Models in Finance - part 12

Master in Actuarial Science

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Continuous time models: preliminary concepts

- $(\Omega, \mathcal{F}, P)$: probability space where P is the real-world probability measure
- S_t (the price process of the risky asset) is adapted (measurable with respect) to the filtration $\mathcal{F}_t$ (given $\mathcal{F}_t$, we know the value of S_u for all $u \leq t$).
- Risk-free cash bond which has a value at time t of B_t.
- We will assume that the risk-free rate of interest is constant $\Longrightarrow B_t$ is deterministic and $B_t = B_0 e^{rt}$.
- Let $\mathcal{F}_t$ be the filtration generated by S_u $(0 \leq u \leq t)$.

Continuous time models: preliminary concepts

- Recall that the market is complete if for any contingent claim X there is a replicating strategy or portfolio (ϕ_t, ψ_t).
- Example of a complete market: the binomial model (we could replicate any derivative payment contingent on the history of the underlying asset price).
- Another example of a complete market is the continuous-time lognormal model for share prices:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma Z_t\right),$$

where Z_t is a standard Brownian motion.

Continuous time models: preliminary concepts

- Two measures P and Q which apply to the same sigma-algebra $\mathcal{F}$ are said to be equivalent if for any event $E \in \mathcal{F} : P(E) > 0$ if and only if $Q(E) > 0$, where $P(E)$ and $Q(E)$ are the probabilities of E under P and Q respectively.
- For the binomial model, for the equivalence of P and Q the only constraint on the real-world measure P is that at any point in the binomial tree the probability of an up move lies strictly between 0 and 1. The only constraint on Q is the same.

Continuous time models: preliminary concepts

- Suppose that Z_t is a standard Brownian motion under P and let $X_t = \gamma t + \sigma Z_t$ be a Brownian motion with drift under P.
- Is there a measure Q under which X_t is a standard Brownian motion and which is equivalent to P ?
- Yes if $\sigma = 1$ but no if $\sigma \neq 1$
- In other words: we can change the drift of the Brownian motion but not the volatility.
- Theorem (Cameron-Martin-Girsanov): Suppose that Z_t is a standard Brownian motion under P and that γ_t is a previsible process. Then there exists a measure Q equivalent to P and where $\widetilde{Z}_t = Z_t + \int_0^t \gamma_s ds$ is a standard Brownian motion under Q. Conversely, if Z_t is a standard Brownian motion under P and if Q is equivalent to P then there exists a previsible process γ_t such that $\widetilde{Z}_t = Z_t + \int_0^t \gamma_s ds$ is a standard Brownian motion under Q.

- Assume that under P (geometric Bm):$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma Z_t\right)$. Then ($e^{-rt} S_t$ is the discounted price):
$$E_P\left[e^{-rt} S_t\right] = e^{(\mu - r)t}$$
and $e^{-rt} S_t$ is not a martingale under P (unless $\mu = r$).
- Take $\gamma_t = \gamma = \frac{\mu - r}{\sigma}$ and define $\widetilde{Z}_t = Z_t + \int_0^t \gamma_s ds = Z_t + \frac{(\mu - r)}{\sigma} t$. Then:
$$\begin{aligned} S_t &= S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma \widetilde{Z}_t - (\mu - r)\, t\right) \\ &= S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t + \sigma \widetilde{Z}_t\right). \end{aligned}$$
By the Cameron-Martin-Girsanov theorem, exists Q equivalent to P such that $\widetilde{Z}_t$ is a Q-standard Bm.

- And clearly, we have (for $u < t$):

$$
\begin{aligned}
&E_Q\left[e^{-rt}S_t|\mathcal{F}_u\right] = \\
&= e^{-rt}S_u E_Q\left[\exp\left(\left(r-\frac{1}{2}\sigma^2\right)(t-u)+\sigma\left(\widetilde{Z}_t-\widetilde{Z}_u\right)\right)\right] \\
&= e^{-ru}S_u E_Q\left[\exp\left(\left(-\frac{1}{2}\sigma^2\right)(t-u)+\sigma\left(\widetilde{Z}_t-\widetilde{Z}_u\right)\right)\right] \\
&= e^{-ru}S_u e^{\left(-\frac{1}{2}\sigma^2\right)(t-u)+\frac{1}{2}\sigma^2(t-u)} = e^{-ru}S_u
\end{aligned}
$$

- Therefore, the discounted price $e^{-rt}S_t$ is a Q-martingale.

Continuous time models: preliminary concepts

- Suppose that X_t is a P-martingale and Y_t is another P-martingale.
- Martingale Representation Theorem (MRT): Exists a unique previsible process ϕ_t such that

$$Y_t = Y_0 + \int_0^t \phi_s dX_s \quad (\text{or: } dY_t = \phi_t dX_t)$$

if and only if there is no other measure equivalent to P under which X_t is a martingale.

5 step method to establish a risk-neutral valuation formula

1. Establish the unique equivalent martingale measure Q under which the discounted asset price process $D_t = e^{-rt}S_t$ is a martingale.
2. Propose the fair price for the derivative $V_t = e^{-r(T-t)}E_Q[X|\mathcal{F}_t]$ and define its discounted value $F_t = e^{-rt}V_t$
3. Under Q, $F_t = e^{-rt}V_t$ is a martingale.
4. By the MRT there exists a previsible process ϕ_t such that $dF_t = \phi_t dD_t$.
5. Define $\psi_t = F_t - \phi_t D_t$. Consider the porfolio (ϕ_t, ψ_t). Show that this portfolio is a replicating portfolio and that it is self-financing.