

The state price deflator approach: n periods

- Important property of A_n is:

$$A_n = A_{n-1} \times e^{-r} \left(\frac{q}{p}\right)^{I_n} \left(\frac{1-q}{1-p}\right)^{1-I_n}$$

where:

$$I_n = \begin{cases} 1 & \text{if } S_n = S_{n-1}u \\ 0 & \text{if } S_n = S_{n-1}d \end{cases}.$$

- Moreover:

$$S_n = S_{n-1} u^{I_n} d^{1-I_n},$$

$$N_n = \sum_{k=0}^{n} I_k.$$

The state price deflator approach: n periods

- The risk-neutral and the state-price deflator approaches give the same price V_0.
- Theoretically, they are the same: only differ in the way that they present the calculation of a derivative price.
- As expected, note that:
  1. if $V_n = 1$ then: $V_0 = E_P[A_n] = e^{-rn}$.
  2. if $V_n = S_n$ then: $V_0 = E_p[A_n \times S_n] = S_0$.
- The state-price-deflator approach can be adapted to price a derivative at any time t and:
$$V_t = \frac{E_p[A_T V_T]}{A_t},$$
where T is the expiry date.