Master in Actuarial Science

Models in Finance

04-02-2020

Time allowed: Two hours (120 minutes)

Solutions and hints for solutions

1. .

   (a) The discounted price is $\widetilde{S}_t = e^{-rt}S_t = e^{-rt}S_0 \exp\left(g(t) + kB_t\right)$. By Itô's lemma (or Itô's formula) applied to $f(t,x) = e^{-rt}\exp(g(t) + kB)$ (it is a $C^{1,2}$ function), after some calculations, we obtain:

$$d\widetilde{S}_t = \left(g'(t) + \frac{1}{2}k^2 - r\right)\widetilde{S}_t dt + k\widetilde{S}_t dB_t.$$

   (b) The discounted price process $\widetilde{S}_t$ is a martingale if and only if the drift coefficient in the SDE is zero, that is, $g'(t) + \frac{1}{2}k^2 - r = 0$. In this case,

$$\widetilde{S}_t = \widetilde{S}_0 + k\int \widetilde{S}_u dB_u,$$

   and the stochastic integral is a martingale. The function that satisfies $g'(t) + \frac{1}{2}k^2 - r = 0$ is

$$g(t) = \left(r - \frac{1}{2}k^2\right)t + C,$$

   but at time 0, we have $S_0 = S_0 \exp\left\{g\left(0\right) + kB_0\right\}$ and therefore

$$g(t) = \left(r - \frac{1}{2}k^2\right)t.$$

   Moreover, if $\widetilde{S}_t$ is a martingale, then $\mathbb{E}\left[S_t\right] = e^{rt}S_0$.

2. .

   (a) Answer is in the slides - check it out.

(b) The put-call parity:

$$c_t + Ke^{-r(T-t)} = p_t + S_t.$$

Now, if we take partial derivatives with respect to S, we can show that we obtain $\Delta_c = \Delta_p + 1$ and $\Gamma_c = \Gamma_p$.
In the dividend case, using the same procedure with the put-call parity with dividends, we can show that we obtain $\Delta_c = \Delta_p + e^{-q(T-t)}$ and $\Gamma_c = \Gamma_p$.

3. .

(a) The delta of a call option can be derived from the Black-Scholes formula and is given by $\Delta = \frac{\partial c_t}{\partial S_t} = \Phi(d_1)$, where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = -0.0798747.$$

Therefore,

$$\Delta = \Phi(-0.0798747) = 0.4682$$

The option price is given by:

$$\begin{aligned} c_t &= S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) = \\ &= 0.6365, \end{aligned}$$

where $d_2 = -0.3035$. The hedging portfolio is: $\Delta \times$number of options $= 4682$ units of stock and $10000 \times 0.6365 - 4682 \times 9 = -35773$€ in cash.

(b) The dynamics of the stock prices S_t under Q is given by the SDE

$$dS_t = r\, S_t\, dt + \sigma\, S_t\, d\overline{W}_t,$$

where $\overline{W}_t$ is a standard Brownian motion under the risk neutral measure (or equivalent martingale measure) Q. By Itô formula applied to $X_t = \ln(S_t)$, we can show that

$$X_T = \ln(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\overline{W}_T - \overline{W}_t\right).$$

The price of the derivative is given by

$$\begin{aligned} F(t, S_t) &= e^{-r(T-t)} E_{\mathbb{Q}}\left[K\mathbf{1}_{\{S_T > e^K\}} + \ln(S_T)\mathbf{1}_{\{S_T \le e^K\}} | \mathcal{F}_t\right] \\ &= e^{-r(T-t)}\left\{E_{\mathbb{Q}}\left[\ln(S_T)\mathbf{1}_{\{S_T \le e^K\}} | S_t\right] + KP_{\mathbb{Q}}\left[S_T > e^K | S_t\right]\right\}. \end{aligned}$$

Moreover, since $Z = \frac{X_T - \ln(S_t) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \sim N(0,1)$ and $X_T \leq K$ is equivalent to $Z \leq K^*$ with $K^* = \frac{K - \ln(S_t) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, we have that

$$E_{\mathbb{Q}}\left[\ln(S_T)\mathbf{1}_{\{S_T \leq e^K\}}|S_t\right] = \int_{-\infty}^{K^*}\left(\ln(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma z\sqrt{T-t}\right) f(z)dz.$$

where $f(z)$ is the pdf of $N(0,1)$.
On the other hand,

$$P_{\mathbb{Q}}\left[S_T > e^K|S_t\right] = P_{\mathbb{Q}}\left[X_T > K|S_t\right] = 1 - \Phi(K^*),$$

where Φ is the cumulative distribution function of $N(0,1)$.
Therefore,

$$\begin{aligned} F(t, S_t) &= e^{-r(T-t)}\int_{-\infty}^{K^*}\left(\ln(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma z\sqrt{T-t}\right) f(z)dz \\ &\quad + e^{-r(T-t)}K\left[1 - \Phi(K^*)\right]. \end{aligned}$$

4. The answer in in the slides. Check it out.

5. .

$$B(t,T) = \exp\left[-\int_t^T f(t,u)\,du\right].$$

Therefore

$$\begin{aligned} B(t,T) &= \exp\left[-\int_t^T \left(r(t) - \alpha(u-t)^2\right) du\right] \\ &= \exp\left[-r(t)(T-t) + \frac{\alpha}{3}(T-t)^3\right] \end{aligned}$$

$$R(t,T) = \frac{-1}{T-t}\log B(t,T) \quad \text{if } t < T$$

and therefore

$$R(t,T) = r(t) - \frac{\alpha}{3}(T-t)^2.$$

Now, let $r(t) = 0.25$, $\alpha = 0.01$ and $T - t = 3$. Then $B(t,T) .= 0.5169$ and $R(t,T) = 0.22$.

6. .

   (a) The answer is in the slides. Check it out.

(b) Let $B(t,T)$ be the price at time t of a zero-coupon bond. We consider a recovery rate δ. Then there exists a risk-neutral measure Q equivalent to P, under which:

$$\begin{aligned} B(t,T) &= e^{-r(T-t)} E_Q \left[Payoff \text{ at } T | \mathcal{F}_t \right] \\ &= e^{-r(T-t)} E_Q \left[1 - (1-\delta) N(T) | \mathcal{F}_t \right]. \end{aligned}$$

One can prove that

$$E_Q \left[N(T) | N(t) = 0 \right] = E_Q \left[1 - \exp\left(- \int_t^T \widetilde{\lambda}(s)\, ds \right) \right].$$

Hence, assuming that $\widetilde{\lambda}(s)$ is deterministic, this implies that:

$$B(t,T) = e^{-r(T-t)} \left[1 - (1-\delta) \left(1 - \exp\left(- \int_t^T \widetilde{\lambda}(s)\, ds \right) \right) \right].$$

Comparing with the zero coupon bond price in our particular case, we have that

$$\int_t^T \widetilde{\lambda}(s)\, ds = T^{3/2} - t^{3/2}$$

and therefore, $\widetilde{\lambda}(s) = \frac{2}{3} s^{1/2}$.