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Arbitrage Pricing Theory (APT)

2. Arbitrage Pricing Theory (APT)

- Deriving APT

2.1 Deriving APT

- Learning objectives
- Arbitrage
- Arbitrage and randomness
- The APT
- APT versus CAPM
- Chen-Roll-Ross
- Questions

Learning objectives

- state the principle of no arbitrage,
- classify and discuss the nature of arbitrages
- state the assumptions and results of the arbitrage pricing theory,
- prove the arbitrage pricing theorem for multi-factor models,
- discuss possible indices for the APT,
- discuss the results of Chen, Roll and Ross (1983) regarding the APT.

Principle of no arbitrage

The principle of no arbitrage is a very powerful tool for determining prices of securities.

It can also be made to apply to portfolio analysis.

- The idea being that if stock prices are as in a multi-factor model but with no idiosyncratic risk,
- then by forming judicious combinations we can eliminate all risk,
- once all the risk has been eliminated, a portfolio must return the risk-free rate.
- Absence of arbitrage gives conditions on *equilibrium* portfolio returns.

Defining arbitrage

Q: How do we formally define arbitrage?

An arbitrage is a trading strategy in a portfolio of assets such that

1. The portfolio is initially of zero value.
2. At some time $T \geq 0$, in the future the portfolio has zero probability of being a negative value.
3. And at the same time T, it has positive probability of positive value.

OBS: We can win but we can't lose!

Classifying arbitrage

There are many different sorts of arbitrage. Here we can define a few.

1. Instant arbitrage: we make an immediate profit at no risk.
2. Static arbitrage: we buy and sell securities at time zero and then wait to some fixed T when we sell everything.
3. Dynamic arbitrage: we continuously buy and sell assets according to what the market does.
4. Statistical arbitrage: we analyze statistical properties of securities and trade in such a way as to make money on average. Not true arbitrage but the term is widely used nevertheless.

OBS:The concept of arbitrage used in APT is that of static arbitrage.

The principle of no arbitrage

- Riskless profits are too good to be true.
- If they do exist, they will be immediately exploited.
- The exploitation will move the market in such a way as to remove the arbitrage opportunity.
- It is possible to compute prices of complicated instruments using the prices of simple instruments together with the principle of no arbitrage.

Assumption (Principle of no arbitrage)

In a market that is in equilibrium, *there are no arbitrage opportunities.*

OBS: This principle, also called the "no free lunch" principle, makes only very weak assumptions, but many powerful consequences.

The law of one price, and replication

- Closely related to the principle of no arbitrage, is a simple consequence: the law of one price.
- This says that: *if we can exactly synthesize the cash-flows of one portfolio with another portfolio then the two portfolios must have the same price today.*
  If not:
  - sell the more expensive portfolio, and buy the cheaper one.
  - All cash-flows net against each other and we have a riskless profit, which contradicts the principle of no arbitrage and therefore cannot occur in equilibrium.
- This leads to pricing by replication: given a new security, its price is the cost of creating the replicating portfolio.

No arbitrage and random variables

- For any interesting asset, the returns are random.
- However, it is sometimes possible to eliminate randomness by holding a mix of long and short positions in such a way to as to cancel out the randomness.
- Example: Suppose X is a random variable.
  - Suppose the risk-free rate is R_f.
  - Suppose asset, A, returns $R_A + X$, and
  - Suppose asset B, returns $R_B + 2X$,
    $R_A, R_b \in \mathbb{R}$, with the same random variable X.
  - Create a portfolio, P, consisting of 2 units of A and -1 units of B. The return on this is

$$2R_A + 2X - R_B - 2X = 2R_A - R_B.$$

    This is a guaranteed return, not an expected return.
  - By the uniqueness of risk-free rate we get: $2R_A - R_B = R_f$.

The multi-factor model again

- Consider a multi-factor model and take a number of uncorrelated indices I_k which are random, to set

$$R_i = a_i + \sum_{k=1}^{K} b_{ik} I_k + c_i.$$

  The numbers a_i and b_{ik} are constants, and c_i are random but uncorrelated with the indices, and have zero mean.
- Suppose that, *in equilibrium*, there are no idiosyncratic terms, i.e. $c_i = 0$ for all i.

$$\Downarrow$$

OBS: We can apply the principle of no arbitrage to the returns.

The APT

- When we have K risk factors, but N assets, with $N > K$, no arbitrage will reduces the set of possible prices.
- In particular, it means we can create riskless portfolios from a collection of risky assets, all with the same *equilibrium* risk-free return.
- This yields relations amongst the possible values of the loadings on the risk factors.
- For $K = 2$, this will imply that there exist terms, λ_k, so that,

$$\bar{R}_i^e = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}.$$

  - The terms b_{ik} are like generalized betas. They reflect the extra compensation achieved for taking risk on the factor I_k.
  - Equilibrium expected returns are in the same plane in the $\mathbb{R}^3$- space where each asset i is represented by the point $(\bar{R}_i^e, b_{i1}, b_{i2})$.

Creating a risk-less portfolio

Suppose we have two portfolios, P, Q, that have the same loadings

$$b_{Pk} = b_{Qk},$$

we want to prove that they have the same expected returns.

i.e. we want to show

Theorem

Under arbitrage pricing theory, if two portfolios have no idiosyncratic risk and the same factor loadings, then they have the same expected return.

Idea of proof: We show that if they have different expected returns then an arbitrage exists.

Proof

- If $\mathbb{E}(R_P) > \mathbb{E}(R_Q)$ we will go long P and short Q.
  [We can swap P and Q to consider the opposite case.]
- Since they have the same randomness, the difference is deterministic and there is arbitrage.
- Concretely, take portfolio, A, consisting of investing 1 euro in P and -1 euro in Q.
  - This portfolio has zero initial value.
  - The expected value of A after a year is $\mathbb{E}(R_P - R_Q) > 0$.
  - The actual value after a year is $R_P - R_Q$ which equals

    $$a_P + \sum b_{Pk} I_k - a_Q - \sum b_{Qk} I_k = a_P - a_Q,$$

    and is deterministic (i.e. riskless). And since it has positive expectation, it is fixed positive number.
  - This means that it will always be positive after a year. And we conclude $A = P - Q$ is an arbitrage strategy.
- So, no arbitrages implies the same expected returns. ■

Equilibrium Relationship

- It can be shown, using these results that there exist constants λ_k such that if

  $$R_i = a_i + \sum_{k=1}^{K} b_{ik} I_k,$$

  then

  $$\bar{R}_i^e = \lambda_0 + \sum_{k=1}^{K} b_{ik} \lambda_k.$$

  where $\lambda_0 = R_f$, and $\lambda_k = \mathbb{E}(I_k) - R_f$ for $k > 0$

- So, equilibrium expected returns are in the same hyperplane in the $\mathbb{R}^{k+1}$- space where each asset i is represented by the point

  $$(\bar{R}_i^e, b_{i1}, b_{i2}, \cdots, b_{ik}).$$

The idiosyncratic terms

- Even if we believe that the market is driven by $K < N$ common factors.
- Each asset will have an idiosyncratic part to its return. i.e. in

  $$R_i = a_i + \sum_{k=1}^{K} b_{ik} I_j + c_i,$$

  we will not have $c_i = 0$, so the no-arbitrage argument does not truly apply.
- Whilst individual securities have residual risk, well-diversified large portfolios will not $\Longrightarrow$ So we can apply the results to large portfolios but not to individual stocks.

OBS: Whilst there is no theoretical justification applying APT to individual stocks, it is often done nevertheless. One argument is that the idiosyncratic terms are diversifiable and so would not attract risk premia.

APT vs CAPM

- APT and CAPM are separate theories.
- In general they will not lead to the same equilibrium expected returns, but they do not necessarily contradict each other.
- If both theories are to agree with one another, then there is some dependence between their parameters.

Example:

- If we have a two-factor APT, then the expected return on asset i satisfies:
$$\bar{R}_i^{APT} = R_f + b_{i1}(\bar{I}_1 - R_f) + b_{i2}(\bar{I}_2 - R_f).$$
- From CAPM we get
$$\bar{R}_i^{CAPM} = R_f + \beta_i[\bar{R}_M - R_f]$$

APT vs CAPM

- We know we can construct diversified portfolios with one unit of exposure to a particular index I_k. Let P_k be such a portfolio.
  - From APT we have:
$$\bar{R}_{P_k}^{APT} = R_f + (\bar{I}_k - R_f).$$
  - From CAPM we have:
$$\bar{R}_{P_k}^{CAPM} = R_f + \beta_k(\bar{R}_M - R_f),$$
  with β_k the beta of P_k.
- For both to hold for portfolios P_k we need
$$\bar{I}_k = R_f + \beta_{I_k}(\bar{R}_M - R_f).$$

APT vs CAPM

- To hold for any individual asset i we need
$$\begin{aligned}\bar{R}_i^{APT} &= R_f + b_{i1}(\bar{I}_1 - R_f) + b_{i2}(\bar{I}_2 - R_f)\\ &= R_f + b_{i1}\left(R_f + \beta_{I_1}(\bar{R}_M - R_f) - R_f\right) + b_{i2}\left(\beta_{I_2}(\bar{R}_M - R_f)\right)\\ &= R_f + \underbrace{(b_{i1}\beta_{I_1} + b_{i2}\beta_{I_2})}_{\beta_i}(\bar{R}_M - R_f) = \bar{R}_i^{CAPM}\end{aligned}$$
- So if,
$$\bar{I}_k = R_f + \beta_{I_k}(\bar{R}_M - R_f) \quad \text{and} \quad \beta_i = b_{i1}\beta_{I_1} + b_{i2}\beta_{I_2}$$
  the two theories are not contradictory!

OBS: Note APT gives us more room for "maneuver".
It can certainly be correct without the CAPM being correct.
It is also possible for CAPM to hold without APT since CAPM makes no assumptions on the number of factors driving movements.

Practical application

A fund manager only has the market and its historical movements. There are two basic approaches to identifying indices:

1. Use economic indicators, e.g. inflation, interest rates, risk-premia, consumer confidence, production. (These would have to be made uncorrelated.)
2. Attempt to statistically extract unobservable factors.

This also relates to testing APT. It says that if returns are given by an $K-$common factor model plus idiosyncratic part, then the expected returns satisfy certain relations. It does not say what the common factor model is. Our test therefore has to identify the putative factors, as well as examine dependence on them.

Chen, Roll and Ross (1983)

If we believe we know the set of true indices, testing becomes easier. Chen, Roll and Ross (1983) examined APT using the indices of

- Inflation,
- Long interest rates minus short interest rates,
- Spread between risky and riskless bonds,
- Industrial production.

Why these indices

- Inflation: Affects the level of interest rates: interest rates minus inflation is fairly constant. It also affects the size of future cash flows in nominal terms, (but not so much in *real* terms.) Also, there is a general belief that inflation spells economic bad times.
- Long interest rates minus short interest rates: The difference affects the value short term cash flows after discounting versus long term cash flows after discounting. Also, there is the belief that the shape of the yield curve reflects economic expectations: a downwards sloping curve generally believed to reflect upcoming recession.
- Bond spreads are affected by the estimated probability of defaults, and the risk aversion of investors. Both of these will directly affect stock prices.
- Industrial production has a direct impact on the economy.

Results

Chen, Roll and Ross found:

- that each of these 4 had strong effects on stock market returns with industrial production having the biggest effect.
- they also found for the period they studied that oil was not a significant factor.

An alternative approach is to use principal components analysis to attempt to statistically find out how many factors affect expected returns. Tests seem to suggest between 3 and 5 factors are significant.

OBS: This is more than one would expect from CAPM!

Theory questions

- What is the principle of no arbitrage?
- What is the "law of one price"?
- State the assumptions for the APT and prove that under these the factor loadings determine returns.
- To what extent are CAPM and the APT compatible?
- What indices did Chen, Roll and Ross use for their work and why? What did they find? Are their results compatible with CAPM?