SIMULATION

- **Introduction**
  - **Basic idea**: To use methods based on random number generation to simulate "complex" processes to get an insight about the random behavior of some important variables.
  - The first "real" use of simulation was related to the "Manhattan project" at Los Alamos at the end of World War 2 (nuclear bomb) and was linked to the names of von Neumann, Stanislaw Ulam and Nicholas Metropolis. Since then, simulation techniques are used in many scientific domains and cover a great variety of applications.
  - The use of simulation to obtain an approximation to the sampling distribution of a statistic illustrates the process. The process can be summarized in three stages:
    a. Simulate the production of random samples from the population.
    b. For each sample generated in the previous stage, get the value of the statistic for which we want to approximate the sampling distribution.
    c. Now we can use the simulated values of the statistic to get an approximation to its sampling distribution (each replica provides an "observed value" of the statistic).
  - Two problems remain: how to generate the random samples and how many replicas should we use.

- **How to generate random samples from a population with a known distribution?**

  - We follow a two-step procedure:
    a. First, we generate pseudorandom numbers, i.e., values that can be considered as independent observations of a random variable with a uniform (0;1) distribution.
    b. Second, using probability theory we transform the pseudorandom numbers to get pseudo random observations from the wanted distribution.
  - In practice, the two stages are merged when we are using a computer. For instance, in R, the command rnorm(50,0,1) will generate 50 “random” observations from a normal distribution with mean 0 and standard deviation 1.

Pseudo random numbers generation

- **Random numbers versus pseudorandom numbers**. A random number is a realization of a continuous random variable following a uniform distribution between 0 and 1. All the generated variables should be independent from each other. As it is not possible to generate these numbers (the best we can get is to generate discrete variables) we will use pseudorandom numbers.
- A **pseudorandom number generator** is an algorithm for generating a sequence of numbers that look like independent observations of a U(0;1) distribution. Obviously, the sequence is not random as it is completely determined by a relatively small set of initial values. However, if we generate a sequence and test it both the independence and the distribution hypotheses will not be rejected.
- Pseudorandom generation is still an area of active research (one main application of random numbers is cryptography). The Mersenne-Twister algorithm (Matsumoto and Nishimura, 1997) replaced the linear congruential generator as the “standard”.
- Testing procedures to test the quality of a random number generator can be found in the literature. The most well-known are Diehart test (Marsaglia, 1995) and TestU01 (L’Ecuyer and Simard, 2007). Both are composed by a battery of tests. The latter webpage is

  Simul.iro.umontreal.ca/testu01/testu01.html

- Example: As the Mersenne -Twister algorithm is far from simple we will illustrate the main idea using the multiplicative linear congruential generator introduced by Lehmer in 1949. The idea is the following

  $n_t = a\, n_{t-1} \text{ mod } c \qquad t = 1, 2, \cdots$

  $u_t = n_t / c$

  where u_t is *t*-th generated random number ($t = 1, 2, \cdots$) and a, c and n_0 are three positive integers with adequate properties. n_0 is called the seed.

  - Example

    Let $a = 123$, $c = 1000$ and $n_0 = 135$. Generate 25 pseudo random numbers. Verify that $n_{20} = n_0$ and comment (Use Excel or another software).

  - As illustrated by the example, the maximum length of the sequence before it begins to repeat can be a major concern. Another important points are to verify if we can assume that the observations **(i)** could have been generated by an independent process and **(ii)** mimic a U(0;1) behavior.

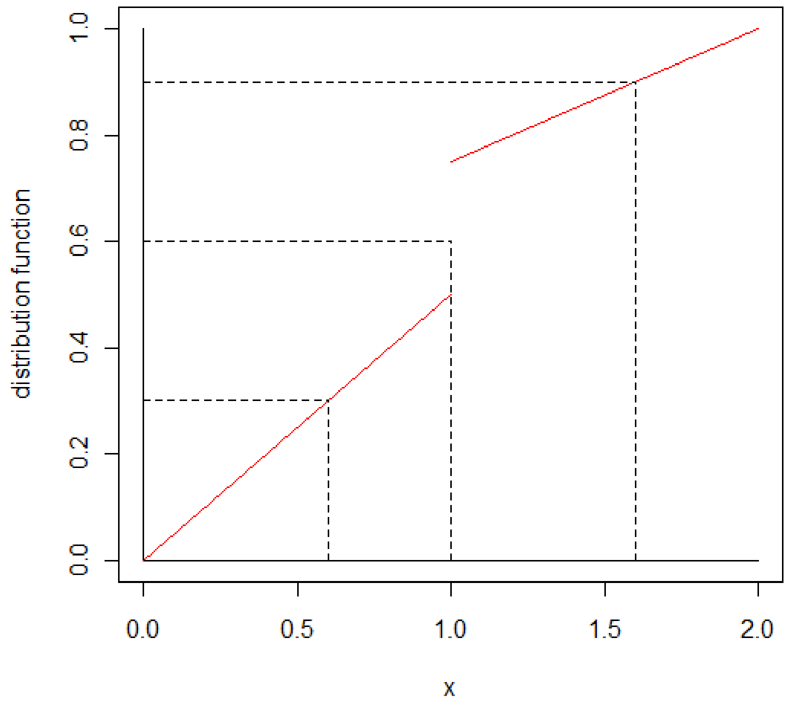
    Parameters a, c and n_0 are chosen to answer these points.

From a uniform distribution to other distributions

- Assuming that we got a sequence of independently distributed uniform variables between 0 and 1, the problem is now how to define functions of these variables that follow a given distribution. We can give a general answer to this problem, but in many cases, we can design more efficient methods.
- The general answer is called the **inverse transform** and is based on the following theorem:
- **Theorem**: Let $Y \sim U(0;1)$ and $F(x)$ be the distribution function of a continuous random variable (we are assuming that F is strictly increasing on the open interval (a,b) where X has positive density – the interval can be unlimited – with $F(a)=0$ and $F(b)=1$). Then the random variable $X = F^{-1}(Y)$ is continuous and $F(x)$ is its distribution function.
- Example: Illustrate the procedure using (i) the exponential distribution (can be done without a computer) and (ii) the normal distribution (needs a computer).
- Extensions:
  - Suppose that the distribution function has a jump at $x=c$, i.e., $F(c^-)=a$ and $F(c)=b$ with $b>a$. If the random number u is such that $a \le u < b$, choose c as the simulated value.
  - Suppose that the distribution function is constant in a given interval, i.e, $F(x)=p$ for $a \le x \le b$. If the random number u is equal to p (abnormal case) choose $x=b$;

- **Example 20.2** – Suppose $F_X(x) = \begin{cases} 0.5x & 0 \leq x < 1 \\ 0.5 + 0.25x & 1 \leq x \leq 2 \end{cases}$.

  Determine the simulated values of x resulting from the uniform numbers 0.3, 0.6 and 0.9.



- **Specific answers -** For many situations one can design a computationally more efficient (and reliable) method to use with specific distributions. The idea is to take advantage of the properties of these distributions.
- **Examples:**
    1. To generate a binomial observation, we can generate *n* Bernoulli observations and sum them.
    2. To generate a discrete mixture: $F_Y(y) = \alpha_1 F_{X_1}(y) + \alpha_2 F_{X_2}(y) + \cdots + \alpha_k F_{X_k}(y)$ we can use a 2 steps procedure (computing the inverse of the distribution function can be challenging)
        - Simulate a discrete random variable where $\Pr(J = j) = a_j,\ j = 1, 2, \cdots, k$.
        - Simulate X_j using an appropriate method
    3. To generate normal random observations, the Box-Muller formula was traditionally used. If (U_1, U_2) are independent uniform variables, then $Z_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2)$ and $Z_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$ are independent $n(0;1)$ random variables.
- If nothing else is said and the desired distribution is available we can use R to generate our pseudo-random observations in just one step.

- **Example 20.1** – generate 10000 pseudo Pareto (with $\alpha = 3$ and $\theta = 1000$) variables and verify that they are indistinguishable from real Pareto observations.

  Procedure (the Pareto distribution is not available in the base R):

  1. Generate 10000 (pseudo) uniforms, u_i
  2. Obtain 10000 (pseudo) Pareto, $x_i = \theta\left((1-u_i)^{-1/\alpha} - 1\right)$
  3. Compare the generated values with the theoretical model:
     - Kolmogorov-Smirnov test
     - Anderson-Darling test
     - Cramer-Von Mises Test (A test similar to the AD test but giving more weight to the core of the distribution
     - χ^2 goodness of fit test (*Loss Models* solution)
     - Other approaches (graphical techniques, …)

A possible solution using R

```
> # generate n Pareto (alpha,theta) variables - inverse method
> n=10000; alpha=3; theta=1000
> u=runif(n); x=theta*((1-u)^(-1/alpha)-1)
>
> Pareto_dist_func=function(x,alpha,theta) 1-(theta/(x+theta))^alpha

> # KS test
> ks.test(x,Pareto_dist_func,alpha=3,theta=1000)

        One-sample Kolmogorov-Smirnov test
data:  x
D = 0.0065653, p-value = 0.7818
alternative hypothesis: two-sided
>
> # AD test
> library(goftest)
> ad.test(x,Pareto_dist_func,alpha=3,theta=1000)
```

```
        Anderson-Darling test of goodness-of-fit
         Null hypothesis: distribution ‘Pareto_dist_func’
         with parameters alpha = 3, theta = 1000
         Parameters assumed to be fixed

data:  x
An = 0.59622, p-value = 0.6516
>
> # Cramer-Von Mises test
> cvm.test(x,Pareto_dist_func,alpha=3,theta=1000)

         Cramer-von Mises test of goodness-of-fit
         Null hypothesis: distribution ‘Pareto_dist_func’
         with parameters alpha = 3, theta = 1000
         Parameters assumed to be fixed

data:  x
omega2 = 0.11035, p-value = 0.5365

>
```

```
> # Qui2 test for large samples - m classes with the same probability
> m=100; aux1=(1:(m-1))/m; aux2=theta*((1-(aux1))^(-1/alpha)-1);
> lb=c(0,aux2); ub=c(aux2,Inf); counts=rep(NA,m)
> for (j in 1:m) counts[j]=sum((x>=lb[j]) & (x<ub[j]))
> expected=rep(n/m,m)
> chi2=((counts-expected)^2)/expected
> chi2.test=sum(chi2)
> p.value=pchisq(chi2.test,m-1,lower.tail=FALSE)
> cat("chi-square statistic = ",chi2.test,"\n","p-value =
",p.value,"\n")
chi-square statistic =  87.48
 p-value =  0.789566
> result=cbind(lb,ub,counts,expected,chi2)
> result
                lb          ub counts expected chi2
  [1,]    0.000000    3.355730     93      100 0.49
  [2,]    3.355730    6.756962     98      100 0.04
  …
 [99,] 2684.031499 3641.588834    113      100 1.69
[100,] 3641.588834         Inf     86      100 1.96
```

- **How many replicas should be used?**
  - The answer depends on the problem we want to solve.
  - Nowadays we can define a **huge number** of replicas for many situations. This is the usual solution.
  - Sometimes, using the Central Limit Theorem we can define an approximate value to the number of replicas to ensure a given precision. Example 21.5 illustrates 3 cases.
  - **Example 20.12** – Use simulation to estimate the mean, $F_X(1000)$ and $\pi_{0.9}$, the 90th percentile of the Pareto distribution with $\alpha = 3$ and $\theta = 1000$. In each case, stop the simulation when you are 95% confident that the answer is within $\pm 1\%$ of the true value.
    - As we know the true values, the simulation is useless but we will behave as these values are unknown ($\mu = 500$, $F_X(1000) = 0.875$, $\pi_{0.9} = 1000 \times \left(0.1^{-1/3} - 1\right) \approx 1154.435$)
    - We will discuss only the first 2 situations.

- **Mean** (μ): The usual estimator is the statistic $T = \bar{X}$ and $\frac{T-\mu}{\sigma/\sqrt{n}} \overset{\circ}{\sim} n(0;1)$

$$\Pr\left(|T-\mu| \le 0.01\mu\right) = \Pr\left(|T-\mu| \le 0.01\mu\right) = \Pr\left(-\frac{0.01\mu}{\sigma/\sqrt{n}} \le \frac{T-\mu}{\sigma/\sqrt{n}} \le \frac{0.01\mu}{\sigma/\sqrt{n}}\right) = 2\Phi\left(\frac{0.01\mu}{\sigma/\sqrt{n}}\right) - 1$$

Then $\Pr\left(|T-\mu| \le 0.01\mu\right) \ge 0.95 \Leftrightarrow \Phi\left(\frac{0.01\mu}{\sigma/\sqrt{n}}\right) \ge 0.975 \Leftrightarrow \frac{0.01\mu}{\sigma/\sqrt{n}} \ge 1.96 \Leftrightarrow n \ge \left(\frac{1.96}{0.01}\right)^2 \times \left(\frac{\sigma^2}{\mu^2}\right)$

As μ and σ^2 are unknown we run a first simulation to estimate them: $\tilde{\mu} = \bar{x}$ and $\tilde{\sigma}^2 = s^2$

Procedure:

- Define m (number of replicas of the first simulation)
- Generate m (pseudo) Pareto observations and compute $\tilde{\mu} = \bar{x}$ and $\tilde{\sigma}^2 = s^2$
- Let n be the smallest integer greater than or equal to $\left(\frac{1.96}{0.01}\right)^2 \times \left(\frac{\tilde{\sigma}^2}{\tilde{\mu}^2}\right)$
- Generate an additional sample with $n-m$ observations and recomputed $\tilde{\mu}$ using all observations (from both samples).

Using R and choosing $m = 10000$:

```
> alpha=3; theta=1000; m=10000;
> u=runif(m); x1=theta*(((1-u)^(-1/alpha))-1)
> tau2_h=var(x1); miu_h=mean(x1)
> n_min=((1.96/0.01)^2)*(tau2_h/(miu_h^2))
> miu_h; tau2_h; n_min
[1] 502.2063
[1] 878943.7
[1] 133877.9
> m2=123878
> u=runif(m2); x2=theta*(((1-u)^(-1/alpha))-1)
> x=c(x1,x2)
> miu_h=mean(x)
> tau2_h=var(x1); n_min=((1.96/0.01)^2)*(tau2_h/(miu_h^2))
> miu_h; tau2_h; n_min
[1] 498.7444
[1] 737651.6
[1] 113921.9
```

- **Distribution Function** ($F_X(1000)$): Let $p = F_X(1000)$. The usual estimator is the ecdf

$T = F_n(1000) = \dfrac{\#\{X_i \le 1000\}}{n}$ and we get $\dfrac{T-p}{\sqrt{p\times(1-p)}/\sqrt{n}} \overset{\circ}{\sim} n(0;1)$

$$\Pr\left(|T-p| \le 0.01p\right) = 2\Phi\left(\frac{0.01p}{\sqrt{p\times(1-p)}/\sqrt{n}}\right) - 1, \text{ then}$$

$$\Pr\left(|T-p| \le 0.01p\right) \ge 0.95 \Leftrightarrow n \ge \left(\frac{1.96}{0.01}\right)^2 \times \left(\frac{1-p}{p}\right)$$

As p is unknown we run a first simulation (m replicas) to estimate it: $\tilde{p} = \#\{x_i \le 1000\}/m$

Procedure:

- Define m (number of replicas of the first simulation)
- Generate m (pseudo) Pareto observations and compute $\tilde{p} = t = \#\{x_i \le 1000\}/m$
- Let n be the smallest integer greater than or equal to $\left(\dfrac{1.96}{0.01}\right)^2 \times \left(\dfrac{1-\tilde{\theta}}{\tilde{\theta}}\right)$
- Generate an additional sample with $n-m$ observations and recomputed $\tilde{p}$ using all observations (from both samples).

Using R and choosing $m = 10000$:

```
> alpha=3; theta=1000; m=10000;
> u=runif(m); x1=theta*(((1-u)^(-1/alpha))-1)
> t=mean(x1<=1000)
> n_min=((1.96/0.01)^2)*(1-t)/t
> t; n_min
[1] 0.881
[1] 5188.994
```

As 10000>5189 we stop.

- **How to use simulation?**
  - Build a model for the random variable for which we want to approximate the distribution, S. This variable can be a function of other random variables.
  - Define the number of replicas to be used, NR
  - For each replica generate as many pseudo-random variables as we need and compute a value for S using the model from step 1. Let us call this value s_j.
  - The cdf of S is then approximated by the ecdf based on $s_1, s_2, \cdots, s_{NR}$. Compute quantities of interest such as the mean, variance, percentiles, probabilities, … from the ecdf.

3.1 – Approximating the sampling distribution of a statistic

Example A – Consider a normal population with mean 10 and standard deviation 3 from which we observe a sample of size n=5 . Using simulation, obtain the sampling distribution of $\bar{X}$ and compare with the theoretical result.

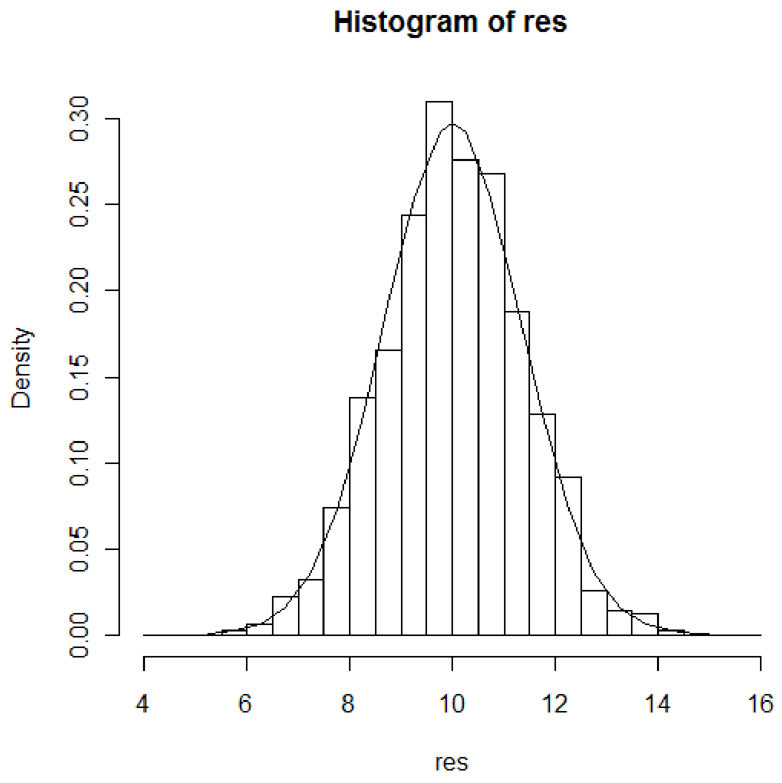
Procedure:

- Choose the number of replicas, NR
- For each replica i, $i = 1, 2, \cdots, NR$
  - Generate 5 random (pseudo) numbers and obtain 5 r.v. with a $n(10; \sigma = 3)$ distribution
  - Compute the sample average and keep this value as element i of the vector res
- Perform a test to check if the values in vector res can be considered as observations of a normal with mean 10 and standard deviation $3/\sqrt{5}$.

```
Solution using R
> NR=1000; n=5; miu=10; sig=3
> res=rep(NA,NR)
> for(i in 1:NR){
+    x=rnorm(n,miu,sig); res[i]=mean(x)
+    }
> mean(res); sd(res) # can also compute skewness and kurtosis
[1] 10.01766
[1] 1.351131
> ks.test(x,"pnorm",10,3/sqrt(5))

        One-sample Kolmogorov-Smirnov test

data:  x
D = 0.4427, p-value = 0.2087
alternative hypothesis: two-sided
> breaks=seq(4,16,0.5)
> points=c(seq(5+0.5/2,16-0.5/2,0.5),10); points=sort(points)
> dens=dnorm(points,miu,sig/sqrt(n))
> hist(res,breaks,prob=TRUE)
> lines(points,dens,type="l")
```


Histogram of res
Density
0.00
0.05
0.10
0.15
0.20
0.25
0.30
4
6
8
10
12
14
16
res

Example B – Consider a Pareto population ($\alpha = 1.5$, $\theta = 100$) from which we observe a sample of size 10. (i) Explain how to use simulation to get an approximation to the sampling distribution of $\bar{X}$. (ii) Perform the simulation with 1000 replicas and compare its results with the normal distribution. (iii) Do you think that increasing the sample size will help?

(i)

Determine NR, the number of replicas to be used.

For each of the NR replicas, $i = 1, 2, \cdots, NR$

- Generate 10 pseudo Pareto distributed variables – we generate 10 uniforms(0,1), u_j, $j = 1, 2, \cdots, 10$ and using the inverse method we get 10 Paretos, $x_j = \theta\left(\left(1 - u_j\right)^{-1/\alpha} - 1\right)$
- Calculate the sample mean, $\bar{x}_i = \sum_{j=1}^{10} x_j$

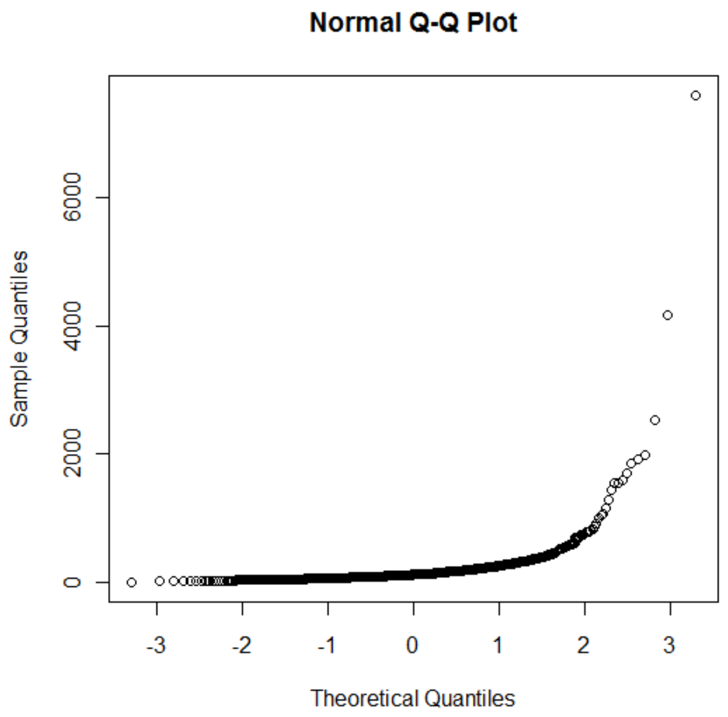
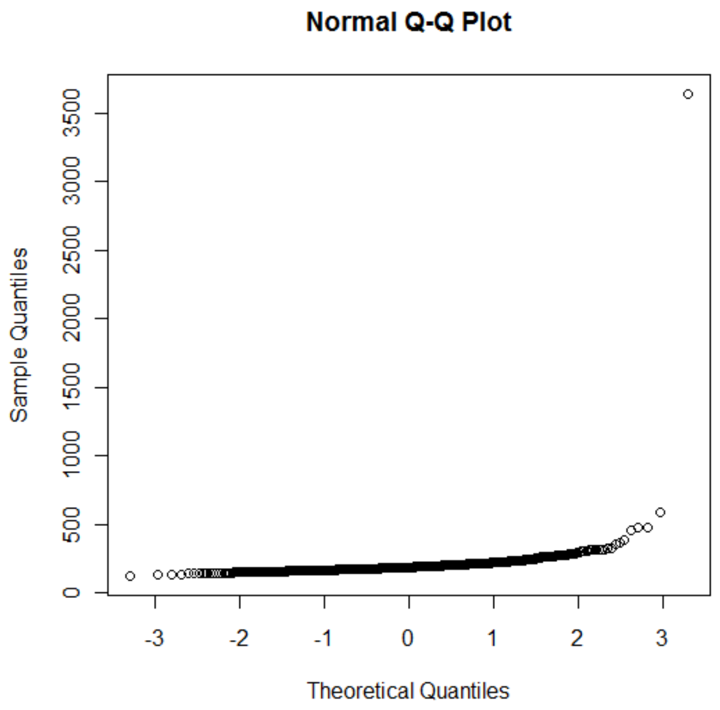
Now we compare the simulated distribution of $\bar{X}$ using our NR pseudo observations $\left(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{NR}\right)$ with a normal distribution and conclude (descriptive statistics, qqplot, ecdf, …)

(ii)

```
> NR=1000; n=10; alpha=1.5; theta=100
> res=rep(NA,NR)
> for(i in 1:NR){
+    u=runif(n); x=theta*((1-u)^(-1/alpha)-1); res[i]=mean(x)
+    }
> library(moments)
> cbind(mean(res),median(res),sd(res),skewness(res),kurtosis(res))
         [,1]     [,2]     [,3]     [,4]    [,5]
[1,] 195.7378 126.9872 339.5334 13.12755 248.533
> qqnorm(res)   # result on next slide
```

(iii) As $\alpha = 1.5 < 2$ the variance of the Pareto distribution does not exist and consequently the CLT does not apply. The sampling distribution of $\bar{X}$ does not converge to a normal distribution. Using the same R program with $n = 1000$ we get

```
> cbind(mean(res),median(res),sd(res),skewness(res),kurtosis(res))
         [,1]     [,2]     [,3]     [,4]    [,5]
[1,] 197.5064 186.5814 115.4582 26.57593 789.149
> > qqnorm(res)   # result on next slide
```

Sample size=10	Sample size=1000
	

Example C – Using simulation determine the p-value of the Kolmogorov-Smirnov test to test if the sample (11.79,11.25,6.83,10.47,13.60,10.60,17.40,10.99,16.45,12.47,8.19,13.46,13.82,11.93,7.47,10.09,14.26, 12.04,12.13,9.66) came from a normal distribution with mean 10 and standard deviation 3.

Procedure

- Compute the value of the test statistic using the observed sample, D
- Determine NR, the number of replicas to be used
- For each of the NR replicas, $i = 1, 2, \cdots, NR$
  - Generate 20 pseudo normal (mean 10 and standard deviation 3) variables, i.e. generate 20 uniforms(0,1), u_j, $j = 1, 2, \cdots, 20$, then, using the Box-Muller method, get 20 $n(0,1)$, z_j, and finally get $x_j = 10 + 3 \times z_j$, $j = 1, 2, \cdots, 20$.
  - Using the generated sample, compute the value of the K-S test statistic, KS_i
- The estimated p-value is given by the proportion of values of KS greater than D

Using R and 10000 replicas we get

```
> x=c(11.79,11.25,6.83,10.47,13.60,10.60,17.40,10.99,16.45,12.47,8.19,
+     13.46,13.82,11.93,7.47,10.09,14.26,12.04,12.13,9.66)
>
> # test if x follows a normal distribution with mean 10 and stdev=3
> a=ks.test(x,"pnorm",10,3)
> D=a$statistic
> a
        One-sample Kolmogorov-Smirnov test
data:  x
D = 0.3122, p-value = 0.03132
alternative hypothesis: two-sided
>
> NR=10000; n=length(x)
> res=rep(NA,NR)
> for(i in 1:NR){
+   y=rnorm(20,10,3); a=ks.test(y,"pnorm",10,3)
+   res[i]=a$statistic
+   }
> p.value=mean(res>=D); p.value
[1] 0.0326
```

3.1 – More complex analysis

Example 20.15 (simplified) – Suppose that we want to approximate the distribution (namely its expected value and variance) of S, the present value of all payments made in respect of a policy issued today and covering loss events that occur in the next year, $S=\sum_{j=1}^{N} X_j V_{T_j}$ where

- N number of loss events during the next year. Losses are ordered according to time of occurrence
- X_j amount of the *j*th loss. Assume that $X \sim Pareto(\alpha=3,\theta=1000)$
- V_t discount rate to be applied to time t. Assume that $V_1=0.06$ (continuous time) – simplified
- T_j time of payment of the *j*th loss. Let $T_j=C_j+L_j$ where C_j is the time of the event and L_j is the time from occurrence to payment. Assume that they are independent.
- Assume that C_j-C_{j-1} (the times between events) are i.i.d. random variables following an exponential distribution with mean 0.2 (year). Let C_0=0. Also assume that C_j is independent of X_j.
- Assume that L_j are independent of each other and that $L_j \mid X_j \sim Weibull\left(\tau=1.5,\theta=(1/6)\times\ln X_j\right)$ if $X_j>1$ and $L_j \mid X_j=0$ if $X_j\le 1$. Corrected

Simulation procedure:

- Define NR, number of replicas
- For $k=1,2,\cdots,NR$ - For each replica we will generate a value for S, s_k

  1. Generate a value for C_j and, consequently, the number of claims, n

     a. Generate u (pseudo-random number) and get $v=-0.2\times\ln(1-u)$

     b. If $v\geq 1$ then $s_k=0$ and begin a new replica. If not, define $j=1$, $c_j=c_1=v$

     c. Generate a new value u and obtain $v=-0.2\times\ln(1-u)$

     d. If $(v+c_j)<1$, compute $c_{j+1}=c_j+v$, $j\leftarrow j+1$, and go back to step c. If not go to step e.

     e. Let $n=j$ and keep array c (length n)

  2. Generate values for X_j and L_j and compute the value of T_j ($j=1,2,\cdots,n$).

     a. Generate x_j, i.e. generate u and compute $x_j=1000\times\left((1-u)^{-1/3}-1\right)$

     b. Generate l_j, i.e. generate u and compute $l_j=\max\left(\frac{1}{6}\ln(x_j)\times\left(-\ln(1-u)\right)^{2/3};0\right)$

     c. Compute $t_j=c_j+l_j$

  3. Compute $s_k=\sum_{j=1}^{n}x_j\times e^{-0.06\times t_j}$ and begin a new iteration.

- Using the NR generated values for S, conclude.

R program

```
NR=10000; res=rep(NA,NR); n.claims.iter=rep(NA,NR)
for(i in 1:NR){
  u=runif(50); v=-0.2*log(1-u)   # over-sized to avoid a while loop
  c=cumsum(v); n=sum(c<1); c=c[1:n]   # hair-cut of the over-sized array
  n.claims.iter[i]=n                  # to check that n<50
  if (n==0) {res[i]=0} else {
    u=runif(n); x=1000*((1-u)^(-1/3)-1)      # claims amount
    u=runif(n); l=pmax((1/6)*log(x)*((-log(1-u))^(2/3)),0) # Lj
    t=c+l;                        # Tj
    pay =x*exp(-0.06*t)           # discounted values for each claim
    res[i]=sum(pay)
    }
  }

mean(res); sd(res); var(res); hist(res)
table(n.claims.iter)
```

Challenging Question: Replace the constant rate of discount by the following hypothesis:

Assume that, for $t > s$, $\dfrac{\ln(V_s / V_t)}{(t-s)}$ has a normal distribution with mean 0.06 and variance $0.0004(t-s)$.

Solution:

Between steps 2 and 3 introduce

- As we must generate the values of the discount rate in order of increasing t_j, sort vector $\mathbf{t}$ (ascending order) and vector $\mathbf{x}$ (according to the values of $\mathbf{t}$). Let $\mathbf{t}^*$ and $\mathbf{x}^*$ be the sorted vectors.
- Compute $\Delta t_j = t_j - t_{j-1}$ where $t_0 = 0$. Δt_j represents the time between payment $j-1$ and payment j
- For $j = 1, 2, \cdots, n$
    - Generate a value for $\dfrac{\ln(V_{t_{j-1}} / V_{t_j})}{(t_j - t_{j-1})}$, i.e. a normal random variable with mean 0.06 and variance $0.0004\Delta t_j$. We can generate a pseudo random value u and use the inverse transform or we can use the Box-Muller transforms. Let y be the generated value.
    - Using $\dfrac{\ln(V_{t_{j-1}} / V_{t_j})}{(t_j - t_{j-1})} = y$, obtain $V_{t_j} = V_{t_{j-1}} \times e^{-y_j (t_j - t_{j-1})}$. Note that $t_0 = 0$ and $V_0 = 1$.
- Now we can return to step 3.

```
NR=30000; res=rep(NA,NR); n.claims.iter=rep(NA,NR)
for(i in 1:NR){
  u=runif(50); v=-0.2*log(1-u)  # over-sized to avoid a while loop
  c=cumsum(v); n=sum(c<1); c=c[1:n]  # hair-cut of the over-sized array
  n.claims.iter[i]=n                 # to check that n<50
  if (n==0) {res[i]=0} else {
    u=runif(n); x=1000*((1-u)^(-1/3)-1)     # claims amount
    u=runif(n); l=pmax((1/6)*log(x)*((-log(1-u))^(2/3)),0) # Lj
    t=c+l;                        # Tj
    t.ord=t[order(t)]; x.ord=x[order(t)]
    delta.t.ord=c(t.ord[1],diff(t.ord))  # computing the differences
    u.norm=rnorm(n,0.06,sqrt(0.0004*delta.t.ord)) # generating normals
    aux1=delta.t.ord*u.norm; aux2=cumsum(aux1)
    V=exp(-aux2)
    pay =x.ord*V         # discounted values for each claim
    res[i]=sum(pay)
    }
  }
mean(res); sd(res); var(res); hist(res)
table(n.claims.iter);
```

Appendix 01 – Proof of the inverse transform theorem

Let us prove the theorem.

Part (a): $X \sim F_X(x) \quad Y = F_X(X) \sim U(0;1)$

$$F_Y(y) = \Pr(Y \le y) = \Pr(F_X(X) \le y) \qquad Y = F_X(X)$$

$$= \Pr(X \le F_X^{-1}(y)) \qquad \text{as } F \text{ is strictly increasing } F^{-1} \text{ exists}$$

$$= F\left(F_X^{-1}(y)\right) = y \qquad 0 < y < 1 \qquad Y \text{ follows a uniform distribution between 0 and 1}$$

Part (b): $Y \sim U(0;1)$ Let us define $X = G(Y)$, where G is a strictly increasing function

$$F_X(x) = \Pr(X \le x) = \Pr(G(Y) \le x) = \Pr(Y \le G^{-1}(x)) \qquad G \text{ is stricLty increasing}$$

$$= F_Y\left(G^{-1}(x)\right) = G^{-1}(x) \qquad \text{as } Y \sim U(0;1)$$

Then, if G is equal to F_X^{-1} we get our proof.

Note that in this case $F_X(x)$ is given and then we have to choose G in accordance.

Appendix 01 – R program

```
>#xx is used to define space in the plot
>x=seq(0,2,length=101);  xx=.5*x
> # type=”n” originates no output
> plot(x,xx,type="n",xlab="x",ylab= "distribution function")
> #internal axes
>lines(c(0,2),c(0,0));lines(c(0,0),c(0,1))
> # distribution functions
> lines(c(0,1),c(0,0.5),col="red");
> lines(c(1,2),c(0.75,1),col="red")
> # u=0.3
> lines(c(0,0.6),c(0.3,0.3),lty=2);
> lines(c(0.6,0.6),c(0,0.3),lty=2)
> # u=0.6
> lines(c(0,1),c(0.6,0.6),lty=2);
> lines(c(1,1),c(0,0.6),lty=2)
> # u=0.9
> lines(c(0,1.6),c(0.9,0.9),lty=2);
> lines(c(1.6,1.6),c(0,0.9),lty=2)
```

Appendix 02 –using R

```
u=0; t=0; lambda=3 # Poisson parameter
x=rep(NA,1000)
for(i in 1:1000){
n=0; tt=0;
repeat{
  u=runif(1); t=-log(1-u)/lambda; tt=tt+t; n=n+1;
  if (tt>=1) {n=n-1; break}
  }
x[i]=n
}

z=rep(1,1000)
tapply(z,x,sum)
```

Appendix 03 – Example 21.17

$h(t) = 0.02$, *then* $H(t) = \int_0^t h(u)\,du = \int_0^t 0.02\,du = 0.02t$ *and* $S(t) = e^{-H(t)} = e^{-0.02t}$.

$$q_x = \Pr(X \le x+1 \mid X > x) = \frac{\Pr(x < X \le x+1)}{\Pr(X > x)} = \frac{S(x) - S(x+1)}{S(x)} = 1 - \frac{e^{-0.02(x+1)}}{e^{-0.02x}} = 1 - e^{-0.02} = 0.0198$$

Last years example

- Among the examples presented in *Loss Models* we will analyze Example 21.17. Skip section 21.2.4 (unless you are familiar with coppulas) and return to section 21.2.6 after bootstrap has been presented.
- **Example 21.17** – An insurance company offers the following product to individuals age 40. A single premium of 10000 is paid (an administrative fee has already been deducted). In return, there are two possible benefits. The 10000 is invested in a mutual fund. If the policyholder dies during the next four years, the fund value is paid to the beneficiary. If not, the fund value is returned to the policyholder at the end of the four years. The policyholder may purchase a guarantee. If the fund has earned less than 5% per year at the time of payment, the payment will be based on a 5% per year accumulation rather than the actual fund value. Determine the 90th percentile of the cost of providing this guarantee. Assume that the force of mortality is constant at 0.02 and that 50000 policies will be sold. Also assume that the annual fund increase has a lognormal distribution with $\mu = 0.06$ and $\sigma = 0.02$.
  We also assume that payments to the beneficiaries (policyholders who died) are made at the end of each year.

Constant force of mortality at 0.02 $\rightarrow q = 1 - e^{-0.02} = 0.0198$

n_0 - number of policies at the beginning of year 1. $n_0 = 50000$

C_0 - Capital value for each policy at the beginning of year 1. $C_0 = 10000$

Let us describe the simulation for replica k

- Years $i = 1, 2, 3$
  - Simulate the number of death m_i from $M_i \sim b(n_{i-1}, 0.0198)$ and calculate the number of survivors $n_i = n_{i-1} - m_i$
  - Simulate x_i, the fund increase during year i from $X_i \sim \text{lnormal}(0.06; 0.02)$ and calculate the value of the policy at the end of year i, $C_i = X_i \times C_{i-1}$
  - If the value of the fund is smaller than the guaranteed capital the insurance fund has to pay the difference to the beneficiaries of the policyholders who died during the year, i.e. $if\,(C_i < C_0 \times 1.05^i)\,then\,P_i = \left(1.05^i\,C_0 - C_i\right)m_i\;else\,P_i = 0$
- Years $i = 4$ (similar to the previous years except that we have to reimbursed all policyholders
  - Simulate x_i, the fund increase during year i from $X_i \sim \text{lnormal}(0.06; 0.02)$ and calculate the value of the policy at the end of year i, $C_i = X_i \times C_{i-1}$

  - If the value of the fund is smaller than the guaranteed capital the insurance fund has to pay the difference to all policyholders (including the beneficiaries of the) i.e.
    $if\,(C_i < C_0 \times 1.05^i)\, then\, P_i = \left(1.05^i\, C_0 - C_i\right) n_{i-1}\, else\, P_i = 0$

- Compute the present value of the payments due to the guarantee. Let us assume that the discount rate is given by the increase of the fund, i.e. $Paym_k = \sum_{i=1}^{k} P_i / v_i$ where $v_i = \prod_{j=1}^{i} x_j$.

Repeat the procedure for $k = 1, 2, \cdots, NR$ where, for instance, we can define $NR = 10000$.
Sort array $Paym_k$ and estimate the 90th percentile using the usual method. We can calculate the corresponding value per policy (just divide by 50000) to price the insurance option. We can also estimate the mean value or others statistics of interest.

See file example21.17.xlsx to see a realization of the simulation.
Challenging question: Can you develop the simulation using R?

```
q=1-exp(-0.02); C0=10000; N0=50000;
G=rep(1.05,4); G=cumprod(G)*C0

NR=1000; pp=rep(NA,NR)
for(j in 1:NR){
  pay=rep(0,4)
  x=rlnorm(4,0.06,0.02); discount=1/cumprod(x); C=C0*cumprod(x)
  NReimb=rep(NA,4)
  n=N0
  for(i in 1:3){
    NReimb[i]=rbinom(1,n,q); n=n-NReimb[i]
    }
  NReimb[4]=n
  pay=NReimb*(G-C)*(G>C)
  pp[j]=sum(pay*discount)
  # G;C;NReimb;pay;
  }

pp_p=pp/N0
mean(pp_p); sd(pp_p); quantile(pp_p,0.9,type=6)
hist(pp_p)
```