Master in Actuarial Science

Models in Finance

06-01-2020

Time allowed: Two hours (120 minutes)

Instructions:

1. This paper contains 6 questions and comprises 4 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all 6 questions.
6. Begin your answer to each of the 6 questions on a new page.
7. Marks are shown in brackets. Total marks: 150.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The Formulae and Tables for Actuarial Examinations may be used.

1. Consider that the share price of a non-dividend paying security is given by a stochastic process S_t which is the solution of the Stochastic Differential Equation (SDE)

$$dS_t = \mu S_t dt + h(t, S_t) dB_t,$$

where:

- B_t is a standard Brownian motion,
- μ is a constant and $h(t, x)$ is a bounded function with continuous and bounded partial derivatives.
- t is the time from now measured in years.

(a) Consider the process $Y_t = g(t, S_t)$, where $g : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a function of class $C^{1,2}\left(\mathbb{R}_0^+ \times \mathbb{R}\right)$ with bounded partial derivatives and such that

$$\frac{\partial g}{\partial t}(t, x) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, x)(h(t, x))^2 + \mu x \frac{\partial g}{\partial x}(t, x) = 0,$$

for all $t \geq 0$ and $x \in \mathbb{R}$. Show that the process Y_t is a martingale with the respect to the filtration generated by the Brownian motion. (16)

(b) Let $\mu = 0.1$ and $h(t, S_t) = 0.25 S_t$. Calculate the probability that the 2-year return will be less than 15% (12)

2. The random variable $(1 + i)$ follows a log-normal distribution with parameters μ and σ^2. The mean of the rate of interest is 0.05 and the variance is 0.004.

(a) Find μ and σ^2. (12)

(b) Consider that the rate of interest, in each year, is log-normally distributed as above and is independent of the rate of interest in any other year. Find the probability that a unit sum of money will accumulate to more than 2 after 10 years and find also the probability that an investor receives a rate of return less than 4% in all the 10 years. (14)

3. Consider European put and call options written on a dividend paying share with maturity T and strike K.

(a) By constructing two portfolios with identical payoffs at the maturity of the options, derive an expression for the put-call parity of European options on a dividend paying share, where the dividends D_1 and D_2 are payable at future dates T_1, T_2 with $t < T_1 < T_2 < T$. More precisely, prove that

$$c_t + D_1 e^{-r(T_1 - t)} + D_2 e^{-r(T_2 - t)} + K e^{-r(T-t)} = p_t + S_t.$$

(16)

(b) Considering the Black-Scholes model, calculate the price of a put option with time until maturity 18 months, strike price 20€, current share price 18€, continuously compounded risk-free interest rate 5% p.a., continuously compounded dividend rate q of 2% p.a. and volatility of 20%. Assume that the values that appear in the Black-Scholes formula are $d_1 = -0.124$ and $d_2 = -0.369$. (12)

4. Consider a 3-period recombining binomial model for the non-dividend paying share with price process S_t such that the price at time $t + 1$ is either $S_t u$ or $S_t d$ with $d = \frac{1}{u}$. Assume that $u = 1.12$ and that the current price of the share is 10€. Assuming that the risk-free interest rate is 5% per year, construct the binomial tree and calculate the price of a derivative composed of a sum of an European put option and an European call option. The put option has strike $K_p = 8.5$ and the call option has a strike $K_c = 12$. The time to maturity of both options is 3 years. (18)

5. Consider an investor with a portfolio of N put options and 50000 shares. Assume that the delta of an individual option is -0.25 and that its gamma is 0.1.

   (a) Calculate the number N of put options in the portfolio such that the portfolio has zero delta. (8)

   (b) Consider that the investor can invest in two other financial derivatives: the derivative X and the derivative Y. These financial derivatives have delta and derivative of delta with respect to the undelrying price satisfying

$$\Delta_X = 0.3, \quad \frac{\partial \Delta_X}{\partial S} = 0.15,$$
$$\Delta_Y = 0.4, \quad \frac{\partial \Delta_Y}{\partial S} = 0.25.$$

   Calculate the number of derivatives X and Y that should be added to the portfolio in order to obtain a total portfolio with both delta and gamma equal to zero. (14)

6. Consider the zero-coupon bond market.

(a) Present the stochastic differential equations (SDE) for the short rate in the Vacisek and CIR models and discuss the critical difference between the two models. (12)

(b) Solve the SDE for the Vasicek model and present the invariant stationary distribution associated to the solution. (16)