Master in Actuarial Science

Models in Finance

03-01-2019

Time allowed: Two and a half hours (150 minutes)

Solutions and hints for the solutions

1. .

   (a) Define $Y_t = \ln(S_t)$. By Itô's formula applied to $f(x) = \ln(x)$, after some calculations, we obtain

   $$S_t = S_0 \exp\left(1 - e^{-t} - \frac{1}{2}\sigma^2 t + \sigma B_t\right).$$

   And after calculating the expected value, one can show that

   $$\mathbb{E}[S_t] = S_0 \exp\left(1 - e^{-t}\right).$$

   (b) We have

   $$S_3 = S_0 \exp(0.8902 + 0.2B_3)$$

   and $X := \log\left(\frac{S_3}{S_0}\right) \sim N(0.8902; 0.12)$. We can calculate $P\left(\frac{S_3}{S_0} \geq 1.20\right) = P(X \geq \ln(1.20)) = 0.9795$.

2. .

   (a) Normality assumption: market crashes appear more often than one would expect from a normal distribution of the log-returns (the empirical distribution has fat tails when compared to the Normal). Moreover, days with very small changes also happen more often than the normal distribution suggests (more peaked distribution). The main advantage of considering the normal distribution is its mathematical tractability.

   The fat tails and jumps justify the consideration of Lévy processes (associated with fat tails) for modelling security prices.

(b) (i) There are good theoretical reasons to suppose that the expected returns per time unit should vary over time. It is reasonable to suppose that investors will require a risk premium on equities relative to bonds. As a result, if interest rates are high, we might expect the expected value of returns to be high as well. However, it is not easy to test this argument empirically.

(ii) Empirical data shows that volatility parameter is not constant in time. The implied volatility obtained from option prices and the examination of historic option prices suggests that volatility expectations fluctuate markedly over time.

(iii) One unsettled empirical question is whether markets are mean reverting, or not. A mean reverting market is one where rises are more likely following a market fall, and falls are more likely following a rise. There appears to be some evidence for this, but the evidence rests heavily on the aftermath of a small number of dramatic crashes. Furthermore, there also appears to be some evidence of momentum effects, which imply that a rise one day is more likely to be followed by another rise the next day.

(i) In the lognormal model, the expected value of returns per time unit, or drift, is constant, which does not agree with the theoretical argument given in (a). However, in this case it is difficult to test empirically if it is really necessary to assume a non-constant drift.

(ii) In the lognormal model, the volatility is assumed to be constant, in contradiction with empirical evidence.

(iii) The lognormal model is not mean reverting. However, there is no strong empirical evidence of mean-reversion effects in stock prices.

One class of models with the feature of non-constant volatility are the ARCH models. Models with non-normal returns or stochastic volatility models also satisfy this property.

3. .

(a) Answer in the slides. Check it out.

(b) Answer in the slides. Check it out.

(c) From the put-call parity (note that $K = S_t = 15€$), we have that $1 + 15e^{-0.04\left(\frac{18}{12}\right)} = p_t + 15$. Therefore $p_t = 0.1265$.

4. .

(a) $u = \frac{1}{d} = 1.087$. In order to obtain an arbitrage free model, we must have $d < e^r < u$. Therefore

$$-0.0834 < r < 0.0834.$$

Since $r = 0.05$, the model is arbitrage free. Binomial tree values: 10; 10.87, 9.2; 11.8157, 10, 8.464; 12.8437, 10.87, 9.2, 7.7869. If $r = 5\%$, then the risk-neutral probability for an up-movement is

$$q = \frac{e^r - d}{u - d} = 0.7861$$

Payoff function of the Financial Derivative:

$$max\left\{exp\left(\frac{S_T}{5}\right) - 8, 0\right\}$$

Payoff: $V_3\left(u^3\right) = 5.0494$, $V_3\left(u^2 d\right) = 0.7934$, $V_3\left(u d^2\right) = 0$, $V_3\left(d^3\right) = 0$

Using the usual backward procedure with $r = 0.05$ and $q = 0.7861$

At time 2: $V_2\left(u^2\right) = \exp(-r)\left[qV_3\left(u^3\right) + (1-q)V_3\left(u^2 d\right)\right] = 3.9372$, $V_2\left(ud\right) = \exp(-r)\left[qV_3\left(u^2 d\right) + (1-q)V_3\left(ud^2\right)\right] = 0.5933$, $V_2\left(d^2\right) = 0$,

At time 1: $V_1\left(u\right) = \exp(-r)\left[qV_2\left(u^2\right) + (1-q)V_2\left(ud\right)\right] = 3.0648$, $V_1\left(d\right) = \exp(-r)\left[qV_2\left(ud\right) + (1-q)V_2\left(d^2\right)\right] = 0.4436$,

At time 0, the price is $V_0 = \exp(-r)\left[qV_1\left(u\right) + (1-q)V_1\left(d\right)\right] = 2.382$.

(b) Answer in the slides. Check it out.

5. .

(a) Answer in the slides. Check it out.

(b) The price is given by $V_t = e^{-r(T-t)}\mathbb{E}_Q\left[\frac{1}{T-t_0}\int_{t_0}^{T} S_u du|\mathcal{F}_t\right]$. The dynamics of the stock prices $S_t$ under $Q$ is given by the SDE

$$\begin{aligned} dS_u &= r\, S_u\, du + \sigma\, S_u\, d\widetilde{Z}_u, \quad u > t \\ S_t &= s \end{aligned}$$

This is a geometric Brownian motion and the solution is such that:

$$S_u = s\exp\left[\left(r - \frac{\sigma^2}{2}\right)(u-t) + \sigma\left(\widetilde{Z}_u - \widetilde{Z}_t\right)\right].$$

After some calculations, one can show that

$$V_t = \frac{e^{-r(T-t)}}{T-t_0}\int_{t_0}^{T} S_t e^{r(u-t)} du = \frac{S_t}{r\left(T-t_0\right)}\left[1 - \exp\left(-r\left(T-t_0\right)\right)\right].$$

6. .

(a) Answer in the slides. Check it out.

(b) The bond price is given by

$$B(t,T) = \exp\left[-\int_t^T f(t,u)\,du\right] = \exp\left[-\int_t^T \left(0.04e^{-0.3(u-t)} + 0.08(1-e^{-0.3(u-t)})\right)du\right]$$

$$B(t,T) = \exp\left[-0.08\,(T-t) - 0.133e^{-0.3(T-t)} + 0.133\right]$$

Moreover, $R(t,T) = 0.08 - 0.133\left[\frac{1-e^{-0.3(T-t)}}{T-t}\right]$.

(c) Answer in the slides. Check it out.

Plot for 6(b):