Master in Actuarial Science

Models in Finance

06-01-2020

Time allowed: Two hours (120 minutes)

Solutions and hints for the solutions

1. .

   (a) By Itô's lemma (or Itô's formula) applied to $g(t,x)$ (it is a $C^{1,2}$ function), after some calculations, we obtain

   $$dg(t,S_t) = 0 + h(t,S_t)\frac{\partial g}{\partial x}(t,S_t)dB_t.$$

   Therefore,

   $$Y_t = Y_0 + \int_0^t h(u,S_u)\frac{\partial g}{\partial x}(u,S_u)dB_u,$$

   and since $h$ and $\frac{\partial g}{\partial x}$ are continuous and bounded, $h(u,S_u)\frac{\partial g}{\partial x}(u,S_u)$ is adapted and the integral of the expected value of the squared process is finite. Hence, the process belongs to the space $L^2_{a,T}$ and therefore $Y_t$ is a martingale (it is a well defined stochastic integral).

   (b) We have

   $$dS_t = 0.1S_tdt + 0.25S_tdB_t,$$

   which is the SDE of a geometric Brownian motion with $\mu = 0.1$ and $\sigma = 0.25$. The solution is (it can be obtained by applying the Itô formula to $f(x) = \log(x)$)

   $$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right]$$

   Therefore

   $$S_t = S_0 \exp\left[0.06875t + 0.25B_t\right].$$

   $$P\left(\frac{S_2}{S_0} \leq 1.15\right) =$$
   $$= P\left(Z \leq \ln(1.15)\right),.$$

   where $Z = 0.1375 + 0.25B_2 \sim N(0.1375; 0, 125)$.

   Therefore: $P\left(\frac{S_2}{S_0} \leq 1.15\right) = 0.5026$.

2. .

(a) $1+i$ has lognormal distribution with parameters $(\mu, \sigma^2)$. We also know that $E[1+i] = 1.05$ and $Var[1+i] = 0.004$. Therefore

$$1.05 = exp(\mu + \sigma^2/2)$$

and

$$0.004 = exp\left(2\mu + \sigma^2\right)\left(exp\left(\sigma^2\right) - 1\right).$$

From these equations, we get $\sigma^2 = 0.003622$ and $\mu = 0.04698$.

(b) We know that in this case we have $ln(S_{10})$ has a normal distribution with mean $10 \times \mu = 0.4698$ and variance $10 \times \sigma^2 = 0.03622$. Therefore, the $P[S_{10} > 2] = P[ln(S_{10}) > ln(2)]$ and this can be calculated as

$$1 - P\left[Z \le \frac{ln(2) - 0.4698}{\sqrt{0.03622}}\right] = 1 - P[Z \le 1.17356] = 0.12.$$

The probability that $ln(1+i) < ln(1.04)$ can be calculated using the normal distribution of $ln(1+i)$ and therefore

$$P[ln(1+i) < ln(1.04)] = P\left[Z < \frac{ln(1.04) - 0.04698}{\sqrt{0.003622}}\right] = 0.4487$$

Since the rates of return are independent, the probability that $1+i < 1.04$ in all the 10 years is simply

$$(0.4487)^{10} = 0.00033.$$

3. .

(a) Let us consider two portfolios. Portfolio $A$: one European call option + cash $D_1 e^{-r(T_1-t)} + D_2 e^{-r(T_2-t)} + Ke^{-r(T-t)}$

Portfolio $B$: one European put option + one dividend paying share.

At time $T$, the value of portfolio $A$ is $S_T - K + D_1 e^{r(T-T_1)} + D_2 e^{r(T-T_2)} + K = S_T + D_1 e^{r(T-T_1)} + D_2 e^{r(T-T_2)}$ if $S_T > K$ and $D_1 e^{r(T-T_1)} + D_2 e^{r(T-T_2)} + K$ if $S_T \le K$.

At time $T$, the value of portfolio $B$ is $0 + S_T + D_1 e^{r(T-T_1)} + D_2 e^{r(T-T_2)}$ if $S_T > K$ and $K - S_T + S_T + D_1 e^{r(T-T_1)} + D_2 e^{r(T-T_2)} = D_1 e^{r(T-T_1)} + D_2 e^{r(T-T_2)} + K$ if $S_T \le K$.

Therefore, the portfolios have the same value at maturity. Then, by the no-arbitrage principle, the porfolios have the same value for any time $t < T$, i.e.,

$$c_t + D_1 e^{-r(T_1-t)} + D_2 e^{-r(T_2-t)} + Ke^{-r(T-t)} = p_t + S_t.$$

(b) For the price of the put option we use the Black-Scholes formula with dividend yield

$$f(t, S_t) = Ke^{-r(T-t)}\Phi(-d_2) - S_t e^{-q(T-t)}\Phi(-d_1). \qquad (1)$$

and use the data given in the problem with $q = 0.2$,$r = 0.05$,$T - t = 1.5$, $\sigma = 0.2$ and $S_t = 18$, $K = 20$,$d_1 = -0.124$ and $d_2 = -0.369$. Using the formula and these values, we obtain

$$price = 2.352.$$

4. .If $r = 5\%$, then the risk-neutral probability for an up-movement is

$$q = \frac{e^r - d}{u - d} = 0.6975.$$

Binomial tree values: 10; 11.2,8.928; 12.544, 10, 7.9709; 14.0493,11.2, 8.928, 7.116436

Payoff function of the derivative (call + put):

$$Payoff = \begin{cases} 8.5 - S_T & \text{if } S_T < 8.5 \\ 0 & \text{if } 8.5 \le S_T \le 12 \\ S_T - 12 & if S_T > 12 \end{cases}.$$

Payoff of the derivative: $C_3(u^3) = 14.0493 - 12 = 2.0493$, $C_3(u^2 d) = 0$, $C_3(ud^2) = 0$, $C_3(d^3) = 8.5 - 7.116436 = 1.383564$

Using the usual backward procedure with $r = 0.05$ and $q = 0.6975$ :

At time 2: $C_2(u^2) = \exp(-r)\left[qC_3(u^3) + (1-q)C_3(u^2 d)\right] = 1.3597$, $C_2(ud) = \exp(-r)\left[qC_3(udu) + (1-q)C_3(ud^2)\right] = 0$, $C_2(d^2) = \exp(-r)\left[qC_3(d^2 u) + (1-q)C_3(d^3)\right] = 0.3981$.

At time 1: $C_1(u) = \exp(-r)\left[qC_2(u^2) + (1-q)C_2(ud)\right] = 0.9021$, $C_1(d) = \exp(-r)\left[qC_2(du) + (1-q)C_2(d^2)\right] = 0.1146$.

The Final price (at time 0) is$C_0 = \exp(-r)\left[qC_1(u) + (1-q)C_1(d)\right] = 0.6315$.

5. .

(a)

In order to have a porfolio with zero delta, $\Delta_p \times N + \Delta_S \times$number of shares= 0. Therefore

$$N = \frac{50000}{0.25} = 200000.$$

(b) We have $\Delta_X = 0.3$, $\Delta_Y = 0.4$, $\Gamma_X = 0.15$, $\Gamma_Y = 0.25$. In order to have a zero delta and a zero gamma portfolio:

$$\begin{cases} 0.3N_X + 0.4N_Y = 0 \\ 200000 \times 0.1 + 0.15N_X + 0.25N_Y = 0 \end{cases}$$

The solution is

$$\begin{cases} N_X = 533333, \\ N_Y = -400000. \end{cases}$$

6. .

(a) Answer in the slides. Check it out.

(b) Answer in the slides. Check it out.