Models in Finance - Part 14

Master in Actuarial Science

João Guerra

ISEG

Black-Scholes model - PDE approach

- idea: use Itô's formula to derive an expression for the price of the derivative as a function $f(S_t)$ of S_t and then construct a risk-free portfolio.
- By Itô's formula:

$$df(t,S_t) = \frac{\partial f}{\partial t}(t,S_t)\,dt + \frac{\partial f}{\partial S_t}(t,S_t)\,dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}(t,S_t)\,(dS_t)^2 . \tag{1}$$

- Recall that $dS_t = S_t(\mu dt + \sigma dZ_t)$ and therefore

$$\begin{aligned}(dS_t)^2 &= S_t^2\left[\mu^2(dt)^2 + \sigma^2(dZ_t)^2 + 2\mu\sigma dt dZ_t\right] \\ &= \sigma^2 S_t^2 dt\end{aligned}$$

(why?)

PDE approach

- Therefore:

$$
\begin{aligned}
df(t, S_t) &= \frac{\partial f}{\partial t}(t, S_t)\, dt + \frac{\partial f}{\partial S_t}(t, S_t)\left[S_t\left(\mu dt + \sigma dZ_t\right)\right] \\
&+ \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}(t, S_t)\,\sigma^2 S_t^2 dt \\
&= \left[\frac{\partial f}{\partial t}(t, S_t) + \mu S_t \frac{\partial f}{\partial S_t}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2}(t, S_t)\right] dt \qquad (2) \\
&+ \sigma S_t \frac{\partial f}{\partial S_t}(t, S_t)\, dZ_t. \qquad (3)
\end{aligned}
$$

PDE approach

- At time t with $0 \leq t < T$, consider you hold the portfolio:
- -1 derivative $+ \frac{\partial f}{\partial S_t}(t, S_t)$ shares
- Let $V(t, S_t)$ be the value of this portfolio:

$$V(t, S_t) = -f(t, S_t) + \frac{\partial f}{\partial S_t}(t, S_t) S_t.$$

- The variation of the portfolio value over the period $(t, t + dt]$ is (by Eq. (2) and (3))

$$\begin{aligned} & -df(t, S_t) + \frac{\partial f}{\partial S_t}(t, S_t) dS_t \\ & = -\left(\frac{\partial f}{\partial t}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2}(t, S_t) \right) dt \end{aligned} \quad (4)$$

PDE approach

- $-df(t, S_t) + \frac{\partial f}{\partial S_t}(t, S_t)\, dS_t$ involves dt but not $dZ_t \Longrightarrow$ instantaneous investment gain in $(t, t+dt]$ is risk-free.
- arbitrage-free market $\Longrightarrow$risk-free rate $= r \Longrightarrow$

$$-df(t, S_t) + \frac{\partial f}{\partial S_t}(t, S_t)\, dS_t = rV(t, S_t)\, dt. \tag{5}$$

- By (4) and (5), we have:

$$\left(\frac{\partial f}{\partial t}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2}(t, S_t)\right) dt = -rV(t, S_t)\, dt$$
$$= -r\left(-f(t, S_t) + \frac{\partial f}{\partial S_t}(t, S_t)\, S_t\right) dt$$

and therefore (substituting $S_t = s$)

$$\frac{\partial f}{\partial t}(t, s) + rs\frac{\partial f}{\partial s}(t, s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2}(t, s) = rf(t, s). \tag{6}$$

- This is the Black-Scholes PDE (partial differential equation).

PDE approach

- The value of the derivative $f(t, S_t)$ is obtained by solving the B-S PDE with appropriate boundary conditions, which are for the call and put:

$$f(T, s) = \max\{s - K, 0\} \quad \text{for the call,}$$
$$f(T, s) = \max\{K - s, 0\} \quad \text{for the put.}$$

- We can try out the solutions given in the proposition:

$$f(t, S_t) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2) \quad \text{for the call,} \tag{7}$$
$$f(t, S_t) = K e^{-r(T-t)} \Phi(-d_2) - S_t \Phi(-d_1) \quad \text{for the put,} \tag{8}$$

and find that they satisfy the PDE and the appropriate boundary conditions.

PDE approach

- Exercise: A forward contract is arranged where an investor agrees to buy a share at time T for an amount K. It is proposed that the fair price of this contract is

$$f\left(t, S_t\right) = S_t - Ke^{-r(T-t)}.$$

Show that this:
(i) Satisfies the appropriate boundary condition.
(ii) Satisfies the Black-Scholes PDE.

- Consider a contingent claim (a financial derivative), with payoff given by
$$X = \Phi\left(S\left(T\right)\right). \tag{9}$$
Its price process is represented by
$$\Pi\left(t\right), \quad t \in \left[0, T\right].$$

Portfolios

- Portfolio $\left(h^0(t), h^*(t)\right)$
- $h^0(t)$: number of bonds (or number of units of the riskless asset) at time t.
- $h^*(t)$: number of of shares of stock in the portfolio at time t.

Portfolios

- Value of the portfolio at time t:

$$V^h(t) = h^0(t) B_t + h^*(t) S_t.$$

- It is supposed that the portfolio is self-financed, that is,

$$dV_t^h = h^0(t) dB_t + h^*(t) dS_t.$$

- In integral form:

$$\begin{aligned} V_t &= V_0 + \int_0^t h^*(s) dS_s + \int_0^t h^0(s) dB_s \\ &= V_0 + \int_0^t \left(\alpha h^*(s) S_s + r h^0(s) B_s\right) ds + \sigma \int_0^t h^*(s) S_s dZ_s. \end{aligned} \tag{10}$$

Replicating portfolio

- Assume that the contingent claim (or financial derivative) has the payoff

$$X = \Phi\left(S\left(T\right)\right). \tag{11}$$

and it is replicated by the portfolio $h = \left(h^0\left(t\right), h^*\left(t\right)\right)$, that is, $V_T^h = X = \Phi\left(S\left(T\right)\right)$ a.s.
Then, the unique price process that is compatible with the no-arbitrage principle is

$$\Pi\left(t\right) = V_t^h, \quad t \in \left[0, T\right]. \tag{12}$$

- Moreover, assume also that

$$\Pi\left(t\right) = V_t^h = F\left(t, S_t\right). \tag{13}$$

where F is a differentiable function of class $C^{1,2}$.

Replicating portfolio

- Applying Itô's formula to (13) and considering $dS_t = \mu S_t dt + \sigma S_t dZ_t$, we obtain

$$dF(t, S_t) = \left(\frac{\partial F}{\partial t}(t, S_t) + \mu S_t \frac{\partial F}{\partial x}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(t, S_t) \right) dt$$
$$+ \left(\sigma S_t \frac{\partial F}{\partial x}(t, S_t) \right) dZ_t.$$

Replicating portfolio

That is,

$$F(t, S_t) = F(0, S_0) + \int_0^t \left(\frac{\partial F}{\partial t}(s, S_s) + AF(s, S_s) \right) ds$$
$$+ \int_0^t \left(\sigma S_s \frac{\partial F}{\partial x}(s, S_s) \right) dZ_s, \quad (14)$$

where

$$Af(t, x) = \mu x \frac{\partial f}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t, x)$$

is the infinitesimal generator associated to the diffusion S_t.

Replicating portfolio

- Comparing (10) and (14), we have

$$\sigma h^{*}(s)\, S_s = \sigma S_s \frac{\partial F}{\partial x}(s, S_s),$$
$$\mu h^{*}(s)\, S_s + r h^{0}(s)\, B_s = \frac{\partial F}{\partial t}(s, S_s) + AF(s, S_s).$$

- Therefore,

$$\frac{\partial F}{\partial x}(s, S_s) = h^{*}(s),$$
$$\frac{\partial F}{\partial t}(s, S_s) + r S_s \frac{\partial F}{\partial x}(s, S_s) + \frac{1}{2}\sigma^2 S_s^2 \frac{\partial^2 F}{\partial x^2}(s, S_s) - rF(s, S_s) = 0.$$

Replicating portfolio

Therefore, we have

- A portfolio h with value $V_t^h = F(t, S_t)$, composed of risky assets with price S_t and riskless assets of price B_t.
- Portfolio h replicates the contingent claim X at each time t, and

$$\Pi(t) = V_t^h = F(t, S_t).$$

- In particular,

$$F(T, S_T) = \Phi(S(T)) = \text{Payoff}.$$

Black-Scholes PDE

- The portfolio should be continuously updated by acquiring (or selling) $h^* (t)$ shares of the risky asset and $h^0 (t)$ units of the riskless asset, where

$$h^* (t) = \frac{\partial F}{\partial x} (t, S_t),$$
$$h^0 (t) = \frac{V_t^h - h^* (t) S_t}{B_t} = \frac{F (t, S_t) - h^* (t) S_t}{B_t}.$$

- The derivative price function satisfies the PDE (Black-Scholes eq.)

$$\frac{\partial F}{\partial t} (t, S_t) + rS_t \frac{\partial F}{\partial x} (t, S_t) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2} (t, S_t) - rF (t, S_t) = 0.$$

Black-Scholes PDE

Theorem

(Black-Scholes eq.) The only pricing function that is consistent with the no-arbitrage principle is the solution F of the following boundary value problem, defined in the domain $[0, T] \times \mathbb{R}^+$:

$$\frac{\partial F}{\partial t}(t,x) + rx\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0, \qquad (15)$$

$$F(T,x) = \Phi(x).$$

The martingale approach

- In the binomial model, we proved that the value of a derivative could be expressed by:

$$V_t = e^{-r(T-t)} E_Q \left[X | \mathcal{F}_t \right],$$

  where X is the value of the derivative at maturity T and Q is the equivalent martingale measure (or risk neutral measure).

- In continuous time, this result can be generalized as:

Proposition: Let X be any derivative payment contingent on $\mathcal{F}_T$, payable at T. Then the value of this derivative at time $t < T$ is

$$V_t = e^{-r(T-t)} E_Q \left[X | \mathcal{F}_t \right]. \tag{16}$$

Proof of the risk neutral valuation

The price function F is solution of the following boundary value problem:

$$\frac{\partial F}{\partial t}(t,x) + rx\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0, \qquad (17)$$

$$F(T,x) = \Phi(x).$$

Applying the Itô formula to $e^{-rs}F(s,X_s)$, where $dX_s = rX_s ds + \sigma X_s dZ_s$, $t \leq s \leq T$ and $X_t = x$, we obtain

$$e^{-rT}F(T,X_T) = e^{-rt}F(t,X_t) +$$

$$+\int_t^T e^{-rs}\left(\frac{\partial F}{\partial s}(s,X_s) + \left(rX_s\frac{\partial}{\partial x} + \sigma^2 X_s^2 \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)F(s,X_s) - rF(s,X_s)\right) ds$$

$$+\int_t^T e^{-rs}\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)\, dZ_s.$$

Proof of the risk neutral valuation

Using (17) and applying the expected value (with $X_t = x$), we obtain

$$E_{t,x}\left[e^{-r(T-t)}F\left(T, X_T\right)\right] = E_{t,x}\left[F\left(t, X_t\right)\right],$$

Therefore

$$F\left(t, x\right) = e^{-r(T-t)}E_{t,x}\left[\Phi(X_T^{t,x})\right],$$

- Note that the process X is not the same as the process S, as the drift of X is rX and not μX.
- idea: change from process X to process S, using the Girsanov (Cameron-Martin-Girsanov) Theorem.

Proof of the risk neutral valuation

- Denote by P the original probability measure ("objective" or "real" probability measure). The P-dynamics of the process S is given in $dS_t = \mu S_t dt + \sigma S_t dZ_t$. Note that this is equivalent to

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma S_t \left(\frac{\mu - r}{\sigma} dt + dZ_t \right) \\
&= rS_t dt + \sigma S_t d \underbrace{\left(\frac{\mu - r}{\sigma} t + Z_t \right)}_{\widetilde{Z}_t}.
\end{aligned}
$$

- By the Girsanov Theorem, there exists a probability measure Q such that, in the probability space $(\Omega, \mathcal{F}_T, Q)$, the process

$$
\widetilde{Z}_t := \frac{\mu - r}{\sigma} t + Z_t
$$

is a Brownian motion, and S has the Q-dynamics:

$$
dS_t = rS_t dt + \sigma S_t d\widetilde{Z}_t. \tag{18}
$$

Proof of the risk neutral valuation

- Consider the following notation: E denotes the expected value with respect to the original measure P, while E_Q denotes the expected value with respect to the new probability measure Q (that comes from the application of the Girsanov theorem). Also, let Z_t denote the original Brownian motion (under the measure P) and $\widetilde{Z}_t$ denote the Brownian motion under the measure Q.
- We represent the solution of the Black-Scholes equation by

$$F(t, s) = e^{-r(T-t)} E_Q \left[X | \mathcal{F}_t \right],$$

where $X = \Phi(S_T)$ represents the payoff, and the dynamics of S under the measure Q is

$$dS_u = rS_u du + \sigma S_u d\widetilde{Z}_u, t \leq u \leq T,$$

$$S_t = s.$$

Delta hedging and martingale approach

- How to determine ϕ_t of the replicating portfolio?
- We can evaluate the price of the derivative $V_t = e^{-r(T-t)} E_Q [X|\mathcal{F}_t]$ using a formula (like the B-S formula) or numerical techniques.
- Then

$$\phi_t = \frac{\partial V}{\partial s} (t, S_t) . \tag{19}$$

- ϕ_t is called the Delta of the derivative:

$$\Delta = \frac{\partial V}{\partial s} (t, S_t) . \tag{20}$$

Delta hedging and martingale approach

If:

- we start at time 0 with V_0 invested in cash and shares,
- we follow a self-financing portfolio strategy,
- we continually rebalance the portfolio to hold exactly $\phi_t = \Delta = \frac{\partial V}{\partial s}(t, S_t)$ units of S_t with the rest in cash,

then we will precisely replicate the derivative payoff.

Example: B-S formula for a call

- Let $X = \max\{S_T - K, 0\}$.
  Then:
  $$V_t = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \tag{21}$$
  where: $d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$ and $\Phi(z)$ is the cumulative distribution function of the standard normal distribution.

Proof:

- Given the information $\mathcal{F}_t$, then under Q, we have:

$$S_T = S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(\widetilde{Z}_T - \widetilde{Z}_t\right)\right]. \qquad (22)$$

Then

$$V_t = e^{-r(T-t)} E_Q\left[\max\left\{S_T - K, 0\right\} | \mathcal{F}_t\right]$$
$$= e^{-r(T-t)}$$
$$\times E_Q\left[\max\left\{S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(\widetilde{Z}_T - \widetilde{Z}_t\right)\right] - K, 0\right\} | \mathcal{F}_t\right]$$
$$= E_Q\left[\max\left\{e^{\alpha+\beta U} - e^{\alpha+\beta u}, 0\right\}\right],$$

where $\alpha = \log(S_t) - \frac{1}{2}\sigma^2(T-t)$, $\beta = \sigma\sqrt{T-t}$, $U \sim N(0,1)$ under Q and $u = \left[\log\left(Ke^{-r(T-t)}\right) - \alpha\right]/\beta$.

Example: B-S formula for a call

Proof:

- Therefore (with $\phi(x)$ the density of the $N(0,1)$ distribution):

$$
\begin{aligned}
V_t &= e^{\alpha+\beta u} \int_u^{\infty} \left(e^{\beta(x-u)} - 1 \right) \phi(x)\, dx \\
&= e^{\alpha} \int_u^{\infty} \frac{1}{\sqrt{2\pi}} e^{\beta x - \frac{1}{2}x^2} dx - e^{\alpha+\beta u} \Phi(-u) \\
&= e^{\alpha + \frac{1}{2}\beta^2} \int_u^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\beta)^2} dx - e^{\alpha+\beta u} \Phi(-u) \\
&= e^{\alpha + \frac{1}{2}\beta^2} \Phi(\beta - u) - e^{\alpha+\beta u} \Phi(-u) = \ldots \\
&= S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2).
\end{aligned}
$$

- Exercise: Prove the B-S formula for the put option, using the same technique.