Models in Finance - Part 17

Master in Actuarial Science

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Term structure of interest rates

- Interest rates are more complicated than share prices because interest rates depend on the current time t and on the term of the investment T.
- $B(t, T)$ - zero-coupon bond price: price at t for 1€ payable at T.
- $r(t)$ - instantaneous risk-free interest rate or "short rate": represents the "force of interest" applied in the market at time t.
- $C(t)$ - unit price for investment at the risk-free rate: accumulated value of at time t of 1 unit invested at time 0.
- $F(t, T, S)$ - forward rate at t for delivery between T and S: represents the force of interest at which we can agree at time t to borrow or lend over the period $[T, S]$.
- $f(t, T)$ - instantaneous forward-rate curve: force of interest at future time T implied by the current market prices at time t.
- $R(t, T)$ - spot-rate (zero-coupon yield) curve: is the constant force of interest applicable over the period $[t, T]$ that is implied by the current market prices at time t.

Relationships between interest rates and bond prices

- Exercise: Suppose that $t = 5$ and that the force of interest has been a constant of 4% p.a. over the last 5 years. Suppose also that the force of interest implied by current market prices is a constant 4% p.a. for the next two years and a constant 6% p.a. thereafter. If $T = 10$ and $S = 15$, write down or calculate each of the six quantities: $B(t, T)$, $r(t)$, $C(t)$, $F(t, T, S)$, $f(t, T)$ and $R(t, T)$.
- Zero-coupon bond prices are related to spot-rate and forward-rate curves by:

$$R(t, T) = -\frac{1}{T-t} \log B(t, T) \quad \text{for } t < T \tag{1}$$

$$\text{or: } B(t, T) = \exp\left[-R(t, T)(T-t)\right]. \tag{2}$$

- Relationship between forward rate and zero-coupon bond price:

$$F(t, T, S) = \frac{1}{S-T} \log\left[\frac{B(t, T)}{B(t, S)}\right] \quad \text{for } t < T < S. \tag{3}$$

Relationships between interest rates and bond prices

- How to deduce this?
- Note that:

$$B(t,S) = B(t,T)\exp\left[-F(t,T,S)(S-T)\right].$$

- Relationship between forward rate and zero-coupon bond price:

$$f(t,T) = \lim_{S\to T} F(t,T,S) = -\frac{\partial}{\partial T}\log B(t,T) \qquad (4)$$

$$\text{or } B(t,T) = \exp\left[-\int_t^T f(t,u)du\right]. \qquad (5)$$

- Exercise: deduce the last expression from $f(t,T) = \lim_{h\to 0} F(t,T,T+h)$.

Relationships between interest rates and bond prices

- Relationship between spot rate, zero-coupon bond price and forward rates:

$$R(t,T) = -\frac{1}{T-t}\log B(t,T) = \frac{1}{T-t}\int_t^T f(t,u)du. \qquad (6)$$

- $C(t)$ can also be called "cash account" and has a SDE:

$$dC(t) = r(t)C(t)dt. \qquad (7)$$

- This means that the investment gain made on a unit of the cash account from t to $t + dt$ is equal to the interest earned at the risk free rate over that short period.
- The solution of the SDE is:

$$C(t) = \exp\left[\int_0^t r(u)du\right]. \qquad (8)$$

Term structure models

- Models which describe the dynamics of $B(t,T)$, $r(t)$, $f(t,T)$ and $R(t,T)$ over time are called term-structure models or models for the term-structure of interest rates or interest-rate models.
- Exercise: Under the term structure model

$$f(t,T) = 0.03e^{-0.1(T-t)} + 0.06(1 - e^{-0.1(T-t)}).$$

Sketch a graph of $f(t,T)$ as a function of T and derive expressions for $B(t,T)$ and $R(t,T)$.

Desirable characteristics of term structure models

- Desirable characteristics of term structure models:
- 1. The model should be arbitrage free.
- In some circumstances this is not essential, but in the majority of modern actuarial applications, this is essential. Otherwise, by dynamic hedging methods, one would immediately exploit arbitrage opportunities.
- 2. In some applications, interest rates should be positive. However, in other applications is important to consider the possibility of negative interest rates.
- Some term-structure models allow interest rates to go negative. Whether-or-not this is a problem depends on the probability of negative interest rates, their magnitude and the particular applications and interest rates that we are considering.

Desirable characteristics of term structure models

- 3. $r(t)$ and other interest rates should be mean-reverting.
- Empirical evidence suggests this behaviour.
- 4. Computational efficiency: we aim for models which either give rise to simple formulae for bond and option prices or which make it straightforward to compute prices using numerical techniques.
- 5. The model should reproduce realistic dynamics for the interest rates and bond prices.
- The model should reproduce features that are similar to what we have seen in the past with reasonable probability and give rise to a full range of plausible yield curves.
- 6. The model, with appropriate parameter estimates, should fit historical interest-rate data.

Desirable characteristics of term structure models

- 7. The model should be easily and accurately calibrated to current market data.
- This is important point when we are attempting to establish the fair value of liabilities.
  If the model cannot fit observed yield curves accurately then it has no chance of providing us with a reliable fair value for a set of liabilities.
- 8. The model should be flexible enough to cope properly with a range of derivative contracts.

Risk-neutral pricing

- Consider one-factor Markov diffusion models for the short rate of interest $r(t)$:

$$dr(t) = a(t, r(t))dt + b(t, r(t))dW_t, \tag{9}$$

where $a(t, r)$ is the drift, $b(t, r)$ is the volatility and W_t is a standard Brownian motion under the real-world measure P.

- Assuming an arbitrage-free model and that the tradeable assets are the zero-coupon bonds with prices $B(t, T)$, we can use an argument similar to the original derivation of the Black-Scholes model in order to prove that

$$B(t, T) = E_Q \left[\exp\left(-\int_t^T r(u)du \right) \middle| r(t) \right], \tag{10}$$

where Q is called the risk-neutral measure.

Risk-neutral pricing

- We have started off with a process for $r(t)$ which is not a tradeable asset. However, since this is a one-factor model, as soon as we introduce one tradeable asset, $B(t, T)$, we will be able to determine what the market price of risk is.
- Assume a Bond with maturity at T_1 with SDE under P:

$$dB(t, T_1) = B(t, T_1)\left[m(t, T_1)dt + S(t, T_1)dW_t\right].$$

where $S(t, T1)$ and $m(t, T1)$ may be stochastic.

- The market price of risk is defined by:

$$\gamma\left(t, T_1\right) = \frac{m(t, T_1) - r(t)}{S(t, T_1)}. \tag{11}$$

Risk-neutral pricing

- $\gamma(t, T_1)$ represents the excess expected return over the risk free rate per unit of volatility.
- The risk premium on the bond is

$$\gamma\left(t, T_1\right) S(t, T_1) = m(t, T_1) - r(t).$$

- We can show that, in a one-factor model, at any time t, we have the same market price of risk $\gamma\left(t\right)$ for bonds of all maturities: that is, $\gamma(t, T) = \gamma\left(t, T_1\right)$ for all $T > t$.

Risk-neutral pricing

We have that, for $t < T$:

$$\begin{aligned} dB(t,T) &= B(t,T)\left[m(t,T)dt + S(t,T)dW_t\right] \\ &= B(t,T)\left[\left(r(t)+\gamma(t)\,S(t,T)\right)dt + S(t,T)dW_t\right] \\ &= B(t,T)\left[r(t)dt + S(t,T)\left(dW_t+\gamma(t)\,dt\right)\right] \\ &= B(t,T)\left[r(t)dt + S(t,T)d\widetilde{W}_t\right], \end{aligned}$$

where $d\widetilde{W}_t = dW_t + \gamma(t)\,dt$ is a standard Brownian motion under Q (change of measure from P to Q that changes $m(t,T)$ in the drift coefficient to $r(t)$).

- Note that when we make this transformation from the SDE under P to Q, the drift under Q (namely $B(t, T)r(t)$) of all tradeable assets must always be equal to the price of the security times the risk-free rate of interest.
- Therefore, two bonds with the same value will experience the same drift, even if they have different terms.