Master in Actuarial Science

Models in Finance

04-02-2020

Time allowed: Two hours (120 minutes)

Instructions:

1. This paper contains 6 questions and comprises 4 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all 6 questions.
6. Begin your answer to each of the 6 questions on a new page.
7. Marks are shown in brackets. Total marks: 150.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The Formulae and Tables for Actuarial Examinations may be used.

1. Consider that the share price of a non-dividend paying security is given by

$$S_t = S_0 \exp\left\{g\left(t\right) + kB_t\right\}$$

where B_t is a standard Brownian motion, k is a constant and $g(t)$ is a deterministic function of class $C^2\left(\mathbb{R}\right)$. Assume that the continuously compounded risk-free interest rate r is constant.

   (a) Deduce the stochastic differential equation (SDE) satisfied by the discounted price process $\widetilde{S}_t$. (15)

   (b) Deduce from (a) what should be the function $g(t)$ such that the discounted price $\widetilde{S}_t$ is a martingale and for that $g(t)$ calculate $\mathbb{E}\left[S_t\right]$. (12)

2. Consider European put and call options written on a share with the same strike and maturity.

   (a) Explain how and why the strike price, the interest rate and the volatility affect the price of the European call and put options. (15)

   (b) Consider that the delta and the gamma the call option are known and are given by Δ_c and Γ_c. Derive formulas for the the delta and gamma of the put option in terms of Δ_c and Γ_c, (i) in the non-dividend case and (ii) in the dividend paying share case. (13)

3. Consider the Black-Scholes model and a European call option written on a non-dividend paying share with expiry date 15 months from now, strike price 10€ and current price 9€. Assume that the (continuously compounded) free-risk interest rate is 5% p.a. and that the volatility is $\sigma = 0.2$.

   (a) Consider that an investor has 10000 call options as defined above. Calculate the corresponding hedging portfolio in shares and cash. (15)

   (b) Consider a financial derivative Φ that has the following payoff at expiry date T depending on the price of the underlying non-dividend paying share at maturity T:

$$Payoff = \begin{cases} K \text{ if } S_T > e^K, \\ \ln\left[S(T)\right] \text{ if } S_T \leq e^K, \end{cases}$$

   where K is positive constant. Show that the price of the derivative at time t is given by (for $t < T$)

$$e^{-r(T-t)} \int_{-\infty}^{K^*} \left(\ln\left(S_t\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma z\sqrt{T-t}\right) f(z)dz$$
$$+ Ke^{-r(T-t)}\left[1 - \Phi\left(K^*\right)\right],$$

where K^* is an appropriate constant, $f(z)$ is the p.d.f. and $\Phi(z)$ is the cumulative distribution function of a certain distribution. (17)

4. Consider a recombining binomial model for the non-dividend paying share with price process S_t such that over each time period δt the stock price can move up by a factor u or down by a factor $d = \frac{1}{u}$. Assume a continuously compounded risk-free interest rate r per year €. This recombining binomial model and the lognormal model should be under the risk-neutral measure Q. In particular, for the lognormal model, we have, under Q, that

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left(\left(r - \frac{1}{2}\sigma^2\right)(t - t_0), \sigma^2 (t - t_0)\right),$$

where r is the interest rate per year. If we calibrate the binomial model in a way that the return and the variance of the log-return over the time interval δt of the binomial model and the lognormal model are equal, derive that (for small δt)

$$q = \frac{e^{r\delta t} - d}{u - d}$$

and

$$u = exp\left(\sigma\sqrt{\delta t}\right).$$

(Hint: assume that $\left\{\mathbb{E}_Q\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right]\right\}^2 \approx 0$ when δt is small.) (17)

5. Consider the zero-coupon bond market for zero-coupon bonds paying 1€ at time T and that the instantaneous forward rate is given by

$$f(t,T) = r(t) - \alpha (T - t)^2,$$

where $\alpha > 0$ is a constant and $r(t)$ is the short rate. Derive expressions for the price of the zero-coupon bond and the spot rate curve at time t and calculate the price of a 3-year zero coupon bond and the value of the spot rate, assuming that $r(t) = 0.25$ (constant) and $\alpha = 0.01$. (16)

6. Consider a two-state model for credit rating with deterministic transition intensity, where the recovery rate is δ and the zero coupon bond price is given by

$$B(t,T) = e^{-r(T-t)}\left[1 - (1-\delta)\left(1 - \exp\left(t^{3/2} - T^{3/2}\right)\right)\right].$$

   (a) Discuss how are defined the transition probabilities in the two-state model for credit rating with deterministic transition intensity and in the Jarrow-Lando-Turnbull model. (14)

(b) State the general formula for the zero coupon bond prices in a two state model for credit ratings and then deduce the implied risk-neutral transition intensity $\widetilde{\lambda}(s)$ for our particular two-state model. (16)