Financial Markets and Investments

Exercises Suggested Solutions

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These are suggested solutions to the exercises in the **Booklet of Exercises**. They are not typo and/or error free.

I would very much appreciate if you would write down a list of typos and/or errors identified during study and hand it in at the exam, for future correction. The same applies to typos in the slides or any other material I have distributed during the course.

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1 Mean–Variance Theory

1.1 Return and Diversification of Risk

Exercise 1.1.

(a) Expected return is the sum of each outcome times its associated probability. Expected returns:
$\bar{R}_1 = 16\% \times 0.25 + 12\% \times 0.5 + 8\% \times 0.25 = 12\%$
$\bar{R}_2 = 6\%$
$\bar{R}_3 = 14\%$
$\bar{R}_4 = 12\%$
Standard deviation of return is the square root of the sum of the squares of each outcome minus the mean times the associated probability. Standard deviations:
$\sigma_1 = \left[(16\% - 12\%)^2 \times 0.25 + (12\% - 12\%)^2 \times 0.5 + (8\% - 12\%)^2 \times 0.25\right]^{\frac{1}{2}} = 2.83\%$
$\sigma_2 = 1.41\%$
$\sigma_3 = 4.24\%$
$\sigma_4 = 3.27\%$

(b) Covariance of return between Assets 1 and 2
$\sigma_{12} = (16 - 12) \times (4 - 6) \times 0.25 + (12 - 12) \times (6 - 6) \times 0.5 + (8 - 12) \times (8 - 6) \times 0.25 = -4$
The variance/covariance matrix for all pairs of assets is:

$$V = \begin{pmatrix} 0.0008 & -0.0004 & 0.0012 & 0 \\ -0.0004 & 0.0002 & -0.0006 & 0 \\ 0.0012 & -0.0006 & 0.0018 & 0 \\ 0 & 0 & 0 & 0.00107 \end{pmatrix}$$

Correlation of return between Assets 1 and 2: $\rho_{12} = \frac{-4}{2.83 \times 1.41} = -1$.
The correlation matrix for all pairs of assets is:

$$\rho = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c)

Portfolio	Expected Return
A	$1/2 \times 12\% + 1/2 \times 6\% = 9\%$
B	13%
C	12%
D	10%
E	13%
F	$1/3 \times 12\% + 1/3 \times 6\% + 1/3 \times 14\% = 10.67\%$
G	10.67%
H	12.67%
I	$1/4 \times 12\% + 1/4 \times 6\% + 1/4 \times 14\% + 1/4 \times 12\% = 11\%$

Portfolio	Variance
A	$(1/2)^2 \times 0.0008 + (1/2)^2 \times 0.0002 + 2 \times 1/2 \times 1/2 \times (-0.0004) = 0.00005$
B	0.00125
C	0.00046
D	0.0002
E	0.0007
F	$(1/3)^2 \times 0.0008 + (1/3)^2 \times 0.0002 + (1/3)^2 \times 0.0018 + 2 \times 1/3 \times 1/3 \times (-0.0004) +$ $+ 2 \times 1/3 \times 1/3 \times 0.0012 + 2 \times 1/3 \times 1/3 \times (-0.0006) = 0.00036$
G	0.0002
H	0.00067
I	$(1/4)^2 \times 0.0008 + (1/4)^2 \times 0.0002 + (1/4)^2 \times 0.0018 + (1/4)^2 \times 0.00107 \times +$ $+ 2 \times 1/4 \times 1/4 \times (-0.0004) + 2 \times 1/4 \times 1/4 \times 12 + 2 \times 1/4 \times 1/4 \times 0 +$ $+ 2 \times 1/4 \times 1/4 \times (-0.0006) + 2 \times 1/4 \times 1/4 \times 0 + 2 \times 1/4 \times 1/4 \times 0 = 0.00027$

(d)-(e)

We can conclude that assets A, B and D are nor efficient, as well as portfolios b and f. In all these cases we can find portfolios with lower or equal risk and higher or equal expected return.

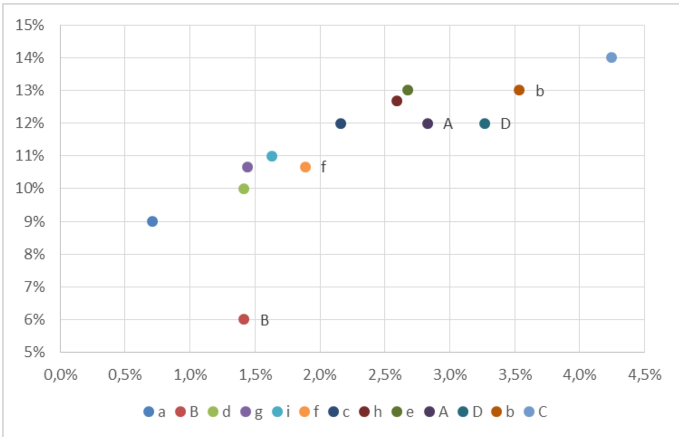


Figure 1: Exercise 1.1 – Representation of the assets and several portfolios in the space $(\sigma_p, \bar{R}_p)$.

Exercise 1.2.

(a) The formula for the variance of an equally weighted portfolio (where $X_i = 1/N \ \forall i = 1, \ldots, N$ securities) is

$$\sigma_H^2 = \frac{1}{N} \underbrace{\left[\sum_{i=1}^{N} \left(\frac{\sigma_i^2}{N} \right) \right]}_{\overline{\sigma_i^2}} + \frac{N-1}{N} \underbrace{\left[\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \frac{\sigma_{ij}}{N(N-1)} \right]}_{\overline{\sigma_{ij}}} = \frac{1}{N} \left(\overline{\sigma_i^2} - \overline{\sigma_{ij}} \right) + \overline{\sigma_{ij}} \quad (1)$$

where $\overline{\sigma_i^2}$ is the average variance across all securities, $\overline{\sigma_{ij}}$ is the average covariance across all pairs of securities, and N is the number of securities. Using the above formula with $\overline{\sigma_i^2} = 0.005$ and $\overline{\sigma_{ij}} = 0.001$ we have:

N	5	10	20	50	100
σ_H^2	0.0018	0.0014	0.0012	0.00108	0.00104

(b) As the number of securities (N) approaches infinity, an equally weighted portfolio's variance (total risk) approaches a minimum equal to the average covariance of the pairs of securities in the portfolio, which is 10. Therefore the risk is $\sigma_{MV} = \sqrt{0.001} = 3.16\%$. Having a risk only 10% higher than the minimum variance portfolio means $\sigma_H \leq 3.16 \times 1.1 = 3.48\% \iff \sigma_H^2 = 0.00121$. To know how many securities a portfolio must have to respect this condition we need to solve the inequality:

$$\sigma_H^2 = \frac{1}{N}\left(\overline{\sigma_i^2} - \overline{\sigma_{ij}}\right) + \overline{\sigma_{ij}} \leq 0.001211$$
$$\frac{1}{N}(0.005 - 0.001) \leq 0.001211 \Leftrightarrow N \geq 19.05$$

Thus, the portfolio must have, at least, 20 securities.

(c) No, the average covariance works as an asymptote to the variance of any portfolio. As N increases the variance of a portfolio converges to that limit, but would only reach it at infinity.

Exercise 1.3.

(a) If the portfolio contains only one security, then the portfolio's average variance is equal to the average variance across all securities, $\bar{\sigma}_j^2$. If instead an equally weighted portfolio contains a very large number of securities, then its variance will be approximately equal to the average covariance of all pairs of securities in the portfolio $\bar{\sigma}_{kj}$. Therefore, the fraction of risk that of an individual security that can be eliminated by holding a large portfolio is expressed by the following ratio:

$$D = \frac{\overline{\sigma_i^2} - \overline{\sigma_{ij}}}{\overline{\sigma_i^2}}$$

The above ratio is equal to 0.6(60%) for Italian securities and 0.8(80%) for Belgian securities.

(b) Setting the above ratio equal to those values and solving for $\overline{\sigma_{ij}}$ gives $\overline{\sigma_{ij}} = 0.4\overline{\sigma_i^2}$ for Italian securities and $\overline{\sigma_{ij}} = 0.2\overline{\sigma_i^2}$ for Belgian securities.

If the average variance of a single security, $\overline{\sigma_i^2}$, in each country equals 0.005, then $\overline{\sigma_{ij}} = 0.4\overline{\sigma_i^2} = 0.4 \times 0.0050 = 0.002$ for Italian securities and $\overline{\sigma_{ij}} = 0.2\overline{\sigma_i^2} = 0.2 \times 0.005 = 0.001$ for Belgian securities. Using Equation (**??**) with $\overline{\sigma_i^2} = 0.005$ and either $\overline{\sigma_{ij}} = 0.002$ for Italy or $\overline{\sigma_{ij}} = 0.001$ for Belgium we have:

Portfolio Size (N securities)	Italian σ_H^2	Belgian σ_H^2
5	0.0026	0.0018
20	0.00215	0.0012
100	0.00203	0.00104

Exercise 1.4.

(a) The diversification ratio measures, in percentage, how much of the average asset variance can be diversified away by building portfolios.

In this case we have

$$D = \frac{\overline{\sigma_i^2} - \overline{\sigma_{ij}}}{\overline{\sigma_i^2}} = \frac{0.0046619 - 0.0007058}{0.0046619} = 84.86\%.$$

(b) The formula for an equally weighted portfolio's variance is

$$\sigma_H^2 = \frac{1}{N}\left(\overline{\sigma_i^2} - \overline{\sigma_{ij}}\right) + \overline{\sigma_{ij}}$$

where $\overline{\sigma_i^2}$ is the average variance across all securities, $\overline{\sigma_{ij}}$ is the average covariance across all securities, and N is the number of securities. The average variance for the securities in the table is 0.0046619 and the average covariance is 0.0007058. We want the volatility to be lower than 2.83%, i.e. the variance $\sigma_H^2 \leq 0.0283^2 = 0.0008$. Using the above equation and solving for N gives:

$$\begin{aligned} 0.0008 \geq \frac{1}{N}(0.0046619 - 0.0007058) + 0.0007058 \\ 0.0000942N \geq 0.0039561 \\ N \geq 41.997 \end{aligned}$$

Since the portfolio's variance decreases as N increases, holding 42 securities will provide a variance less than 0.0008, so 42 is the minimum number of securities required.

1.2 Investment Opportunity Sets and Efficient Frontiers

Exercise 1.5.

(a) We know that $\sigma_A = 9\%$ and $\sigma_B = 15\%$. We also know that securities A and B are combined in order to override the portfolio risk, which is only possible when $\rho = -1$. Therefore, the weight of each asset in portfolio of zero risk is given the equation system

$$\begin{cases} x_A + x_B = 1 \\ \sigma_A^2 x_A^2 + \sigma_B^2 x_B^2 + 2x_A x_B \sigma_{AB} = 0 \end{cases} \Leftrightarrow \begin{cases} x_B = 1 - x_A \\ \sigma_A^2 x_A^2 + \sigma_B^2 (1 - x_A)^2 + 2x_A (1 - x_A)\sigma_{AB} = 0 \end{cases}$$

$$\begin{cases} x_B = 1 - x_A \\ 0.0081x_A^2 + 0.225\left(1 - 2x_A + x_A^2\right)^2 + 2x_A(1 - x_A)(-0.0135) = 0 \end{cases} \Leftrightarrow$$

$$x_A = 0.625 \qquad x_B = 1 - 0.625 = 0.375$$

Therefore, $x_A = 62.5\%$ and $x_B = 37.5\%$.

(b) If the null risk portfolio has a return of 7.5%, we know its composition is

$$7.5\% = 5\% x_A + \bar{R}_B(1 - x_A)$$

and from (a) we also know that $x_A = 62.5\%$. Thus,

$$\bar{R}_B = \frac{0.075 - 0.05 \times 0.625}{0.375} = 11.67\%$$

(c) The statement is TRUE. Asset B is the one with the highest expected return and risk. From above we see the zero-risk portfolio requires a positive investment in asset B (of 37.5%). Any portfolio with lower weight in B has a negative Sharpe ratio (slope in mean-variance space). Thus, short selling of asset B to invest more than 100% in asset A is also necessarily inefficient.

Given the perfect negative correlation, we know that geometrically the investment opportunity set (IOS) is defined by two segments of lines each passing by each of the two risky securities and with a common y-cross at the zero risk portfolio. When shortselling is forbidden the drashed portions are not feasible.

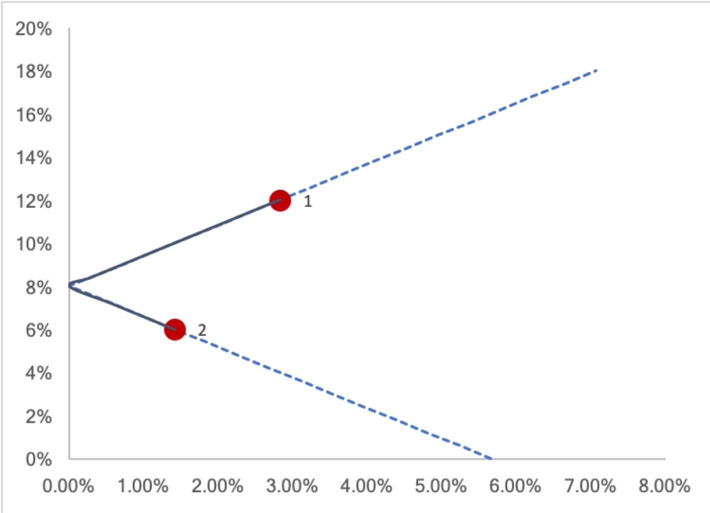


Figure 2: Exercise 1.6 – two risky assets $\rho = -1$. IOS with (full + dashed lines) and without (full) shortselling.

Exercise 1.6.

(a)-(b) From Exercise 1.1 we know $\bar{R}_1 = 12\%$, $\bar{R}_2 = 6\%$, $\sigma_1 = 2.83\%$, $\sigma_2 = 1.41\%$ and $\rho_{12} = -1$.

We can get the IOS analytical expression to the equations by: (i) first finding the expected return of the combination with zero risk, and then (ii) using the basic assets 1 and 2 to find the slopes of the two lines.

(i) The minimum variance portfolio is the one without risk, $\sigma_p = 0$. Analytically,

$$0 = \sigma_1^2 x_1^2 + \sigma_2^2 (1 - x_1)^2 + 2x_1 (1 - x_1) \sigma_{12}$$

$$x_1 = \frac{\sigma_p + \sigma_2}{\sigma_1 + \sigma_2} = \frac{\sigma_2}{\sigma_1 + \sigma_2} = \frac{\sqrt{2}}{\sqrt{8} + \sqrt{2}} = \frac{1}{3} \qquad \Rightarrow \qquad x_2 = \frac{2}{3}$$

Thus, the portfolio has 33.33% of security 1 and 66.67% of security 2. The expected return is

$$\bar{R}_{MV} = \sum x_i \bar{R}_i = \frac{1}{3} \times 12\% + \frac{2}{3} \times 6\% = 8\%$$

(ii) The slopes of the two lines are given by $\frac{\bar{R}_1 - 8\%}{\sigma_1} = 1.41$ and $\frac{\bar{R}_2 - 8\%}{\sigma_2} = -1.41$, respectively.

So, the IOS is given by $\bar{R}_p = \begin{cases} 8\% + 1.41\sigma_p & \sigma_p \leq 2,83\% \\ 8\% - 1.41\sigma_p & \sigma_p \leq 1.41\% \end{cases}$.

(c) All portfolios in the segment line with positive slope dominate those in the negative slope segment line, since risk averse investors will prefer from a set of two portfolios with the same risk, the one with highest return. Therefore, the efficient frontier in the positive slope segment line, i.e. $\bar{R}_p = 8\% + 1.41\sigma_p$ for $\sigma_p \leq 2.83\%$.

(d) If shortselling is allowed the derivations in (a)-(b) still stand, the only different is that in the representation of the IOS the entire lines should be considered. I.e. in the above figure the dashed segments would also be feasible.

The efficient set would, thus, be represented by the entire upper line.

$$\text{IOS:} \quad \bar{R}_p = 8\% \pm 1.41\sigma_p \quad \text{and} \quad \text{EF:} \quad \bar{R}_p = 8\% + 1.41\sigma_p \ .$$

In particular, all combinations of 1 and 2 that require shortselling of asset 2 to invest more than 100% in 1 are efficient.

Exercise 1.7.

(a) We start by determining expected returns, variances and covariances of the two assets.

$$\bar{R}_1 = \mathbb{E}(R_1) = \frac{1}{3} \times (0.2 + 0.14 + 0.08) = 14\%$$
$$\bar{R}_2 = \mathbb{E}(R_2) = \frac{1}{3} \times (0.16 + 0.12 + 0.08) = 12\%$$
$$\sigma_1^2 = \mathbb{E}\left[\left(R_{1t} - \bar{R}_1\right)^2\right] = \frac{1}{3} \times \left[(0.2 - 0.14)^2 + (0.14 - 0.14)^2 + (0.08 - 0.14)^2\right] = 0.0024$$
$$\sigma_1 = \sqrt{0.0024} = 4.90\%$$
$$\sigma_2^2 = \mathbb{E}\left[\left(R_{2t} - \bar{R}_2\right)^2\right] = \frac{1}{3} \times \left[(0.16 - 0.12)^2 + (0.12 - 0.12)^2 + (0.08 - 0.12)^2\right] = 0.001067$$
$$\sigma_2 = \sqrt{0.001067} = 3.236\%$$
$$\sigma_{12} = \mathbb{E}\left[\left(R_{1t} - \bar{R}_1\right)\left(R_{2t} - \bar{R}_2\right)\right] = \frac{1}{3} \times \left[(0.2 - 0.14)(0.16 - 0.12) + (0.08 - 0.14)(0.08 - 0.12)\right]$$
$$= 0.0016$$
$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} = \frac{0.0016}{\sqrt{0.0024}\sqrt{0.001067}} = +1$$

Thus, the returns of the two securities are perfectly positively correlated, thus, the investment opportunity set (IOS), when shortselling is allowed, is given by two lines: one connecting the two risky securities, and the line with symmetric slope.

$$\text{IOS (i)}: \qquad \bar{R}_p = \underbrace{\bar{R}_2 - \frac{\bar{R}_1 - \bar{R}_2}{\sigma_1 - \sigma_2}\sigma_2}_{\text{y-cross}} + \underbrace{\frac{\bar{R}_1 - \bar{R}_2}{\sigma_1 - \sigma_2}}_{\text{slope}} \sigma_p$$
$$= 0.12 - \frac{0.14 - 0.12}{0.049 - 0.03236} 0.03236 + \frac{0.14 - 0.12}{0.049 - 0.03236} \sigma_p$$
$$= 0.08 + 1.2247\sigma_p$$

and

$$\bar{R}_p = 0.08 - 1.2247\sigma_p \ .$$

Although the negative slope line is not efficient it still belongs to the IOS.

When shortselling is not allowed, the IOS is only the segment of the line that passes by the two risky assets

$$\text{IOS (ii)}: \qquad \bar{R}_p = 0.08 + 1.2247\sigma_p \quad \text{for} \quad 1.41\% \leq \sigma_p \leq 3.236\%$$

(b) The minimum variance portfolio, when shortselling is forbidden – scenario (ii) – involves placing all funds in the lower risk security (asset 2). Consequently, the expected return is $\bar{R}_{MV} = \bar{R}_2 = 12\%$ and risk is $\sigma_{MV} = \sigma_2 = 3.236\%$.

If short sales were allowed – scenario (i) – than $\sigma_p = 0$ and $\bar{R}_p = 8\%$. Moreover, the weights of the MV portfolio is,

$$x_1 = \frac{\sigma_p - \sigma_2}{\sigma_1 - \sigma_2} = -200\% \Rightarrow x_2 = 1 - x_1 = 1 - (-2) = 300\%$$

(c) As known, the efficient frontier is the investment opportunity set and investor are considered to be risk averse. For the two scenarios we have:

$$\text{EF (i):} \qquad \bar{R}_p = 8\% + 1.2247\sigma_p$$
$$\text{EF (ii)} = \text{IOS (ii)}: \qquad \bar{R}_p = 0.08 + 1.2247\sigma_p \quad \text{for} \quad 1.41\% \le \sigma_p \le 3.236\%$$

(d) If we have a riskless asset with $R_f = 10\%$ then The investment opportunity set becomes:

- IOS (i): When shortselling is allowed without any bound, the theoretical answer would be the entire space $(\sigma_p, \bar{R}_p)$.
  In a real life situation, there will be an extreme combination, E, where one takes the highest possible shortselling position in asset 2. In that case the IOS would be the *entire area bellow the straight line* $\bar{R}_p = 0.1 + \dfrac{\bar{R}_E - 0.1}{\sigma_E}\sigma_p$.
- IOS (ii): When shortselling is not allowed the IOS is *the cone limited by* the lines $\bar{R}_p = 0.1 \pm \dfrac{0.14 - 0.1}{0.049}\sigma_p$.

The efficient frontier (EF) becomes:

- When shortselling is allowed – scenario (i) – the efficient frontier would be the straight line that has y-cross at 10% and has the highest possible slope.
  In a real life situation ,where eventually there would be a limit to how much one can shortsell of asset 2, it would be combinations of the riskless asset with the portfolio with that extreme, E portfolio,
  EF (i): $\bar{R}_p = 0.1 + \dfrac{\bar{R}_E - 0.1}{\sigma_E}\sigma_p$.
- When shortseliing is not allowed – scenario (ii) the efficient frontier would be given by combinations of the riskless asset with asset 1,
  EF (ii): $\bar{R}_p = 0.1 + \dfrac{0.14 - 0.1}{0.049}\sigma_p$.

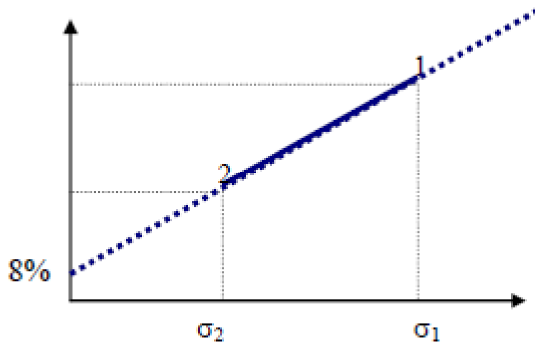


Figure 3: Exercise 1.7 – Two perfectly correlated assets. Efficient frontier with (full + drashed) and without (full) shortselling.

Exercise 1.8.

(a) Similar to (b) but with $\rho = -1$ (see slides).

(b) As discussed in Exercise **??**, the investments opportunity set generated by two assets with perfect negative correlation is given by two line segments. An alternative to the solution presented there is to deduce directly the equation(s) $\bar{R}_p = f(\sigma_p)$.

Starting with some transformation in σ_p equation:

$$\begin{aligned}\sigma_p &= \sqrt{x_1^2\sigma_1^2 + x_2^2\sigma_2^2 - 2x_1x_2\sigma_1\sigma_2} \\ &= \sqrt{(x_1\sigma_1 - x_2\sigma_2)^2} \\ &= \pm\left|x_1\sigma_1 - x_2\sigma_2\right| \\ &= \pm\left|x_1\sigma_1 - (1-x_1)\,\sigma_2\right|\end{aligned}$$

With additional transformations, we get an equation to x_1

$$\sigma_p \pm \sigma_2 = \pm x_1(\sigma_1 - \sigma_2) \Leftrightarrow x_1 = \pm\frac{\sigma_p + \sigma_2}{\sigma_1 + \sigma_2}$$

Replacing x_1 in the expected return equation for a two assets portfolio, we get

$$\begin{aligned}\bar{R}_p &= x_1\bar{R}_1 + (1-x_1)\,\bar{R}_2 \\ &= \pm\frac{\sigma_p + \sigma_2}{\sigma_1 + \sigma_2}\bar{R}_1 + \left(1 \pm \frac{\sigma_p + \sigma_2}{\sigma_1 + \sigma_2}\right)\bar{R}_2 \\ &= \bar{R}_2 + \frac{\pm\sigma_p\bar{R}_1 + \sigma_2\bar{R}_1 \pm \sigma_p\bar{R}_2 - \sigma_2\bar{R}_2}{\sigma_1 + \sigma_2} \\ &= \left(\bar{R}_2 + \frac{\bar{R}_1 - \bar{R}_2}{\sigma_1 - \sigma_2}\sigma_2\right) \pm \left(\frac{\bar{R}_1 - \bar{R}_2}{\sigma_1 - \sigma_2}\right)\sigma_p\end{aligned}$$

The first term in the right side in the intersection in the y's axis and the second term is the slope, which can be positive or negative, giving origin to the two expected line segments.

(c) Equal to (d), replacing the generic ρ by 0 (see slides).

(d) Solved in class.

(e) Real life correlations are not perfect, so real life correlations are $\rho \neq -1$ or $\rho \neq +1$. Returns laso tend to be correlate with one another, so $\rho \neq 0$. All other values may occur, but for financial assets positive correlations are more common than negative.

Exercise 1.9.

(a) The investment opportunity sets are represented in the Figure **??** below.

(b)
- When $\rho = +1$, the least risky "combination" of securities 1 and 2 is security 2 held alone (assuming no short sales). This requires $x_1^{MV} = 0$ and $x_2^{MV} = 1$, where the $x's$ are the investment weights. The standard deviation of this "combination" is equal to the standard deviation of security 2: $\sigma_{MV} = \sigma_2 = 2\%$.
- When $\rho = -1$, we can always find a combination of the two securities that will completely eliminate risk, and we this combination can be found by solving $x_1^{MV} = \frac{\sigma_2}{\sigma_1+\sigma_2}$. So, $x_1^{MV} = \frac{2\%}{5\%+2\%} = \frac{2}{7}$, and since the investment weights must sum to 1, $x_2^{MV} = 1 - x_1 = 1 - \frac{2}{7} = \frac{5}{7}$. So a combination of $\frac{2}{7}$ invested in security 1 and $\frac{5}{7}$ invested in security 2 will completely eliminate risk when ρ equals -1, and σ_{MV} will equal 0.
- When $\rho = 0$ e, the minimum-risk combination of two assets can be found by solving $x_1^{MV} = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$. So, $x_1^{MV} = \frac{4\%}{25\%+4\%} = \frac{4}{29}$, and $x_2^{MV} = 1 - x_1 = 1 - \frac{4}{29} = \frac{25}{29}$. When ρ equals 0, the expression for the standard deviation of a two-asset portfolio is

$$\sigma_p = \sqrt{x_1^2\sigma_1^2 + (1-x_1)^2\,\sigma_2^2}$$

Substituting $\frac{4}{29}$ for x_1 in the above equation, we have

$$\sigma_{MV} = \sqrt{\left(\frac{4}{29}\right)^2 \times 0.0025 + \left(\frac{25}{29}\right)^2 \times 0.0004} = 1.86\% \ .$$

(c) (i) Both for $\rho = -1$ nd $\rho = 0$ the minimum variance portfolios remain the same. However, for $\rho = +1$ if shortselling is allowed we can fully eliminate risk. Since, in the case, we have $\sigma_p = |x_1\sigma_1 + (1 - x_1)\sigma_2|$, setting $\sigma_{MV} = 0$, we obtain

$$0 = x_1^{MV} \times 0.05 + (1 - x_1^{MV})0.02 \quad \Leftrightarrow \quad x_1^{MV} = -\frac{0.02}{0.03} = -66.67\%,\ x_2^{MV} = 166.67\%.$$

(ii) When we have $\rho = \pm 1$ there is a combination of 1 and 2 that fully eliminates risk, thus there is a risk-free investment or a "ficticious" riskless asset. The return of the zero risk combinations give us the appropriate risk-free return R_f.

$$\begin{aligned}
\rho = -1: \qquad R_f &= x_1^{MV}\bar{R}_1 + x_2^{MV}\bar{R}_2 \\
&= \frac{2}{7} \times 10\% + \frac{5}{7} \times 4\% = 5.71\% \\
\rho = +1: \qquad R_f &= x_1^{MV}\bar{R}_1 + x_2^{MV}\bar{R}_2 \\
&= -0.6667 \times 10\% + 1.6667 \times 4\% = 0\% \ .
\end{aligned}$$

In the case of $\rho = 0$ the minimum risk combination portfolio has positive volatility $sigma_M V = 1.86\%$, thus there is no risk free investment.

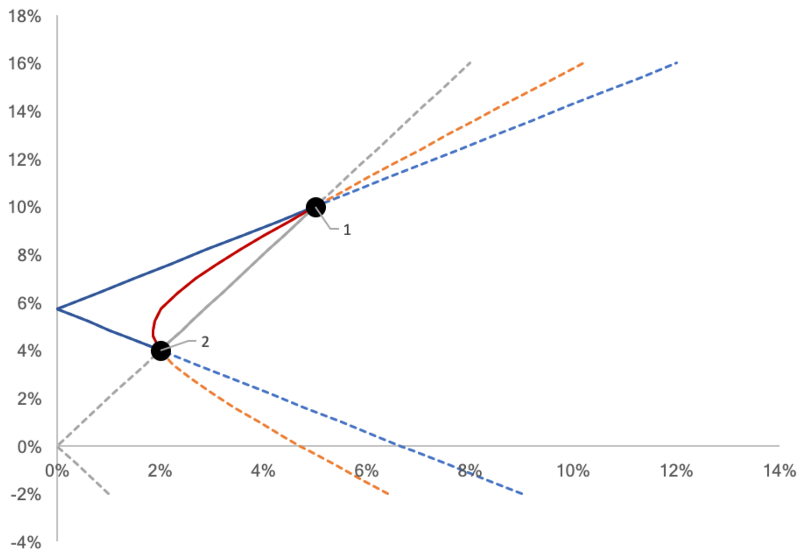


Figure 4: Exercise 1.9 – blue line $\rho = -1$, red line $\rho = 0$ and grey line $\rho + 1$, full lines (no shortselling), dashed lines (shortselling required).

Exercise 1.10. If the risk-less rate is 10%, then the risk-free asset dominates both risky assets both in terms of risk and return. It offers as much or higher return than each of the risky assets, for zero risk. Assuming the investor prefers more to less and is risk averse, the only efficient investment is 100% investment in the risk-free asset.

Exercise 1.11.

(a) When there is a risk-free asset that can be used for both lending and borrowing we know the efficient frontier is a straight line tangent to the investment opportunity set of risky assets. Thus, there is only one efficient portfolio made only of risky assets - the so called *tangent portfolio*. See Figure **??**.

To find this unique efficient portfolio we need to maximize Sharpe's Ratio of all portfolios formed with assets A, B and C. From the first order conditions of this maximisation problem, result the following equation system:

$$\begin{cases} 0.11 - R_f = 0.0004z_A + 0.001z_B + 0.0004z_C \\ 0.14 - R_f = 0.0010z_A + 0.0036z_B + 0.003z_C \\ 0.17 - R_f = 0.0004z_A + 0.003z_B + 0.0081z_C \end{cases}$$

The Z-vector, for each given value for R_F and the unrestricted tangent portfolios are:

	$R_f = 6\%$	$R_f = 8\%$	$R_f = 10\%$
z_A	351.0067	185.2348	19.4631
z_B	-104.3624	-52.6845	-1.0070
z_C	34.8993	21.4765	8.0537
x_A	124.67%	120.26%	73.42%
x_B	-37.07%	-34.20%	-3.80%
x_C	12.40%	13.94%	30.38%
Tangent Portfolio			
Expected Return	10.63%	10.81%	12.71%
Standard Deviation	1.28%	1.35%	3.20%
Sharpe ratio	3.611	2.081	0.8474
Efficient Frontier	$\bar{R}_p = 0.06 + 3.611\sigma_p$	$\bar{R}_p = 0.08 + 2.081\sigma_p$	$\bar{R}_p = 0.1 + 0.8474\sigma_p$

(b) If there is no credit to invest in risky assets nothing changes in the efficient frontier for risk levels lower or equal to σ_T, however for $\sigma_p > \sigma_T$ the efficient thing to do are the combinations on the envelop hyperbola.

The hyperbola delimiting the IOS of the risky assets is given by

$$\sigma_p^2 = \frac{A\bar{R}_p^2 - 2B\bar{R}_p + C}{AC - B^2} \qquad \text{where} \qquad \begin{matrix} A = \mathbb{1}'V^{-1}\mathbb{1} \\ B = \bar{R}'V^{-1}\mathbb{1} \\ C = \bar{R}'V^{-1}\bar{R} \end{matrix}$$

For our concrete example we get

$$\sigma_p^2 = 1.6450\bar{R}_p^2 - 0.3426\bar{R}_p + 0.018 \ .$$

The efficient frontier is, thus, given by

$$\begin{cases} \bar{R}_p = R_f + SR_T \times \sigma_p & \sigma_p \leq \sigma_T \\ \sigma_p^2 = 1.6450 \ \bar{R}_p^2 - 0.3426\bar{R}_p + 0.018 & \sigma_p > \sigma_T \end{cases}$$

where in the expression above we should replace for the appropriate values of R_f, SR_T and σ_T, according to each scenario.

(c) If shortselling is forbidden we know we are not going to invest in asset B, since the optimal would be to short sell it.

We can solve the problem numerically, imposing $x_i \geq 0$ to all $i = A, B, C$, or, in this case the problem reduces to a two-asset case and find the two-asset tangent portfolios.

Either way, we get

	$R_f = 6\%$	$R_f = 8\%$	$R_f = 10\%$
x_A	93.77%	89.61%	68.83%
x_C	6.23%	10.39%	31.17%
Tangent Portfolio			
Expected Return	11.37%	11.62%	12.87%
Standard Deviation	2.07%	2.2%	3.39%
Sharpe ratio	2.592	1.648	0.8471
Efficient Frontier	$\bar{R}_p = 0.06 + 2.592\sigma_p$	$\bar{R}_p = 0.08 + 1.648\sigma_p$	$\bar{R}_p = 0.1 + 0.8471\sigma_p$

(d)

(i) We use the same Z-vectors as in the unrestricted case, to get the Lintner portfolios.

	$R_f = 6\%$	$R_f = 8\%$	$R_f = 10\%$
x_A	71.60%	71.41%	68.24%
x_B	-21.29%	-20.31%	-3.53%
x_C	7.11%	8.28%	28.24%
Lintner Portfolios			
$\sum x_i$	57.42%	59.38%	92.95%
x_f	42.58%	40.62%	7.05%
Expected Return	8.66%	9.67%	12.52%
Standard Deviation	0.74%	0.80%	2.97%
Sharpe ratio	3.611	2.081	0.8474

Note that Lintner portfolios have the same Sharpe ratios as unrestricted tangent portfolios. They can always be interpreted as a combination of deposit with the (unrestricted) tangent portfolio.

(ii) Since, none the original tangent portfolios requires more than 50% shortselling, they all satisy this restriction.

(ii) For the case of $R_f = 10\%$ this limit is satisfied and nothing changes.

For $R_f = 6\%$ and $R_f = 8\%$ the limit is not satisfied by the original tangent portfolios, thus, we know that we will now get $x_B = -25\%$. The remaining weight we can get numerically (for instance using excel solver).

The table below show the results.

	$R_f = 6\%$	$R_f = 8\%$	$R_f = 10\%$
x_A	115.27%	112.82 %	73.42%
x_B	-25.00%	-25.00%	-3.80%
x_C	9.73%	12.18%	30.38%
Tangent Portfolios (limited 25% shortselling)			
Expected Return	10.83%	10.98%	12.71%
Standard Deviation	1.42%	1.47%	3.20%
Sharpe ratio	3.413	2.021	0.8474

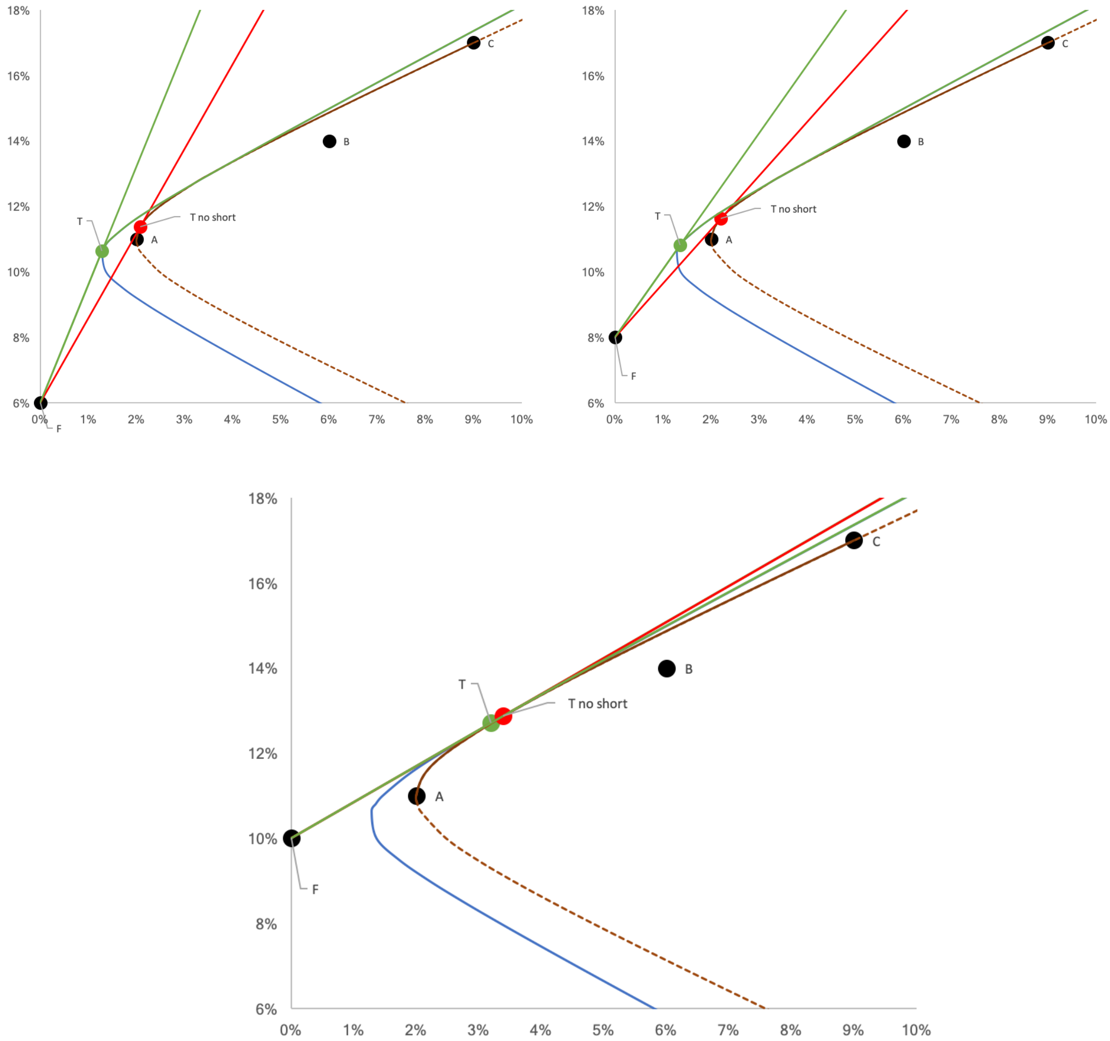


Figure 5: Exercise 1.11 – Efficient Frontiers (green) when shortselling is allowed (with and without borrowing). Outer hyperbola (blue) is the envelop hyperbola when we consider investment without constraints in the three assets A, B, C. Inner hyperbola is the two-assets hyperbola for assets A and C where the full line represents the no shortselling segment and the dashed line the portfolios that require shortselling. Top left image: $R_f = 6\%$. Top right: $R_f = 8\%$. Bottom image: $R_f = 10\%$

Exercise 1.12.

(a) Since the given portfolios, A and B, are on the efficient frontier, and we know $\rho_{AB} = +5/6 = 0.8333$. We can determine their covariance $\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B = 0.002$ and the MV portfolio can be obtained by finding the minimum-risk combination of the two portfolios:

$$x_A^{MV} = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}} = \frac{0.0016 - 0.002}{0.0036 + 0.0016 - 2 \times 0.002} = -\frac{1}{3}$$
$$x_B^{MV} = 1 - x_A^{MV} = 1 - \left(-\frac{1}{3}\right) = \frac{4}{3}$$

This gives $\bar{R}_{MV} = 7.33\%$ and $\sigma_{MV} = 3.83\%$.

Also, since the two portfolios are on the efficient frontier, the entire efficient frontier can then be traced by using various combinations of the two portfolios, starting with the MV portfolio and moving up along the efficient frontier (increasing the weight in portfolio A and decreasing the weight in portfolio B).

Since $x_B = 1 - x_A$ the efficient frontier equations are:

$$\begin{cases} \bar{R}_p = x_A\bar{R}_A + (1 - x_A)\,\bar{R}_B \\ \sigma_p^2 = x_A^2\sigma_A^2 + (1 - x_A)^2\,\sigma_B^2 + 2x_A\,(1 - x_A)\,\sigma_{AB} \end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases} \bar{R}_p = 0.10x_A + 0.08 \times (1 - x_A) \\ \sigma_p^2 = 0.0036x_A^2 + 0.0016\,(1 - x_A)^2 + 2x_A\,(1 - x_A)\,0.002 \end{cases}$$
$$\Leftrightarrow$$
$$\sigma_p^2 = 3\bar{R}_p^2 - 0.44\bar{R}_p + 0.0176 \qquad \text{(hyperbola equation)}$$

Since short sales are allowed, the efficient frontier will extend beyond portfolio A and out toward infinity. The efficient frontier appears as shown in Figure **??** (full blue line).

(b) If there is a risk-free asset that can be used for both deposit and borrowing, then we know the efficient frontier is a straight line passing by the risk-free asset and tangent to the hyperbola given by combinations of any two efficient portfolios. So, Its equations is given by $\bar{R}_p = R_f + SR_T\sigma_p$ where SR_T is the Sharpe ratio of the tangent portfolio.

The tangent portfolio is the combination of the two efficient portfolios that has the highest Sharpe ratio. From the FOC we find

$$Z = V^{-1}(\bar{R} - R_f\mathbb{1}) = \begin{pmatrix} 4.5455 \\ 31.8182 \end{pmatrix} \qquad \Rightarrow \qquad X_T = \begin{pmatrix} 12.5\% \\ 87.5\% \end{pmatrix}$$

and we get

$$\bar{R}_T = X_T'\bar{R} = \begin{pmatrix} 12.5\% & 87.5\% \end{pmatrix} \begin{pmatrix} 10\% \\ 8\% \end{pmatrix} = 8.25\%$$
$$\sigma_T^2 = X_T'VX_T = \begin{pmatrix} 12.5\% & 87.5\% \end{pmatrix} \begin{pmatrix} 0.0036 & 0.002 \\ 0.002 & 0.0016 \end{pmatrix} \begin{pmatrix} 12.5\% \\ 87.5\% \end{pmatrix} = 0.00171875$$
$$\Rightarrow \sigma_T = 4.15\%$$
$$SR_T = \frac{\bar{R}_T - R_f}{\sigma_T} = \frac{0.0825 - 0.02}{0.0415} = 1.5076\ .$$

So the efficient frontier in this case is given by

$$\bar{R}_p = 0.02 + 1.5076\sigma_p\ ,$$

the straight green line in Figure **??**.

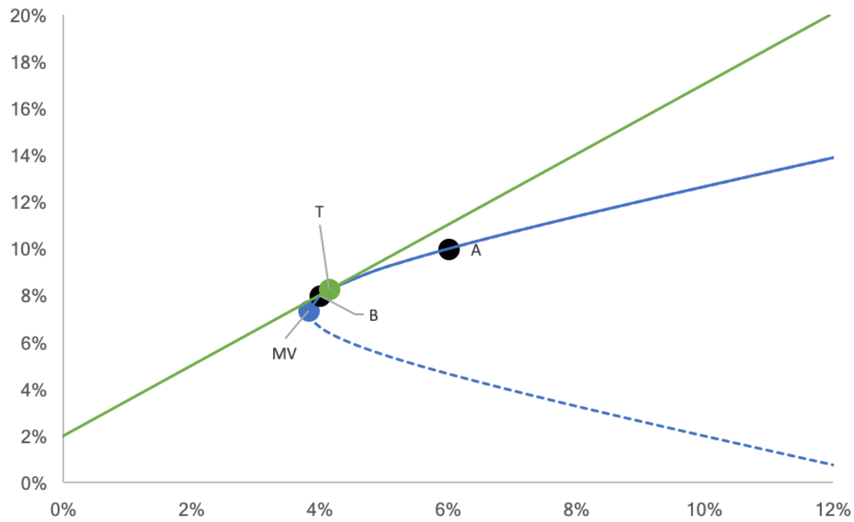


Figure 6: Exercise 1.12 – Efficient Frontier with (green line) and without the risk-free asset (full blue line on the upper part of the hyperbola.

(c) Since the tangent portfolio does not require shortselling the Lintner portfolio is the tangent portfolio itself and noting changes. When shortselling is limited *a la* Lintner nothing changes in term

  (i) If A and B are still feasible this means they are portfolios without any shortselling position. In addition, since the tangent portfolio does not require shortselling of A nor B, we have the guarantee it remains feasible. The minimum variance portfolio, however, requires short-selling of asset A and it would be no longer feasible. In this case the MV portfolio would be portfolio B itself. The envelop hyperbola would be the same but limited below by B above by A. Beyond these points the efficient frontier would be composed by parts of sets of hyperbolas in the interior of the general envelop hyperbola.

  (ii) If one of the original efficient portfolios is not longer feasible, that means that portfolio would require shortselling of some risky asset. Without two efficient portfolios we would not be able to derive the envelop hyperbola equation. Since we have no information about the basic risky assets in this market, we cannot derive the new efficient frontier, but it would be contained in the interior of the previously derived hyperbola.

1.3 Portfolio Protection

Exercise 1.13.

(a) (i) Since $R_f = 8\%$ we know that for $R_L = 6\%$ to minimize the probability of returns lower than 6% is equivalent to set that probability to zero, i.e. to deposit 100% of our wealth.

(ii) If we have $R_f = R_L = 8\%$ and Gaussian returns, all efficient portfolio have the same probability of returns lower than 8% .

(iii) If we have $R_L = 10\% > R_f = 8\%$ and Gaussian returns the optimal turns out to be to leverage up as much as possible (borrowing as much as possible) to invest more than our wealth in the tangent portfolio.

(b) Portfolios with the highest return-at-risk (RaR) are also the safest portfolio according to Kataoka. We, thus, are interested in portfolio returns with the probabilities lower or equal to 10%, i.e. in the worst 10% scenarios,

$$\begin{aligned}
&\Pr\left(R_p \leq R_L\right) \leq 10\% \\
&\Pr\left(\frac{R_p - \bar{R}_p}{\sigma_p} \leq \frac{R_L - \bar{R}_p}{\sigma_p}\right) \leq 10\% \\
&\Phi\left(\frac{R_L - \bar{R}_p}{\sigma_p}\right) \leq 10\% \\
&\frac{R_L - \bar{R}_p}{\sigma_p} \leq \Phi^{-1}(10\%) \\
&\bar{R}_p \geq R_L - \Phi^{-1}(10\%)\sigma_p \\
&\bar{R}_p \geq R_L + 1.2816\sigma_p
\end{aligned}$$

So the Kataoka lines are given by $\bar{R}_p = R_L + 1.2816\sigma_p$ and the goal is to maximize R_L along the EF.

From Exercise **??** recall that for $R_f = 8\%$, the EF is given by $\bar{R}_p = 0.08 + 2.081\sigma_p$. Since the slope of the EF is higher than the slope of the Kataoka lines, the maximum R_L will be the highest expected return portfolio. That is, the lowest RaR portfolio turns out to require extreme leverage (borrowing as much as possible) to invest more than our wealth in the tangent portfolio.

(c) Following the exact same steps as in (b) we get

$$\Pr\left(R_p \leq 10\%\right) \leq 10\% \qquad \Leftrightarrow \qquad \bar{R}_p \geq 0.1 + 1.2816\sigma_p \ .$$

Since the EF has a lower y-cross and a higher slope $\bar{R}_p = 0.08 + 2.081\sigma_p$, we need to find the crossing point.

$$0.1 + 1.2816\sigma_p = 0.08 + 2.081\sigma_p \quad \Leftrightarrow \quad \sigma_p = \frac{0.1 - 0.08}{2.081 - 1.2816} = 2.5\% \quad \Rightarrow \quad \bar{R}_p = 13.21\%.$$

(d) In (a) we deal with the Roy criterion, in (b) with the Kataoka criterion and in (c) with the Telser criterion. See Figure **??** for a graphical representation of the previous answers.

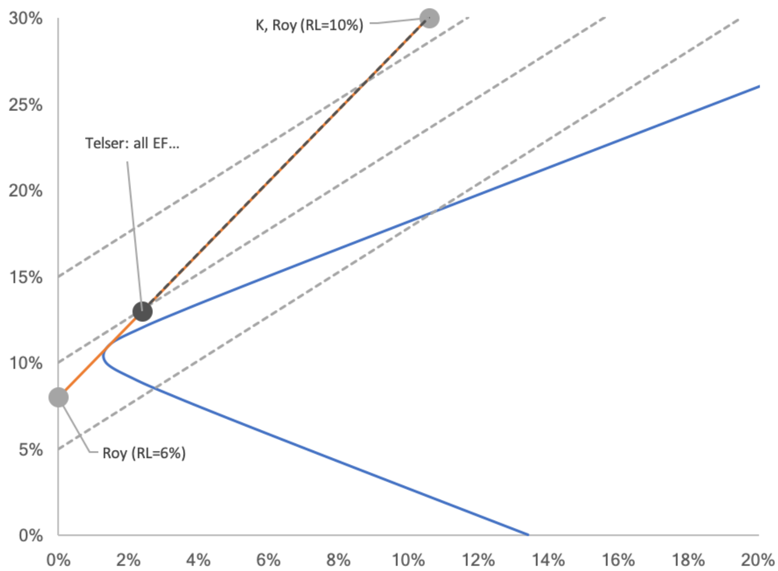


Figure 7: Exercise **??** – Efficient Frontier (red) and safety first portfolios determined in (a)–(c). Dashed grey lines are Kataoka lines for different R_L. All portfolios in the efficient frontier has the have probability of returns lower than $R_L = 8\%$. Dark grey dashed segment of line on the EF identify all portfolios that satisfy the Telser restriction.

Exercise 1.14.

(a) All combinations of A and B satisfy

$$\Pr\left(R_p \leq 5\%\right) = \Pr\left(\frac{R_p - \bar{R}_p}{\sigma_p} \leq \frac{5\% - \bar{R}_p}{\sigma_p}\right) = \Phi\left(\frac{5\% - \bar{R}_p}{\sigma_p}\right)$$

since, it must be less or equal than 15% we have

$$\Phi\left(\frac{5\% - \bar{R}_p}{\sigma_p}\right) \leq 15\% \quad \Leftrightarrow \quad \bar{R}_p \geq 5\% - \Phi^{-1}\left(15\%\right)\sigma_p \quad \Leftrightarrow \quad R_p \geq 5\% + 1.0364\,\sigma_p\ .$$

From Exercise **??** we also know all combinations of A and B are given by the hyperbola

$$\sigma_p^2 = 3\bar{R}_p^2 - 0.44\bar{R}_p + 0.0176.$$

From Figure **??** we clearly see that there is no combination of A and B that satisfies the safety condition.

(b) The combination that maximizes the likelihood of getting returns above 5% is the one that minimizes the probability of returns lower or equal to 5%, i.e. it is the Roy portfolio with $R_L = 5\%$. The Roy portfolio can be determine as a tangent portfolio, where R_L acts as a ficticious risk-free rate. In this case we get

$$Z = V^{-1}\left[\bar{R} - R_L\mathbb{1}\right] = \begin{pmatrix} 909.091 & -1136.364 \\ -1136.364 & 2045,454545 \end{pmatrix}\begin{pmatrix} 5\% \\ 3\% \end{pmatrix} = \begin{pmatrix} 11.3636 \\ 4.5455 \end{pmatrix}$$

$$\Downarrow$$

$$X^{\text{Roy}} = \begin{pmatrix} 71.43\% \\ 28.57\% \end{pmatrix}$$

The Roy portfolio is a concrete combination of A adnd B, so it belongs to the hyperbola. It has $\bar{R}^{\text{Roy}} = 9.43\%$ and $\sigma^{\text{Roy}} = 5.28\%$. Thus its probability of returns lower than 5% is

$$\Phi\left(\frac{5\% - 9.43\%}{5.28\%}\right) = \Phi\left(-0.894\right) = 20.1\% \ .$$

See the representation of the Roy portfolio in Figure **??**.

(c) The combination with the highest 15% quantile is the Kataoka portfolio for an $\alpha = 15\%$. For a fixed α we have lines with a fixed slope equal to $-\Phi^{-1}(\alpha)$. In this case we have $-\Phi^{-1}(\alpha) = 1.0364$. For find the Kataoka portfolio we need to find the hyperbola point with the exact same slope.

From the Kataoka lines we get

$$\bar{R}_p = R_L + 1.0364\ \sigma_p \qquad \Rightarrow \qquad \left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{Kataoka} = 1.0364$$

From the hyperbola equation, $\sigma_p^2 = 3\bar{R}_p^2 - 0.44\bar{R}_p + 0.0176$, and considering only its upper part (the efficient part, we have

$$\bar{R}_p = \frac{+0.44 + \sqrt{0.44^2 - 4 \times 3 \times (0.0176 - \sigma_p^2)}}{6}$$

and differentiating w.r.t. σ_p

$$\left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{hyperbola} = \frac{1}{6} \times \frac{1}{2}\left(0.44^2 - 4 \times 3 \times (0.0176 - \sigma_p^2)\right)^{-\frac{1}{2}} \times (-2\sigma_p)$$

Matching the slopes of the Kataoka lines with the hyperbola slope

$$\begin{aligned}
\left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{Kataoka} &= \left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{hyperbola} \\
1.0364 &= -\frac{1}{6}\left(0.44^2 - 4 \times 3 \times (0.0176 - \sigma_p^2)\right)^{-\frac{1}{2}} \sigma_p \\
1.0364^2 &= \frac{1}{36}\left(0.44^2 - 4 \times 3 \times (0.0176 - \sigma_p^2)\right)^{-1} \sigma_p^2 \\
1.0741\left(0.44^2 - 12(0.0176 - \sigma_p^2)\right) &= \frac{1}{36}\sigma_p^2 \\
\sigma_p^2 &= 0.002203 \qquad \Rightarrow \qquad \sigma^{\text{Kataoka}} = 4.05\%
\end{aligned}$$

Which implies and expecte return of

$$\bar{R}_p = \frac{+0.44 + \sqrt{0.44^2 - 4 \times 3 \times (0.0176 - 0.002203)}}{6} = 8.9\%.$$

Finally, in terms of composition we have $X^{\text{Kataoka}} = \begin{pmatrix} 45\% \\ 55\% \end{pmatrix}$.

(d) In (a) we deal with the Telser criterion, but in this case there was no feasible portfolio satisfiying the safety condition. In (b) we address Roy's safety criterion and in (c) Kataoka's.

(e) If the returns were not Gaussian we could do the same type of computations but using the correct distribution function.

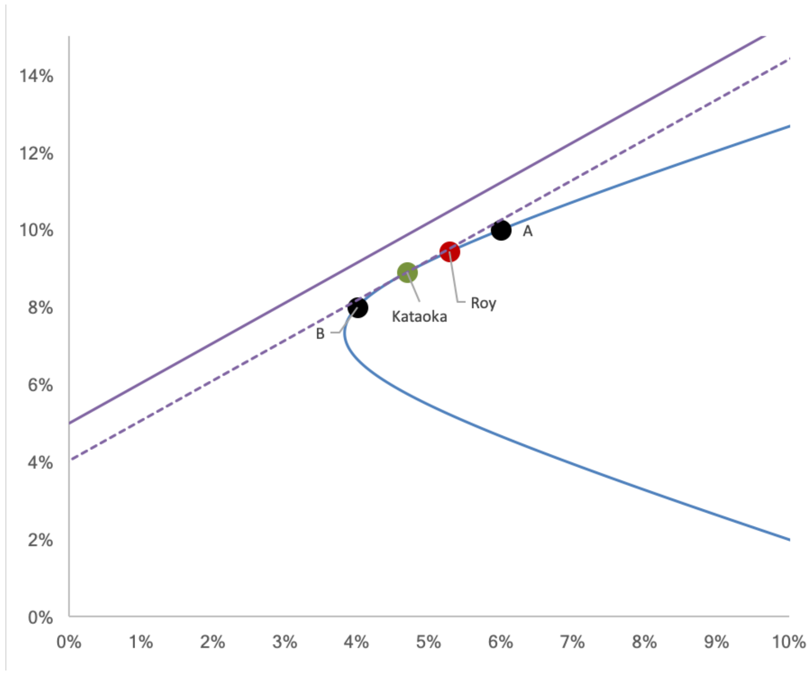


Figure 8: Exercise **??** – No Telser portfolio feasible. Representation of the two efficient portfolios A and B and the Roy and Kataoka portfolios.

1.4 International Diversification

Exercise 1.16. Diversification means combine different assets with different risk profiles such that we can manage to decrease our risk exposure while maintaining our return. Of course, diversification is only possible if the assets in the portfolio are not perfectly positively correlated ($\rho = 1$). Actually, the most idyllic scenario would perfectly negatively correlation ($\rho = -1$) among assets since it would allow us to cancel an important portion of portfolio's risk: the specific or idiosyncratic risk. Let,

$$\sigma_P^2 = \sum_{i=1}^{N} x_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} x_i x_j \sigma_{ij}$$

If $x_i = \frac{1}{N}$ then

$$\sigma_H^2 = \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{1}{N}\right)\left(\frac{1}{N}\right) \sigma_{ij}$$

Factoring out $1/N$ from the first summation and $(N-1)/N$ from the second and simplifying yields

$$\begin{aligned} \sigma_H^2 &= \frac{1}{N} \sum_{i=1}^{N} \left[\frac{\sigma_i^2}{N}\right] + \frac{(N-1)}{N} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \left[\frac{\sigma_{ij}}{N(N-1)}\right] \\ &= \frac{1}{N}\bar{\sigma}_i^2 + \frac{N-1}{N}\bar{\sigma_{ij}} = \frac{1}{N}\left(\bar{\sigma}_i^2 - \bar{\sigma}_{ij}\right) + \bar{\sigma}_{ij} \end{aligned}$$

This is a quite realistic representation of what occur when we invest in a portfolio of assets. The contribution to the portfolio variance of the variance of the individual securities goes to zero as N gets very large. However, the contribution of the covariance terms approaches the average covariance as N gets large. Actually, if we let $N \to \infty$, it cames

$$\lim_{N\to\infty} \sigma_H^2 = \lim_{N\to\infty} \frac{1}{N}\left(\bar{\sigma}_i^2 - \bar{\sigma}_{ij}\right) + \bar{\sigma}_{ij} = \bar{\sigma}_{ij}$$

Thus, as said before, the individual risk of securities can be diversified ways. Of course the higher the number os securities in the portfolio, the better the diversification. If we only consider a domestic market, the available number of tradable securities is lower than when we also consider external markets. Therefore, the major effect of diversification is to allow for a better diversification. However, this is at a price, which is exchange rate risk.

Exercise 1.17.

(a) The return due to exchange-rate changes (R_X) is equal to $fx_t/fx_{t-1} - 1$, where fx_t is the foreign exchange rate at time t expressed in terms of the investor's home currency per unit of foreign currency. Let fx_t be the exchange rate expressed in terms of dollars and fx_t^* be the exchange rate expressed in terms of pounds. These two rates are simply reciprocals, i.e., $fx_t^* = 1/fx_t$. So from the table in the problem we have:

Period	$(1+R_X)$ (for US investor)	$(1+R_X^*)$ (for UK investor)
1	$2.5/3 = 0.833$	$3/2.5 = 1.200$
2	$2.5/2.5 = 1.000$	$2.5/2.5 = 1.000$
3	$2/2.5 = 0.800$	$2.5/2 = 1.250$
4	$1.5/2 = 0.750$	$2/1.5 = 1.333$
5	$2.5/1.5 = 1.667$	$1.5/2.5 = 0.600$

The total return to a U.S. investor from a U.K. investment is

$$(1 + R_{US}) = (1 + R_X)(1 + R_{UK})$$

And the total return to a U.K. investor from a U.S. investment is

$$(1 + R_{UK}) = (1 + R_X)(1 + R_{US})$$

So,

– Return to US investor

Period	From US investment	From UK investment
1	10%	$(0.833)(1.05) - 1 = -12.5\%$
2	15%	$(1)(0.95) - 1 = -5.0\%$
3	−5%	$(0.8)(1.15) - 1 = -8.0\%$
4	12%	$(0.75)(1.08) - 1 = -19.0\%$
5	6%	$(1.667)(1.1) - 1 = 83.3\%$
Average	7.6%	7.76%

– Return to UK investor

Period	From UK investment	From US investment
1	5%	$(1.2)(1.1) - 1 = 32.0\%$
2	−5%	$(1)(1.15) - 1 = 15.0\%$
3	15%	$(1.25)(0.95) - 1 = 18.75\%$
4	8%	$(1.333)(1.12) - 1 = 49.3\%$
5	10%	$(0.6)(1.06) - 1 = -36.4\%$
Average	6.6%	15.73%

(b) The standard deviation of return is given by

$$\sigma = \sqrt{\sum_{i=1}^{N} \frac{\left(R_i - \bar{R}_i\right)^2}{N}}$$

Thus,

- For US investor

$$\sigma_{US} = \sqrt{\frac{(10-7.6)^2 + (15-7.6)^2 + (-5-7.6)^2 + (12-7.6)^2 + (6-7.6)^2}{5}}$$
$$= 6.95\%$$

$$\sigma_{UK} = \sqrt{\frac{(-12.5-7.76)^2 + (-5-7.76)^2 + (-8-7.76)^2 + (-19-7.76)^2 + (83.3-7.76)^2}{5}}$$
$$= 38.06\%$$

- For UK investor

$$\sigma_{UK} = \sqrt{\frac{(5-6.6)^2 + (-5-6.6)^2 + (15-6.6)^2 + (8-6.6)^2 + (10-6.6)^2}{5}}$$
$$= 6.65\%$$

$$\sigma_{US} = \sqrt{\frac{(32-15.73)^2 + (15-15.73)^2 + (18.75-15.73)^2 + (49.3-15.73)^2 + (-36.4-15.73)^2}{5}}$$
$$= 38.06\%$$

Exercise 1.18. In general, we should hold non-domestic (N) securities instead of domestic securities(D) when foreign investment is more attractive than domestic investment. What happens when the following inequality holds

$$\frac{\bar{R}_N - R_F}{\sigma_N} > \frac{\bar{R}_D - R_F}{\sigma_D} \rho_{N,D}$$

Specifically, for an US investor

$$\frac{\bar{R}_N - R_F}{\sigma_N} > \frac{\bar{R}_{US} - R_F}{\sigma_{US}} \rho_{N,US}$$

$\bar{R}_{US}$ and $\bar{R}_N$, σ_N and $\sigma_{N,US}$ for the foreign countries are given in the problem and summarized below:

	$\bar{R}_N$ (%)	σ_N	$\sigma_{N,US}$
Austria	14	24.50	0.281
France	16	17.76	0.534
Japan	14	25.70	0.348
UK	15	15.59	0.646

We also know that $\bar{R}_{US} = 20\%$, $\sigma_{US} = 13.59$ and $R_F = 6\%$. Thus, we have

	$\frac{\bar{R}_N - R_F}{\sigma_N}$	$\frac{\bar{R}_{US} - R_F}{\sigma_{US}} \rho_{N,US}$
Austria	0.327	0.289
France	0.563	0.550
Japan	0.311	0.358
UK	0.577	0.665

For Austria and France, the above inequality holds, so a US investor should consider those foreign markets as attractive investments; for Japan and the UK, the above inequality does not hold, so a US investor should not consider those foreign markets as attractive investments.

Exercise 1.19. The formula to find the minimum-risk portfolio of two assets is get by taking the first derivative of the portfolio variance w.r.t. x_1 and equal 0, which gives

$$x_1^{MV} = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{1,2}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{1,2}}$$

where x_1 is the investment weight for asset 1 and $x_2 = 1 - x_1$.

(a) For equities, $\sigma_{US} = 13.59\%$, $\sigma_N = 16.70\%$ and $\rho_{N,US} = 0.423$. So the minimum-variance portfolio is:

$$x_{US}^{MV} = \frac{(19.0\%)^2 - 15.39\% \times 19.0\% \times 0.423}{15.39\%)^2 + (19.0\%)^2 - 2 \times 15.39\% \times 19.0\% \times 0.423} = 0.6771$$
$$x_N^{MV} = 1 - x_{US}^{MV} = 0.3266$$

(b) For bonds, $\sigma_{US} = 6.92\%$, $\sigma_N = 12.875\%$ and $\rho_{N,US} = 0.527$. So the minimum-variance portfolio is:

$$x_{US}^{MV} = \frac{(12.875\%)^2 - 6.92\% \times 12.875\% \times 0.527}{(6.92\%)^2 + (12.875\%)^2 - 2 \times 6.92\% \times 12.875\% \times 0.527} = 0.9924$$
$$x_N^{MV} = 1 - x_{US}^{MV} = 0.0076$$

(c) For T-bills, $\sigma_{US} = 1.068\%$, $\sigma_N = 10.057\%$ and $\rho_{N,US} = -0.220$. So the minimum-variance portfolio is:

$$x_{US}^{MV} = \frac{(10.057\%)^2 + 1.068\% \times 10.057\% \times 0.220}{(1.068\%)^2 + (10.057\%)^2 + 2 \times 1.068\% \times 10.057\% \times 0.220} = 0.9673$$
$$x_N^{MV} = 1 - x_{US}^{MV} = 0.0327$$

2 Portfolio Selection Models

2.1 Constant Correlation Model

Exercise 2.1.

(a)-(b) The only assumption of the Constant Correlation Model is that the correlation between any pair of securities is constant, such that $\rho_{ij} = \rho^* \; \forall i, j$. This is an unrealistic assumption that may lead to introduction of model risk. On the other hand, it allows us to decrease the number of parameters one needs to estimate to use MVT. So, the use of CCM may lead to a considerable reduction in estimation risk. It also allows us to use cut-off methods to find tangent portfolios.

Exercise 2.2.

(a) Yes, since all pairwise correlations are the same, this is the ideal scenario to use CCMs. In this case we have zero model risk.

(b) If short sales are allowed, all securities will be included in the optimal portfolio. Assuming constant correlation we can apply the cut-off method that consists in

1. Rank all securities accordingly to Sharpe's Ratio
2. Calculate the Cut-Off point
3. Compute Z and the weights X.

In Table **??** below, given that the riskless rate equals 4%, the securities are ranked in descending order by their excess return over standard deviation. To calculate the cut-off point C^* we need a general expression that give us C_i. This expression is

$$C_i = \frac{\rho}{1 - \rho + i\rho} \sum_{i=1}^{N} \frac{\bar{R}_i - R_F}{\sigma_i}$$

where ρ is the correlation coefficient - assumed constant for all securities. The subscript i indicates that C_i is calculated, using data on the first i securities. Each C_i is calculated as follows

$$C_1 = \frac{\rho}{1 - \rho + 1\rho} \sum_{i=1}^{1} \frac{\bar{R}_{10} - R_F}{\sigma_1 0} = \frac{0.5}{1 - 0.5 + 1 \times 0.5} \times \frac{12 - 4}{2} = 2$$

$$C_2 = \frac{\rho}{1 - \rho + 2\rho} \sum_{i=1}^{2} \frac{\bar{R}_3 - R_F}{\sigma_3} = \frac{0.5}{1 - 0.5 + 2 \times 0.5} \left(\frac{12 - 4}{2} + \frac{12 - 4}{4} \right) = 2$$

$$\vdots$$

Since short-sales are allowed, we include all securities, which implies that the cut-off rate is given by the C rate of the last security. In this exercise, $C^* = C^{10^{th}} = 1.41$.

The last step to find the optimal portfolio is to calculate Zs, which is given by

$$z_i = \frac{1}{(1 - \rho)\,\sigma_i} \left(\frac{\bar{R}_i - R_F}{\sigma_i} - C^* \right)$$

Security	Rank i	$\bar{R}_i - R_F$	$\frac{\bar{R}_i - R_F}{\sigma_i}$	$\sum_{i=1}^{N} \frac{\bar{R}_i - R_F}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	C	z_i	x_i
10	1	8	4.00	4.00	0.50	2.00	2.59	189.22%
3	2	8	2.00	6.00	0.33	2.00	0.30	21.68%
6	3	5	1.67	7.67	0.25	1.92	0.17	12.69%
9	4	6	1.50	9.17	0.20	1.83	0.05	3.44%
4	5	10	1.43	10.6	0.17	1.77	0.01	0.48%
1	6	6	1.20	11.8	0.14	1.69	-0.08	−6.00%
5	7	2	1.00	12.8	0.13	1.60	-0.41	−29.59%
7	8	1	1.00	13.8	0.11	1.53	-0.81	−59.17%
8	9	4	1.00	14.8	0.10	1.48	-0.20	−14.79%
2	10	4	0.67	15.47	0.09	1.41	-0.25	−17.97%

Table 1: Exercise **??**- Efficient Portfolio

Then,

$$z_1 = \frac{1}{(1-\rho)\,\sigma_{10}} \left(\frac{\bar{R}_{10} - R_F}{\sigma_{10}} - C^* \right) = \frac{1}{(1-0.5)\,2} \left(\frac{12-4}{2} - 1.41 \right) = 2.59$$

$$z_2 = \frac{1}{(1-\rho)\,\sigma_3} \left(\frac{\bar{R}_3 - R_F}{\sigma_3} - C^* \right) = \frac{1}{(1-0.5)\,4} \left(\frac{12-4}{4} - 1.41 \right) = 0.30$$

$$\vdots$$

Finally, to find the weights Xs and since $x_i = \frac{Z_i}{\sum_{i=1}^{N} z_i}$, we have

$$x_1 = \frac{z_1}{\sum_{i=1}^{10} z_i} = \frac{2.59}{1.37} = 1.8922$$

$$x_2 = \frac{z_2}{\sum_{i=1}^{10} z_i} = \frac{0.3}{1.37} = 0.2168$$

$$\vdots$$

Table **??** presents all previous calculations and the efficient portfolio, T^A.

(e) The efficient portfolio, T^A found in part (b) is the unique efficient portfolio we have with a risk-free rate of 4%, being the tangent portfolio between the capital market line and the efficient frontier of risky assets. Applying the formulas for portfolio's expected return and risk, we have $\bar{R}_T = 18.907\%$ and $\sigma_T = 3.297\%$. Now, we can draw the capital market line, which is the efficient frontier in this case (see Figure **??**)

Exercise 2.3.

(a) The efficient frontier is the line from R_F and is tangent to the efficient frontier of risky assets. It is similar to Figure **??**.

(b)

(i) In Table **??**, given that the riskless rate equals 5%, the securities are ranked in descending order by their excess return over standard deviation. To calculate the

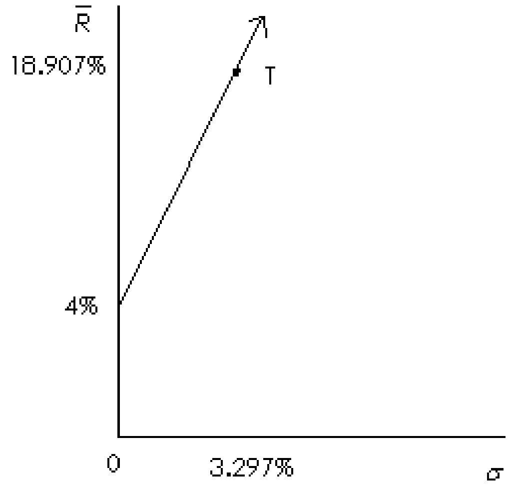


Figure 9: Exercise **??** - Efficient Frontier

cut-off point C^* we need a general expression that give us C_i. This expression is

$$C_i = \frac{\rho}{1-\rho+i\rho}\sum_{i=1}^{N}\frac{\bar{R}_i - R_F}{\sigma_i}$$

where ρ is the correlation coefficient - assumed constant for all securities. The subscript i indicates that C_i is calculated, using data on the first i securities. Each C_i is calculated as follows

$$C_1 = \frac{\rho}{1-\rho+1\rho}\sum_{i=1}^{1}\frac{\bar{R}_1 - R_F}{\sigma_1} = \frac{0.5}{1-0.5+1\times 0.5}\times\frac{15-5}{10} = 0.5$$

$$C_2 = \frac{\rho}{1-\rho+2\rho}\sum_{i=1}^{2}\frac{\bar{R}_2 - R_F}{\sigma_2} = \frac{0.5}{1-0.5+2\times 0.5}\left(\frac{15-5}{10}+\frac{20-5}{15}\right) = 0.6667$$

$$\vdots$$

With no short sales, we only include those securities for which $\frac{\bar{R}_i - R_F}{\sigma_i} > C_i$. Thus, only securities 1, 2, 5 and 6 (the four highest ranked securities in the above table) are in the optimal (tangent) portfolio. We could have stopped our calculations after the first time we found a ranked security for which $\frac{\bar{R}_i - R_F}{\sigma_i} < C_i$, (in this case the fifth highest ranked security, security 4), but we did not so that we could demonstrate that $\frac{\bar{R}_i - R_F}{\sigma_i} < C_i$ for all of the remaining lower ranked securities as well.
Since security 6 (the fourth highest ranked security, where i = 4) is the last ranked security in descending order for which $\frac{\bar{R}_i - R_F}{\sigma_i} > C_i$, we set $C^* = C_4 = 0.78$
The last step to find the optimal portfolio is to calculate Zs, which is given by

$$z_i = \frac{1}{(1-\rho)\,\sigma_i}\left(\frac{\bar{R}_i - R_F}{\sigma_i} - C^*\right)$$

Security	Rank i	$\bar{R}_i - R_F$	$\frac{\bar{R}_i - R_F}{\sigma_i}$	$\sum_{i=1}^{N} \frac{\bar{R}_i - R_F}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	C	z_i	x_i
1	1	10	1.00	1.00	0.5000	0.5000	0.0440	0.2375
2	2	15	1.00	2.00	0.3333	0.6667	0.0293	0.1581
5	3	5	1.00	3.00	0.2500	0.7500	0.0880	0.4749
6	4	9	0.90	3.90	0.2000	0.7800	0.0240	0.1295
4	5	7	0.70	4.60	0.1667	0.7668	-	-
3	6	13	0.65	5.25	0.1429	0.7502	-	-
7	7	11	0.55	5.80	0.1250	0.7250	-	-

Table 2: Exercise **??**(b)(i) – Efficient Portfolio (short-selling not allowed)

Then,

$$z_1 = \frac{1}{(1-\rho)\,\sigma_1}\left(\frac{\bar{R}_1 - R_F}{\sigma_1} - C^*\right) = \frac{1}{(1-0.5)\,10}\left(\frac{15-5}{10} - 0.78\right) = 0.0440$$

$$z_2 = \frac{1}{(1-\rho)\,\sigma_2}\left(\frac{\bar{R}_2 - R_F}{\sigma_2} - C^*\right) = \frac{1}{(1-0.5)\,15}\left(\frac{20-5}{15} - 0.78\right) = 0.0293$$

$$\vdots$$

Finally, to find the weights Xs and since $x_i = \frac{z_i}{\sum_{i=1}^{N} z_i}$, we have

$$x_1 = \frac{z_1}{\sum_{i=1}^{4} z_i} = \frac{0.0440}{0.1853} = 0.2375$$

$$x_2 = \frac{z_2}{\sum_{i=1}^{4} z_i} = \frac{0.0293}{0.1853} = 0.1581$$

$$\vdots$$

Table **??** presents all previous calculations and the efficient portfolio without short-selling. Since $i = 1$ for security 1, $i = 2$ for security 2, $i = 3$ for security 5 and $i = 4$ for security 6, the tangent portfolio when short sales are not allowed consists of 23.75% invested in security 1, 15.81% invested in security 2, 47.49% invested in security 5 and 12.95% invested in security 6.

(ii) When short-selling is allowed, we set the cut-off rate to $C^* = 0.725$ ir order to include all securities in our efficient portfolio (see Table **??**). The Zs and the weights Xs are calculated as before. Concretely we have

$$z_1 = \frac{1}{(1-\rho)\,\sigma_1}\left(\frac{\bar{R}_1 - R_F}{\sigma_1} - C^*\right) = \frac{1}{(1-0.5)\,10}\left(\frac{15-5}{10} - 0.725\right) = 0.0550$$

$$\vdots$$

$$z_5 = \frac{1}{(1-\rho)\,\sigma_2}\left(\frac{\bar{R}_2 - R_F}{\sigma_2} - C^*\right) = \frac{1}{(1-0.5)\,10}\left(\frac{12-5}{10} - 0.725\right) = -0.0050$$

$$\vdots$$

And we can determine all weights $x_i = \frac{z_i}{\sum_{i=1}^{N} z_i}$. For this concrete case we have $\sum_{i=1}^{N} z_i = 0.2061$. See Table **??** for concrete weight values.

Security	Rank i	$\bar{R}_i - R_F$	$\frac{\bar{R}_i - R_F}{\sigma_i}$	$\sum_{i=1}^{N} \frac{\bar{R}_i - R_F}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	C	z_i	x_i
1	1	10	1.00	1.00	0.5000	0.5000	0.0550	0.2661
2	2	15	1.00	2.00	0.3333	0.6667	0.0367	0.1776
5	3	5	1.00	3.00	0.2500	0.7500	0.1100	0.5322
6	4	9	0.90	3.90	0.2000	0.7800	0.0350	0.1703
4	5	7	0.70	4.60	0.1667	0.7668	-0.0050	-0.0242
3	6	13	0.65	5.25	0.1429	0.7502	-0.0075	-0.0363
7	7	11	0.55	5.80	0.1250	0.7250	-0.0175	-0.0847

Table 3: Exercise **??**(b)(ii) - Efficient Portfolio (short-selling allowed - Standard Definition)

(iii) When shotselling is allowed, but limited *a la* Linter, the Zs are the same as above (compare z_i in Tables **??** and **??**, however the weights are determined as $x_i = \frac{z_i}{\sum_{i=1}^{N} |z_i|}$,

$$x_1 = \frac{z_1}{\sum_{i=1}^{7} |z_i|} = \frac{0.05500}{0.2667} = 0.2062$$

$$\vdots$$

$$x_5 = \frac{z_2}{\sum_{i=1}^{7} |z_i|} = \frac{-0.0050}{0.2667} = -0.0187$$

$$\vdots$$

Table **??** presents all previous calculations and the efficient portfolio with short-selling (Lintner definition).

Security	Rank i	$\bar{R}_i - R_F$	$\frac{\bar{R}_i - R_F}{\sigma_i}$	$\sum_{i=1}^{N} \frac{\bar{R}_i - R_F}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	C	z_i	x_i
1	1	10	1.00	1.00	0.5000	0.5000	0.0550	0.2062
2	2	15	1.00	2.00	0.3333	0.6667	0.0367	0.1376
5	3	5	1.00	3.00	0.2500	0.7500	0.1100	0.4124
6	4	9	0.90	3.90	0.2000	0.7800	0.0350	0.1312
4	5	7	0.70	4.60	0.1667	0.7668	-0.0050	-0.0187
3	6	13	0.65	5.25	0.1429	0.7502	-0.0075	-0.0281
7	7	11	0.55	5.80	0.1250	0.7250	-0.0175	-0.0656

Table 4: Exercise **??**(b)(iii) - Efficient Portfolio (short-selling allowed - Lintner Definition)

(c) If the risk-free asset does not exist, their are an infinite number of efficient portfolios of risky assets. Determine all these portfolios imply the calculation of the efficient frontier, which can be done using pretty sophisticated matricial equations, which are outside the scope of this course. Nevertheless, we have a different and easier way to do this calculation. We just need to assume the existence of a fictitious risk-free rate of return to find an efficient portfolio. Then we assume a second fictitious frontier to have a second efficient portfolio. Now, with these two portfolios we can find any other portfolio applying the Efficient Portfolios Theorem and we can, also, derive the representative equation of the hyperbole that corresponds to the efficient frontier.

2.2 Single-Index Model

Exercise 2.4.

(a) The β of Security A is lower than 1 therefore it is considered a defensive stock. On the other side, security B has a β higher than 1, so that it is an aggressive stock.

(b) (i) To compute the portfolio's β we proceed as follows

$$\beta_p = \sum x_i\beta_i \Leftrightarrow \beta_p = x_A\beta_A + x_B\beta_B = 0.25 \times 0.75 + 0.75 \times 2 = 1.6875$$

(ii) Using the single-index model (SIM), the portfolio's risk is

$$\sigma_p^2 = \beta_p^2\sigma_m^2 + \sum x_i^2\sigma_{ei}^2 = 1.6875^2 \times 0.25^2 + \left[(0.25)^2 \times 0.02 + (0.75)^2 \times 0.03\right]$$

$$\sigma_p^2 = 0.177978 + 0.018125 = 0.1961 \qquad \sigma_p = \sqrt{0.1961} = 44.28\%.$$

(c) A portfolio with A and B, which risk equals the market risk is a portfolio, which risk equals the market risk, thus $\sigma_p^2 = \beta_p^2\sigma_m^2 = 0.25^2 = 0.0625$. To calculate the weight of stock A (X_A)we need to solve the portfolio variance equation in order to X_A. To do so we us first need to compute the return's variance for stock A and B and the covariance between this returns using the single-index model:

$$\sigma_A^2 = \beta_A^2\sigma_m^2 + \sigma_{eA}^2 = 0.75^2 \times 0.25^2 + 0.02 = 0.0552$$

$$\sigma_B^2 = \beta_B^2\sigma_m^2 + \sigma_{eB}^2 = 2^2 \times 0.25^2 + 0.03 = 0.28$$

$$\sigma_{AB} = \beta_A \times \beta_B \times \sigma_M^2 = 0.75 \times 2 \times 0.25^2 = 0.09375$$

Then,

$$\begin{aligned}
\sigma_p^2 &= x_A^2\sigma_A^2 + (1-x_A)^2\sigma_B^2 + 2x_A(1-x_A) \times \sigma_{AB}^2 \\
0.25^2 &= 0.0552x_A^2 + 0.28(1-x_A)^2 + 2 \times 0.75 \times 2 \times 0.25^2 x_A(1-x_A) \\
0.0625 - 0.028 &= (0.0552 + 0.28 - 2 \times 0.09375)x_A^2 + 2 \times (0.09375 - 0.28)x_A \\
0 &= 0.1477x_A^2 - 0.3725x_A + 0.2175 \\
x_A &= \frac{0.3725 \pm \sqrt{0.3724^2 - 4 \times 0.1477 \times 0.2175}}{2 \times 0.1477} \Leftrightarrow x_A = 160.39\% \vee x_A = 91.85\%
\end{aligned}$$

There are two possible solutions to x_A, nevertheless just one makes sense, since just one is efficient. Such solution is $x_A = 91.85\%$. The β of this portfolio is $\beta_P = x_A\beta_A + (1 - x_A)\beta_B = 0.9185 \times 0.75 + 0.0815 \times 2 \approx 0.85$.

(d) In part (c) we calculated the stocks variance using SIM. When we compare these results with the new data we realize that $\sigma_A^{2SIM} = 0.0552 \neq 0.1$ and $\sigma_B^{2SIM} = 0.28 \approx 0.3$. Thus, the SIM does not seems to hold when we use it with stock A, despite it seems to be a good approximation when applied to stock B.

Exercise 2.5.

(a) The covariance between stock B and the market portfolio is $\sigma_{BM} = \beta_B\beta_M\sigma_M^2 = 1.125 \times 1 \times 0.4^2 = 0.18$

(b) If the Single Index Model (SIM) holds, the portfolio variance is as follows

$$\sigma_p^2 = \underbrace{\beta_p^2\sigma_m^2}_{\text{systematic variance}} + \underbrace{\sum_{i=1}^{n} x_i^2\sigma_{\varepsilon i}^2}_{\text{residual variance}}$$

Thus, the residual variance in this homogenous portfolio (in a homogenous portfolio each security weight is given by $1/N$, where N is the number of securities, in this case $x_i = 1/2 = 0.5$) is

$$\sigma_{eH}^2 = \sum_{i=1}^{n} x_i^2\sigma_{\varepsilon i}^2 = 0.5^2 \times 0.1 + 0.5^2 \times 0.15 = 0.0625$$

(c) Since the covariance between the residual variances of security A and B are not zero, the single-index model does not apply. Therefore, the residual variance calculated in part (b) is not the effective residual variance of a homogeneous portfolio, which is given by the modern portfolio's theory. Thus, for two securities, the variance is

$$\begin{aligned}\sigma_{e_H}^2 &= x_A^2\sigma_{e_B}^2 + x_B^2\sigma_{e_B}^2 + 2x_Ax_B\sigma_{e_Ae_B}\\ &= 0.5^2 \times 0.1 + 0.5^2 \times 0.15 + 2 \times 0.5 \times 0.5 \times 0.1\\ &= 0.1125\end{aligned}$$

(d) As seen in part (b), the systematic risk, under SIM, is $\sigma_{e_{Syst}}^2 = \beta_p^2\sigma_m^2$, so $\beta_H = \sum_{i=1}^{2} x_i\beta_i = 0.5 \times 0.875 + 0.5 \times 1.125 = 1$. Thus,

$$\sigma_{\text{syst}_H}^2 = \beta_H^2\sigma_m^2 = 1^2 \times 0.4^2 = 0.16$$

(e) (i) Total risk for each individual security calculated with SIM or with Portfolio Theory is the same as long as SIM's assumptions hold, namely that $\sigma_{e_iM} = 0$. In this case nothing is said about this, therefore anything definitive can be said.

(ii) In the general case, total risk for a portfolio computed under SIM or Markowitz assumptions is the same, as long as SIM's assumptions hold, namely that $\sigma_{e_ie_j} = 0$. However, this is not the case when we use securities A and B to construct a portfolio, since $\sigma_{e_Ae_B} = 0.1$. Actually, under Markowitz total variance is $\sigma_p^2 = \beta_p^2\sigma_m^2 + x_A^2\sigma_{e_B}^2 + x_B^2\sigma_{e_B}^2 + 2x_Ax_B\sigma_{e_Ae_B} = 0.16 + 0.1125 = 0.2725$ and under SIM total variance is $\sigma_p^2 = \beta_p^2\sigma_m^2 + \sum_{i=1}^{n} x_i^2\sigma_{\varepsilon i}^2 = 0.16 + 0.0625 = 0.2225$. Thus, their total risk is also different.

Exercise 2.6.

(a) This exercise is based on the single-index model, more precisely in the market model, which a positive correlation between any given stock returns and the market returns, such that the return on a stock can be written as

$$R_i = a_i + \beta_iR_m$$

The term a_i represents that component of return insensitive to the return on the market, i.e. it represents specific risk. The term a_i can be broken into two components: $alpha_i$ that denotes the expected value for a_i; and ε_i representing the random element of a_i, which expected value is zero ($\mathbb{E}\varepsilon_i = 0$). Then $a_i = \alpha_i + \varepsilon_i$ and

$$R_i = \alpha_i + \beta_iR_m + \varepsilon_i$$

Note that both ε_i and R_m are random variables with standard deviations denoted by σ_{ε_i} and σ_m, respectively. The term $\beta_i R_m$ represent the systematic risk and measure how sensitivity the stock's return is to the market's return.

The model's main assumptions are:

- ε_i is uncorrelated with R_m, such that the model ability to explain stock returns is independent of what the return on the market happens to be. More formally $cov(\varepsilon_i R_m) = \mathbb{E}\left[(\varepsilon_i - 0)\left(R_m - \bar{R}_m\right)\right] = 0$
- ve_i is independent of e_j for all values of i and j. which implies that the only reason stocks vary together, systematically, is because of a common co-movement with the market. More formally $\mathbb{E}(\varepsilon_i \varepsilon_j) = 0$

(b) (i) The expected return is given by $\bar{R}_i = a_i + \beta_i \bar{R}_m$. Thus,

$$\begin{cases} \bar{R}_A = a_A + \beta_A \bar{R}_m \\ \bar{R}_B = a_B + \beta_B \bar{R}_m \\ \bar{R}_C = a_C + \beta_C \bar{R}_m \\ \bar{R}_D = a_D + \beta_D \bar{R}_m \end{cases} \Leftrightarrow \begin{cases} \bar{R}_A = 2\% + 1.5 \times 8\% \\ \bar{R}_B = 3\% + 1.3 \times 8\% \\ \bar{R}_C = 1\% + 0.8 \times 8\% \\ \bar{R}_D = 4\% + 0.9 \times 8\% \end{cases} \Leftrightarrow \begin{cases} \bar{R}_A = 14\% \\ \bar{R}_B = 13.4\% \\ \bar{R}_C = 7.4\% \\ \bar{R}_D = 11.2\% \end{cases}$$

(ii) The security variance is given by $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2$. Therefore,

$$\begin{cases} \sigma_A^2 = \beta_A^2 \sigma_m^2 + \sigma_{\varepsilon_A}^2 \\ \sigma_B^2 = \beta_B^2 \sigma_m^2 + \sigma_{\varepsilon_B}^2 \\ \sigma_C^2 = \beta_C^2 \sigma_m^2 + \sigma_{\varepsilon_C}^2 \\ \sigma_D^2 = \beta_D^2 \sigma_m^2 + \sigma_{\varepsilon_D}^2 \end{cases} \Leftrightarrow \begin{cases} \sigma_A^2 = 1.5^2 \times 0.0025 + 0.0009 \\ \sigma_B^2 = 1.3^2 \times 0.0025 + 0.0001 \\ \sigma_C^2 = 0.8^2 \times 0.0025 + 0.0004 \\ \sigma_D^2 = 0.9^2 \times 0.0025 + 0.0016 \end{cases} \Leftrightarrow \begin{cases} \sigma_A^2 = 0.006525 \\ \sigma_B^2 = 0.004325 \\ \sigma_C^2 = 0.0020 \\ \sigma_D^2 = 0.003625 \end{cases}$$

(iii) The covariance is given by $\sigma_{ij} = \beta_i \beta_j \sigma_m^2$. Therefore,

$$\begin{cases} \sigma_{AB} = \beta_A \beta_B \sigma_m^2 \\ \sigma_{AC} = \beta_A \beta_C \sigma_m^2 \\ \sigma_{AD} = \beta_A \beta_D \sigma_m^2 \\ \sigma_{BC} = \beta_B \beta_C \sigma_m^2 \\ \sigma_{BD} = \beta_B \beta_D \sigma_m^2 \\ \sigma_{CD} = \beta_C \beta_D \sigma_m^2 \end{cases} \Leftrightarrow \begin{cases} \sigma_{AB} = 1.5 \times 1.3 \times 0.0025 \\ \sigma_{AC} = 1.5 \times 0.8 \times 0.0025 \\ \sigma_{AD} = 1.5 \times 0.9 \times 0.0025 \\ \sigma_{BC} = 1.3 \times 0.8 \times 0.0025 \\ \sigma_{BD} = 1.3 \times 0.9 \times 0.0025 \\ \sigma_{CD} = 0.8 \times 0.9 \times 0.0025 \end{cases} \Leftrightarrow \begin{cases} \sigma_{AB} = 0.004875 \\ \sigma_{AC} = 0.0030 \\ \sigma_{AD} = 0.003375 \\ \sigma_{BC} = 0.0026 \\ \sigma_{BD} = 0.002925 \\ \sigma_{CD} = 0.0018 \end{cases}$$

The covariance matrix Σ is

$$\begin{pmatrix} 0.006525 & 0.004875 & 0.0030 & 0.003375 \\ 0.004875 & 0.004325 & 0.0026 & 0.002925 \\ 0.0030 & 0.0026 & 0.0020 & 0.0018 \\ 0.003375 & 0.002925 & 0.003625 & 0.0016 \end{pmatrix}$$

(c) A homogenous portfolio is a portfolio where each security weight is given by $1/n$, where n denotes the number of security. Now, $n = 4$, thus each security weight is $1/4 = 0.25$.

(i) The portfolio's β is the weighted average β of all , i.e $\beta_p = \sum_{i=1}^{N} x_i \beta_i$,

$$\beta_H = 1.5 \times 0.25 + 1.3 \times 0.25 + 0.8 \times 0.25 + 0.9 \times 0.25 = 1.125$$

(ii) Like β_P, α_p is given by the weighted average α of all securities, $\alpha_p = \sum_{i=1}^{N} x_i \alpha_i$

$$\alpha_H = 2\% \times 0.25 + 3\% \times 0.25 + 1\% \times 0.25 + 4\% \times 0.25 = 2.5\%$$

(iii) The portfolio's variance is $\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} x_i^2 \sigma_{\varepsilon_i}^2$. Thus,

$$\begin{aligned}\sigma_H^2 &= \beta_H^2 \sigma_m^2 + \left(\frac{1}{4}\right)^2 \sum_{i=1}^{4} x_i^2 \sigma_{\varepsilon_i}^2 \\ &= 1.125^2 \times 25 + 0.25^2 (0.0009 + 0.0001 + 0.0004 + 0.0016) \\ &= 0.003352\end{aligned}$$

(iv) To find the portfolio's expected return we apply the market model using the portfolio's α and β. Therefore,

$$\bar{R}_P = \alpha_P + \beta_P \bar{R}_m = 2.5\% + 1.125 \times 8\% = 11.5\%$$

(d) Using the suggested adjustment to find the β of the following period, we have

$$\beta_{2A} = 0.343 + 0.677\beta_{1A} = 0.343 + 0.677 \times 1.5 = 1.3585$$

$$\beta_{2B} = 0.343 + 0.677\beta_{1B} = 0.343 + 0.677 \times 1.3 = 1.2231$$

$$\beta_{2C} = 0.343 + 0.677\beta_{1C} = 0.343 + 0.677 \times 0.8 = 0.8846$$

$$\beta_{2D} = 0.343 + 0.677\beta_{1D} = 0.343 + 0.677 \times 0.9 = 0.9523$$

(e) Applying the Vasiček technique with the provided data and knowing the Vasiček β is given by

$$\beta_{2i} = \frac{\sigma_{\beta_{1i}}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1i}}^2} \bar{\beta}_1 + \frac{\sigma_{\bar{\beta}_1}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1i}}^2} \beta_{1i}$$

we have

$$\beta_{2A} = \frac{\sigma_{\beta_{1A}}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1A}}^2} \bar{\beta}_1 + \frac{\sigma_{\bar{\beta}_1}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1A}}^2} \beta_{1A} = \frac{0.0441}{0.0441 + 0.00625} \cdot 1 + \frac{0.0625}{0.0441 + 0.0625} \cdot 1.5 = 1.2932$$

$$\beta_{2B} = \frac{\sigma_{\beta_{1B}}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1B}}^2} \bar{\beta}_1 + \frac{\sigma_{\bar{\beta}_1}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1B}}^2} \beta_{1B} = \frac{0.1024}{0.1024 + 0.0625} \cdot 1 + \frac{0.0625}{0.1024 + 0.0625} \cdot 1.3 = 1.1137$$

$$\beta_{2C} = \frac{\sigma_{\beta_{1C}}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1C}}^2} \bar{\beta}_1 + \frac{\sigma_{\bar{\beta}_1}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1C}}^2} \beta_{1C} = \frac{0.0324}{0.0324 + 0.0625} \cdot 1 + \frac{0.0625}{0.0324 + 0.0625} \cdot 0.8 = 0.8683$$

$$\beta_{2D} = \frac{\sigma_{\beta_{1D}}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1D}}^2} \bar{\beta}_1 + \frac{\sigma_{\bar{\beta}_1}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1D}}^2} \beta_{1D} = \frac{0.04}{0.04 + 0.0625} \cdot 1 + \frac{0.0625}{0.04 + 0.0625} \cdot 0.9 = 0.9390$$

Exercise 2.7.

(a) The covariance between any two securities can be written as

$$\sigma_{ij} = \mathbb{E}\left[\left(R_i - \bar{R}_i\right)\left(R_j - \bar{R}_j\right)\right]$$

Substituting for R_i, $\bar{R}_i$, R_j and $\bar{R}_j$ yields

$$\sigma_{ij} = \mathbb{E}\left\{\left[(\alpha_i + \beta_i R_m + \varepsilon_i) - \left(\alpha_i + \beta_i \bar{R}_m + \varepsilon_i\right)\right]\left[(\alpha_j + \beta_j R_m + \varepsilon_j) - \left(\alpha_j + \beta_j \bar{R}_m + \varepsilon_j\right)\right]\right\}$$

Simplifying by canceling the α's and combining the terms involving β's yields

$$\sigma_{ij} = \mathbb{E}\left\{\left[\beta_i\left(R_m - \bar{R}_m\right) + \varepsilon_i\right]\left[\beta_j\left(R_m - \bar{R}_m\right) + \varepsilon_j\right]\right\}$$

Carrying out the multiplication

$$\sigma_{ij} = \beta_i\beta_j\mathbb{E}\left(R_m - \bar{R}_m\right)^2 + \beta_j\mathbb{E}\left[\varepsilon_i\left(R_m - \bar{R}_m\right)\right] + \beta_i\mathbb{E}\left[\varepsilon_j\left(R_m - \bar{R}_m\right)\right] + \mathbb{E}\left(\varepsilon_i\varepsilon_j\right)$$

From the single-index model assumptions we know

$$\mathbb{E}\left(R_m - \bar{R}_m\right)^2 = \sigma_m^2$$

$$\mathbb{E}\left[\varepsilon_i\left(R_m - \bar{R}_m\right)\right] = 0$$

$$\mathbb{E}\left[\varepsilon_j\left(R_m - \bar{R}_m\right)\right] = 0$$

And from the data in the problem

$$k = cov\left(\varepsilon_i\varepsilon_j\right) = \mathbb{E}\left[\varepsilon_i\varepsilon_j\right] - \underbrace{\mathbb{E}\left[\varepsilon_i\right]}_{0}\underbrace{\mathbb{E}\left[\varepsilon_j\right]}_{0} = \mathbb{E}\left[\varepsilon_i\varepsilon_j\right]$$

Thus,

$$\sigma_{ij} = \beta_i\beta_j\sigma_m^2 + k$$

(b) The general equation for the portfolio variance is

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} X_iX_j\sigma_{ij} \tag{2}$$

From the Single-Index Model we know that

$$\sigma_i^2 = \beta_i^2\sigma_m^2 + \sigma_{\varepsilon_i}^2 \tag{3}$$

and

$$\sigma_{ij} = \beta_i\beta_j\sigma_m^2$$

However, in this case, the covariance among the returns residuals is K and, therefore,

$$\sigma_{ij} = \beta_i\beta_j\sigma_m^2 + k \tag{4}$$

as calculates in part b. Applying (**??**) and (**??**) in (**??**) we get

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2\left(\beta_i^2\sigma_m^2 + \sigma_{\varepsilon_i}^2\right) + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} X_iX_j\left(\beta_i\beta_j\sigma_m^2 + k\right)$$

Doing some transformations we finally have

$$\begin{aligned}
\sigma_p^2 &= \sum_{i=1}^{N} X_i^2 \left(\beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2\right) + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j \left(\beta_i \beta_j \sigma_m^2 + k\right) \\
&= \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\varepsilon_i}^2 + + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j k \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\varepsilon_i}^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j k \\
&= \underbrace{\left(\sum_{i=1}^{N} X_i \beta_i\right)}_{\beta_P} \underbrace{\left(\sum_{i=1}^{N} X_i \beta_i\right)}_{\beta_P} \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\varepsilon_i}^2 + k \left(\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j\right) \\
&= \beta_P^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\varepsilon_i}^2 + k \left(\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j\right)
\end{aligned}$$

Exercise 2.8.

(a) This is a standard portfolio selection exercise, in which we have to choose the tangent portfolio between the capital market line and the efficient frontier of risky assets. The solution for this problem involves solving the following system of simultaneous equations in order to Z_i, $\forall i \geqslant 0$

$$\begin{cases}
\bar{R}_1 - R_F = Z_1 \sigma_1^2 + Z_2 \sigma_{12} + Z_3 \sigma_{13} + \cdots + Z_N \sigma_{1N} \\
\bar{R}_2 - R_F = Z_1 \sigma_{21} + Z_2 \sigma_2^2 + Z_3 \sigma_{23} + \cdots + Z_N \sigma_{2N} \\
\bar{R}_3 - R_F = Z_1 \sigma_{31} + Z_2 \sigma_{32} + Z_3 \sigma_3^2 + \cdots + Z_N \sigma_{3N} \\
\vdots \\
\bar{R}_N - R_F = Z_1 \sigma_{NN}^2 + Z_2 \sigma_{N2} + Z_3 \sigma_{N3} + \cdots + Z_N \sigma_N^2
\end{cases}$$

which can be written using matricial notation

$$Z = V^{-1} \left(R - R_F 1\right)$$

where Σ^{-1} is the inverse covariance matrix, R is a column vector with the securities returns, R_F is a scalar and 1 is a column vector of 1s. The Zs are proportional to the optimum amount to invest in each security. Then the optimum proportions to invest in stock k is X_k, where

$$X_k = \frac{Z_k}{\sum\limits_{i=1}^{N} Z_i}$$

Thus, we need to calculate the covariance matrix and then invert it. To find each pair of covariances we can use the variance and covariance definitions used in the Single-Index Model $\left(\sigma_i^2 = \beta^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2 \text{ and } \sigma_{ij} = \beta_i \beta_j \sigma_m^2\right)$. Thus, for security 1 and for the pair 1, 2 it comes

$$\sigma_i^2 = \beta^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2 = 1^2 \times 10 + 30 = 40$$

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2 = 1 \times 1.5 \times 10 = 15$$

Proceeding similarly for the other securities we arrive to the covariance matrix

$$V = \begin{pmatrix} 0.004 & 0.0015 & 0.002 & 0.0008 & 0.001 & 0.0015 \\ 0.0015 & 0.00425 & 0.003 & 0.0012 & 0.0015 & 0.00225 \\ 0.002 & 0.003 & 0.008 & 0.0016 & 0.002 & 0.003 \\ 0.0008 & 0.0012 & 0.0016 & 0.00164 & 0.0008 & 0.0012 \\ 0.001 & 0.0015 & 0.002 & 0.0008 & 0.003 & 0.0015 \\ 0.0015 & 0.00225 & 0.003 & 0.0012 & 0.0015 & 0.00325 \end{pmatrix}$$

Then the inverse matrix is

$$V^{-1} = \begin{pmatrix} 317.109 & -36.505 & -24.337 & -38.939 & -24.337 & -73.010 \\ -36.505 & 417.863 & -54.758 & -87.613 & -54.758 & -164.274 \\ -24.337 & -54.758 & 213.495 & -58.408 & -36.505 & -109.516 \\ -38.939 & -87.613 & -58.408 & 906.547 & -58.408 & -175.225 \\ -24.337 & -54.758 & -36.505 & -58.408 & 463.495 & -109.516 \\ -73.010 & -164.274 & -109.516 & -175.225 & -109.516 & 671.453 \end{pmatrix}$$

And $\bar{R} - R_F 1$ is

$$\bar{R} - R_F 1 = \begin{pmatrix} 15\% \\ 12\% \\ 11\% \\ 8\% \\ 9\% \\ 14\% \end{pmatrix} - 5\% \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.07 \\ 0.06 \\ 0.03 \\ 0.04 \\ 0.09 \end{pmatrix}$$

Finally,

$$Z = V^{-1}(R - R_F 1) = \begin{pmatrix} 18.983 \\ 2.711 \\ -6.526 \\ -4.410 \\ -1.526 \\ 25.422 \end{pmatrix}$$

Since short sales are not allowed, we need to use the cut-off method to know how many securities will show up in the tangent portfolio.
We known we will not invest in any security with a negative "Z" and we may even not invest in some of the securities with positive Z.
In this case it turns out the optimal portfolio will have 3 securities, i.e., securities 1, 2 and 6 (which are the first three in the ranking and for which we have $z_i > 0$). Thus, summing over the Zs from these three securities we get $\sum_{i=1}^{3} z_i = 47.116$ and the weights to invest in each security are

$$z_1 = \frac{18.983}{47.116} = 0.4029 \ , \ z_2 = \frac{2.711}{47.116} = 0.0575 \ , \ z_6 = \frac{25.422}{47.116} = 0.5396$$

(b) If short sales are allowed, using the standard definition, $\sum_{i=1}^{6} Z_i = 34.623$ and the weights to invest in each security are

$$x_1 = \frac{18.983}{34.623} = 0.5483 \qquad x_2 = \frac{2.711}{34.623} = 0.0783$$
$$x_3 = -\frac{6.526}{34.623} = -0.1885 \qquad x_4 = -\frac{4.41}{34.623} = -0.1283$$
$$x_5 = -\frac{1.526}{34.623} = -0.0441 \qquad x_6 = \frac{25.422}{34.623} = 0.7343$$

Using Lintner definition, $\sum_{i=1}^{6} |z_i| = 59.609$ and the weights to invest in each security are

$$x_1 = \frac{18.983}{59.609} = 0.3185 \qquad x_2 = \frac{2.711}{59.609} = 0.0485$$
$$x_3 = -\frac{6.526}{59.609} = -0.1095 \qquad x_4 = -\frac{4.41}{59.609} = -0.0745$$
$$x_5 = -\frac{1.526}{59.609} = -0.0256 \qquad x_6 = \frac{25.422}{59.609} = 0.4265 \; .$$

Summing up all weights we can conclude only 58.08% is imvested in risky assets, which means the remaining 41.92% is invested in the riskless asset.

(c) If the risk-free asset does not exist, their are an infinite number of efficient portfolios of risky assets. Determine all these portfolios imply the calculation of the efficient frontier, which can be done using pretty sophisticated matricial equations, which are outside the scope of this course. Nevertheless, we have a different and easier way to do this calculation. We just need to assume the existence of a fictitious risk-free rate of return to find an efficient portfolio. Then we assume a second fictitious frontier to have a second efficient portfolio. Now, with these two portfolios we can find any other portfolio applying the Efficient Portfolios Theorem and we can, also, derive the representative equation of the hyperbole that corresponds to the efficient frontier.

Exercise 2.9. We know β_i can be written as σ_{im}/σ_m^2. We also know that $\sigma_{im} = \rho_{im}\sigma_i\sigma_m$. Then,

$$\beta_i = \frac{\rho_{im}\sigma_i\sigma_m}{\sigma_m^2} = \frac{\rho_{im}\sigma_i}{\sigma_m} \tag{5}$$

Since we have constant correlation ρ^* between each pair of securities we should be to express ρ_{im} as a function of ρ^*. If the Single-Index Model holds, then $\sigma_{ij} = \beta_i\beta_j\sigma_m^2$ that can be rewritten as follows

$$\sigma_{ij} = \beta_i\beta_j\sigma_j^2 = \frac{\rho_{im}\sigma_i\sigma_m}{\sigma_m^2} \times \frac{\rho_{jm}\sigma_j\sigma_m}{\sigma_m^2} \times \sigma_m^2 = \rho_{im}\rho_{jm}\sigma_i\sigma_j$$

From statistics we have $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$. If we let correlations to be constant, then $\sigma_{ij} = \rho^*\sigma_i\sigma_j$. If correlations are constant and the Single-Index Model holds, we have

$$\rho^*\sigma_i\sigma_j = \rho_{im}\rho_{jm}\sigma_i\sigma_j$$
$$\rho^* = \rho_{im}\rho_{jm}$$

As the correlation is constant between each pair of securities we must have $\rho_{im} = \rho_{jm}$. Then,

$$\rho^* = \rho_{im}\rho_{im} = \rho_{im}^2$$

and,

$$\rho_{im} = \sqrt{|\rho^*|} \tag{6}$$

Finally, using (**??**) into (**??**), we have

$$\beta_i = \frac{\sqrt{|\rho_*|}}{\sigma_m}\sigma_i$$

So, if correlations are constant and equal to ρ^*, then, under the Single-Index Model assumptions, each security β is a proportion of its volatility. This proportion is constant and equal to all securities and defined as $\frac{\sqrt{|\rho_*|}}{\sigma_m}$.

Exercise 2.10. Accordingly to the Single-Index Model, the expected return and risk are given by

$$\bar{R}_i = \alpha_i + \beta_i \bar{R}_m$$

$$\sigma_i^2 = \underbrace{\beta_A^2 \sigma_M^2}_{\text{Systematic Variance}} + \underbrace{\sigma_{\varepsilon_A}^2}_{\text{Specific Variance}}$$

$$\sigma_i = \sqrt{\beta_A^2 \sigma_M^2 + \sigma_{\varepsilon_A}^2}$$

Therefore, the table can be filled using the equations. Notice that to calculate systematic risk we assume specific risk to be zero. On the other hand, when we calculate the specific risk we assume that systematic risk is zero.

Invest	Expected Return	Systematic Risk	Specific Risk	Total Risk
A	$\alpha_A + \beta_A \bar{R}_m$	$\sqrt{\beta_A^2 \sigma_M^2}$	$\sqrt{\sigma_{\varepsilon_A}^2}$	$\sqrt{\beta_A^2 \sigma_m^2 + \sigma_{\varepsilon_A}^2}$
B	$\alpha_B + \beta_B \bar{R}_m$	$\sqrt{\beta_B^2 \sigma_M^2}$	$\sqrt{\sigma_{\varepsilon_B}^2}$	$\sqrt{\beta_B^2 \sigma_m^2 + \sigma_{\varepsilon_B}^2}$
C	$\alpha_C + \beta_C \bar{R}_m$	$\sqrt{\beta_C^2 \sigma_M^2}$	$\sqrt{\sigma_{\varepsilon_C}^2}$	$\sqrt{\beta_C^2 \sigma_m^2 + \sigma_{\varepsilon_C}^2}$
Port K	$\alpha_K + \beta_K \bar{R}_m$	$\sqrt{\beta_K^2 \sigma_M^2}$	σ_{ε_K}	$\sqrt{\beta_K^2 \sigma_m^2 + \sigma_{\varepsilon_K}^2}$

which give us

Invest	Expected Return	Systematic Risk	Specific Risk	Total Risk
A	$2\% + 1.5 \times 20\% = 32\%$	$\sqrt{1.5^2 \times (10\%)} = 15\%$	4%	$\sqrt{15\%^2 + 4\%^2} = 15.52\%$
B	$4\% + 0.8 \times 20\% 20\%$	$\sqrt{0.8^2 \times (10\%)^2} = 8\%$	3%	$\sqrt{8\%^2 + 3\%^2} = 8.54\%$
C	$6\% + 0.4 \times 20\% = 14\%$	$\sqrt{0.4^2 \times (10\%)^2} = 4\%$	2%	$\sqrt{4\%^2 + 2\%^2} = 4.47\%$
Port K	$4.6 + 0.74 \times 20 = 19.4$	$\sqrt{0.74^2 \times (10\%)^2} = 7.4\%$	1.56%	$\sqrt{7.4\%^2 + 1.56\%^2} = 7.56\%$

For the portfolio K, α_K, β_K and σ_{ε_K} are as follows

$$\alpha_K = \sum_{i=1}^{3} x_i \alpha_i = 2\% \times 0.2 + 4\% \, times 0.3 + 6\% \times 0.5 = 4.6\%$$

$$\beta_K = \sum_{i=1}^{3} x_i \beta_i = 1.5 \times 0.2 + 0.8 \times 0.3 + 0 - 4 \times 0.5 = 0.74$$

$$\sigma_{\varepsilon_K} = \sqrt{\sum_{i=1}^{3} x_i^2 \sigma_{\varepsilon_i}^2} = \sqrt{4\%^2 \times 0.2^2 + 3\%^2 \times 0.3^2 + 2\%^2 \times 0.5^2} = 1.56\%$$

2.3 Multi-Index Model

Exercise 2.11. Let us start with multi-index model with 3 correlated indexes I_1^*, I_2^* and I_3^*:

$$R_i = a_i^* + b_{i1}^* \times I_1^* + b_{i2}^* \times I_2^* + b_{i3}^* \times I_3^* + c_i \tag{7}$$

To reduce a general three-index model to a three-index model with orthogonal indexes we need first to set $I_1^* = I_1$. Then, since I_1^* and I_2^* are correlated, we can express I_2^* in terms of I_1, defining an index I_2 which is orthogonal to I_1 as follows

$$I_2^* = \gamma_0 + \gamma_1 \times I_1 + d_t$$

The part from I_2^* that is independent of I_1^* and adds new information to it is given by the residuals in the linear regression, such that $I_2 = d_t$. Thus

$$I_2 = d_t = I_2^* - (\gamma_0 + \gamma_1 \times I_1)$$

$$I_2^* = \gamma_0 + \gamma_1 \times I_1 + I_2$$

Substituting the above expression into equation **??** and rearranging we get:

$$R_i = (a_i^* + b_{i2}^* \times \gamma_0) + (b_{i1}^* + b_{i2}^* \times \gamma_1) \times I_1 + b_{i2}^* \times I_2 + b_{i3}^* \times I_3^* + c_i$$

The first term in the above equation is a constant, which we can define as a_i' . The coefficient in the second term of the above equation is also a constant, which we can define as b_{i1}'. We can then rewrite the above equation as:

$$R_i = a_i' + b_{i1}' \times I_1 + b_{i2}^* \times I_2 + b_{i3}^* \times I_3^* + c_i \qquad (8)$$

This model is equivalent to equation **??**, but with two orthogonal indexes, I_1 and I_2, and a third index I_3^* that can be explained by I_1 and I_2, through a linear regression

$$I_3^* = \theta_0 + \theta_1 \times I_1 + \theta_2 \times I_2 + e_t$$

As before, all new information due to I_3^* is captured by the residuals e_t. Therefore,

$$I_3 = e_t = I_3^* - (\theta_0 + \theta_1 \times I_1 + \theta_2 \times I_2)$$

$$I_3^* = \theta_0 + \theta_1 \times I_1 + \theta_2 \times I_2 + I_3$$

Substituting the above expression into equation **??** and rearranging we get:

$$R_i = (a_i' + b_{i3} \times \theta_0) + (b_{i1}' + b_{i3} \times \theta_2) + (b_{i2}^* + b_{i3} * \theta_2) \times I_2 + b_{i3}^* \times I_3 + c_i$$

In the above equation, the first term and all the coefficients of the new orthogonal indices are constants, so we can rewrite the equation as follows, getting a three-index model with orthogonal indexes:

$$R_i = a_i + b_{i1} \times I_1 + b_{i2} \times I_2 + b_{i3} \times I_3 + c_i$$

Where $a_i = a_i^* + b_{i2}^* \times \gamma_0 + b_{i3} \times \theta_0$, $b_{i1} = b_{i1}^* + b_{i2}^* \times \gamma_1 + b_{i3}^* \times \theta_1$, $b_{i2} = b_{i2}^* + b_{i3}^* \times \theta_2$ and $b_{i3} = b_{i3}^*$.

Exercise 2.12.

(a) In a three-index model we have:

$$R_i = a_i + b_{i1} \times I_1 + b_{i2} \times I_2 + b_{i3} \times I_3 + c_i$$

Since $\mathbb{E}[C_i] = 0$, we

$$\mathbb{E}[R_i] = a_i + b_{i1} \times \mathbb{E}[I_1] + b_{i2} \times \mathbb{E}[I_2] + b_{i3} \times \mathbb{E}[I_3]$$

(b) To derive the variance we need to recall three assumptions of a multi-index model

1. the indexes are uncorrelated: $\mathbb{E}[I_i I_j] = \mathbb{E}[I_i]\,\mathbb{E}[I_j]$

2. the specific factors of each security are independent: $\mathbb{E}[c_i c_j] = 0$
3. For any security, each index factors are independent of the specific factors of that same security: $\mathbb{E}[I_i c_i] = 0$
4. $\mathbb{E}[c_i]^2 = \sigma_{ci}^2$

Now we can apply the variance formula $\sigma_i^2 = \mathbb{E}\left[\left(R_i - \bar{R}_i\right)^2\right]$, such that

$$\begin{aligned}\sigma_i^2 &= \mathbb{E}\left[\left(a_i + b_{i1} \times I_1 + b_{i2} \times I_2 + b_{i3} \times I_3 + c_i - \left(a_i + b_{i1} \times \bar{I}_1 + b_{i2} \times \bar{I}_2 + b_{i3} \times \bar{I}_3\right)\right)^2\right] \\ &= \mathbb{E}\left[\left(b_{1i}\left(I_1 - \bar{I}_1\right) + b_{2i}\left(I_2 - \bar{I}_2\right) + b_{3i}\left(I_3 - \bar{I}_3\right)\right)^2\right]\end{aligned}$$

Carrying out the squaring, noting that the indices are all orthogonal with each other and using the stated assumptions gives us

$$\sigma_i^2 = b_{i1}^2 \sigma_{I1}^2 + b_{i2}^2 \sigma_{I2}^2 + b_{i3}^2 \sigma_{I3}^2 + \sigma_{c_i}^2$$

(c) Here we apply exactly the same reasoning that we used in part b. Covariance is given by $\sigma_{ij} = \mathbb{E}\left[\left(R_i - \bar{R}_i\right)\left(R_j - \bar{R}_j\right)\right]$.Thus,

$$\begin{aligned}\sigma_{ij} &= \mathbb{E}\left[\begin{array}{l}\left(a_i + b_{i1} \times I_1 + b_{i2} \times I_2 + b_{i3} \times I_3 + c_i - \left(a_i + b_{i1} \times \bar{I}_1 + b_{i2} \times \bar{I}_2 + b_{i3} \times \bar{I}_3\right)\right) \times \\ \times \left(a_i + b_{i1} \times I_1 + b_{i2} \times I_2 + b_{i3} \times I_3 + c_j - \left(a_j + b_{j1} \times \bar{I}_1 + b_{j2} \times \bar{I}_2 + b_{j3} \times \bar{I}_3\right)\right)\end{array}\right] \\ &= \mathbb{E}\left[\begin{array}{l}\left(b_{1i}\left(I_1 - \bar{I}_1\right) + b_{2i}\left(I_2 - \bar{I}_2\right) + b_{3i}\left(I_3 - \bar{I}_3\right)\right) \times \\ \times \left(b_{1j}\left(I_1 - \bar{I}_1\right) + b_{2j}\left(I_2 - \bar{I}_2\right) + b_{3j}\left(I_3 - \bar{I}_3\right)\right)\end{array}\right]\end{aligned}$$

Carrying out the squaring, noting that the indices are all orthogonal with each other and using the stated assumptions gives us

$$\sigma_{ij} = b_{i1} b_{j1} \sigma_{I1}^2 + b_{i2} b_{j2} \sigma_{I2}^2 + b_{i3} b_{j3} \sigma_{I3}^2$$

Exercise 2.14. To build such model, we can use all kind of economic explanatory factors, such as, GDP growth rate, inflation rate, interest rate, or firms characteristics that proxies risk factors as size, book to market ratio, sales/equity ratio, price/earnings or a market factor. For example, Fama and French (1992 and 2003) developed in the early 90s a three factor model, whose factors were variables built to capture size, the relation between book-value and market-value and the market return. Earlier, late 80s, Burmeister, McElroy (1987 and 1988) and other found that five variables are sufficient to describe security returns: two variables were related to the discount rate used to find the present value of cash flows; one related to both size of the cash flows and discount rates; one related only to cash flows; and a remaining variable that captures the impact of the market not incorporated in the first four variables.

Exercise 2.15.

(a) By definition the risk-free asset does not have any risk, so that the sensitivity to risk factors must be zero. Thus, $b_{F1} = 0 \wedge b_{F2} = 0$

(b) From the presented two-index model we know the expected return of any security is

$$\bar{R}_i = a_i + b_{i1} \bar{R}_{I_1} + b_{i2} \bar{R}_{I_2}$$

The above model is valid for any security including security B that is explained by factor 2, since $b_{i1} = 0$. Thus, we have

$$\bar{R}_B = a_B + b_{B2} \bar{R}_{I_2}$$

$$9.5 = -0.1 + 1.2 \bar{R}_{I_2}$$

$$\bar{R}_{I_2} = \frac{9.6}{1.2} = 8$$

(c) The expected return of security A is

$$\begin{aligned}\bar{R}_A &= a_A + b_{A1}\bar{R}_{I_1} + b_{A2}\bar{R}_{I_2}\\ &= 0.2 + 1.2 \times 15 - 0.15 \times 8\\ &= 17\end{aligned}$$

(d) Total risk as measured by standard deviation is

$$\sigma_i = \sqrt{b_{i1}^2\sigma_{I1}^2 + b_{i2}^2\sigma_{I2}^2 + \sigma_{c_i}^2}$$

And the systematic risk is measure by

$$\sigma_i = \sqrt{b_{i1}^2\sigma_{I1}^2 + b_{i2}^2\sigma_{I2}^2}$$

Thus, the risk of A, B and C is

$$\sigma_A = \sqrt{b_{A1}^2\sigma_{I1}^2 + b_{A2}^2\sigma_{I2}^2} = \sqrt{1.2^2 \times 25^2 - 0.15^2 \times 5^2} = 30$$

$$\sigma_B = \sqrt{b_{B1}^2\sigma_{I1}^2 + b_{B2}^2\sigma_{I2}^2} = \sqrt{0.8^2 \times 25^2 + 0^2 \times 5^2} = 20$$

$$\sigma_C = \sqrt{b_{C1}^2\sigma_{I1}^2 + b_{C2}^2\sigma_{I2}^2} = \sqrt{0^2 \times 25^2 + 1.2^2 \times 5^2} = 6$$

(e) Variance and covariance are measured, respectively, by

$$\sigma_i^2 = b_{i1}^2\sigma_{I1}^2 + b_{i2}^2\sigma_{I2}^2 + \sigma_{c_i}^2$$

$$\sigma_i^2 = b_{i1}b_{j1}\sigma_{I1}^2 + b_{i2}b_{j2}\sigma_{I2}^2$$

Applying the data in the exercise,

$$\sigma_A^2 = b_{A1}^2\sigma_{I1}^2 + b_{A2}^2\sigma_{I2}^2 + \sigma_{c_A}^2 = 1.2^2 \times 25^2 - 0.15^2 \times 5^2 + 5^2 = 925.56$$

$$\sigma_B^2 = b_{B1}^2\sigma_{I1}^2 + b_{B2}^2\sigma_{I2}^2 + \sigma_{c_B}^2 = 0.8^2 \times 25^2 + 0^2 \times 5^2 + 2^2 = 404$$

$$\sigma_B^2 = b_{C1}^2\sigma_{I1}^2 + b_{C2}^2\sigma_{I2}^2 + \sigma_{c_C}^2 = 0^2 \times 25^2 + 1.2^2 \times 5^2 + 1^2 = 37$$

$$\sigma_{AB} = \sigma_{BA} = b_{A1}b_{B1}\sigma_{I1}^2 + b_{A2}b_{B2}\sigma_{I2}^2 = 1.2 \times 0.8 \times 25^2 - 0.15 \times 0 \times 5^2 = 600$$

$$\sigma_{AC} = \sigma_{CA} = b_{A1}b_{C1}\sigma_{I1}^2 + b_{A2}b_{C2}\sigma_{I2}^2 = 1.2 \times 0 \times 25^2 - 0.15 \times 1.2 \times 5^2 = -4.5$$

$$\sigma_{BC} = \sigma_{BC} = b_{B1}b_{C1}\sigma_{I1}^2 + b_{B2}b_{C2}\sigma_{I2}^2 = 0.8 \times 0 \times 25^2 + 0 \times 1.2 \times 5^2 = 0$$

So that, the covariance matrix is

$$\begin{pmatrix} 925.56 & 600 & -4.5 \\ 600 & 404 & 0 \\ -4.5 & 0 & 37 \end{pmatrix}$$

(f) (i) To find the minimum variance portfolio (mvp) we need to take the derivative and equal to 0 of the portfolio variance in order to X_B, which is the weight of security B in the mvp. Since securities B and C are not correlated and, therefore, $\rho_{BC} = 0$, we have

$$\sigma_V^2 = X_B^2\sigma_B^2 + (1 - X_B)^2\,\sigma_C^2$$

Taking the derivative, equaling 0 and solving for X_B

$$\frac{\partial \sigma_V^2}{\partial X_B} = 2X_B\sigma_B^2 + 2\left(1 - X_B\right)\left(-1\right)^2\sigma_C^2 = 0$$

$$X_B = \frac{\sigma_C^2}{\sigma_B^2 + \sigma_C^2}$$

Consequently,

$$X_B = \frac{\sigma_C^2}{\sigma_B^2 + \sigma_C^2} = \frac{37}{404 + 37} = 0.084$$
$$X_C = 1 - X_B = 1 - 0.084 = 0.916$$

Finally the portfolio's risk is

$$\sigma_V = \sqrt{X_B^2\sigma_B^2 + X_C^2\sigma_C^2} = \sqrt{0.084^2 \times 404 + 0.916^2 \times 37} = 0.0582$$

(ii) If we could invest in a risk-free security, the mvp would be 100% composed with the risk-free security, since, of course, it is impossible to build a portfolio with less risk then the risk-free security.

(g) (i) This is a standard portfolio selection exercise, in which we have to choose the tangent portfolio between the capital market line and the efficient frontier of risky assets. The solution for this problem involves solving the following system of simultaneous equations in order to Z_i, $\forall i = A,\ B,\ C$

$$\begin{cases} \bar{R}_A - R_F = Z_A\sigma_A^2 + Z_B\sigma_{AB} + Z_C\sigma_{AC} \\ \bar{R}_B - R_F = Z_A\sigma_{BA} + Z_B\sigma_B^2 + Z_C\sigma_{BC} \\ \bar{R}_C - R_F = Z_A\sigma_{CA} + Z_B\sigma_{CB} + Z_C\sigma_C^2 \end{cases}$$

Applying the data in the problem,

$$\begin{cases} 17 - 5 = 925.56Z_A + 600Z_B - 4.5Z_C \\ 12.5 - 5 = 600Z_A + 404Z_B \\ 9.5 - 5 = -4.5Z_A + 37Z_C \end{cases} \Leftrightarrow \begin{cases} Z_A = 0.041525 \\ Z_B = -0.04311 \\ Z_C = 0.12667 \end{cases}$$

Then, $\sum_{i=A}^{C} Z_i = 0.12509$. Therefore, the weights of the tangent portfolio are

$$X_A = \frac{Z_A}{\sum_{i=A}^{C} Z_i} = \frac{0.041525}{0.12509} = 0.332$$
$$X_B = \frac{Z_B}{\sum_{i=A}^{C} Z_i} = \frac{-0.04311}{0.12509} = -0.3446$$
$$X_C = \frac{Z_C}{\sum_{i=A}^{C} Z_i} = \frac{0.12793}{0.12509} = 1.0126$$

Finally, the portfolio's expected return is

$$\bar{R}_T = \sum_{i=A}^{C} X_i\bar{R}_i = 0.332 \times 17 - 0.3446 \times 12.5 + 1,0126 \times 9.5 = 10.96$$

The portfolio's variance is

$$\sigma_T^2 = X'\Sigma X$$
$$= \begin{pmatrix} 0.332 & -0.3446 & 1.0126 \end{pmatrix} \begin{pmatrix} 925.56 & 600 & -4.5 \\ 600 & 404 & 0 \\ -4.5 & 0 & 37 \end{pmatrix} \begin{pmatrix} 0.332 \\ -0.3446 \\ 1.0126 \end{pmatrix}$$
$$= 47.61$$

And portfolio's risk is

$$\sigma_T = 6.9$$

(ii) The capital market line is

$$\begin{aligned} \bar{R}_i &= R_F + \frac{R_T - R_F}{\sigma_T}\sigma_i \\ &= 5 + \frac{10.96 - 5}{6.9}\sigma_i \\ &= 5 + 0.86\sigma_i \end{aligned}$$

3 Selecting the Optimal Portfolio

3.1 Expected Utility Theory

Exercise 3.1. A *fair game* is a game where the initial investment equals the expected value of the payoff, i.e., where we have $\mathbb{E}(W) = W_0$.

We also know the utility functions of risk neutral investors are linear, while utility functions of risk averse are concave and of risk lovers are convex functions. See general shapes of utility function in Figure **??**

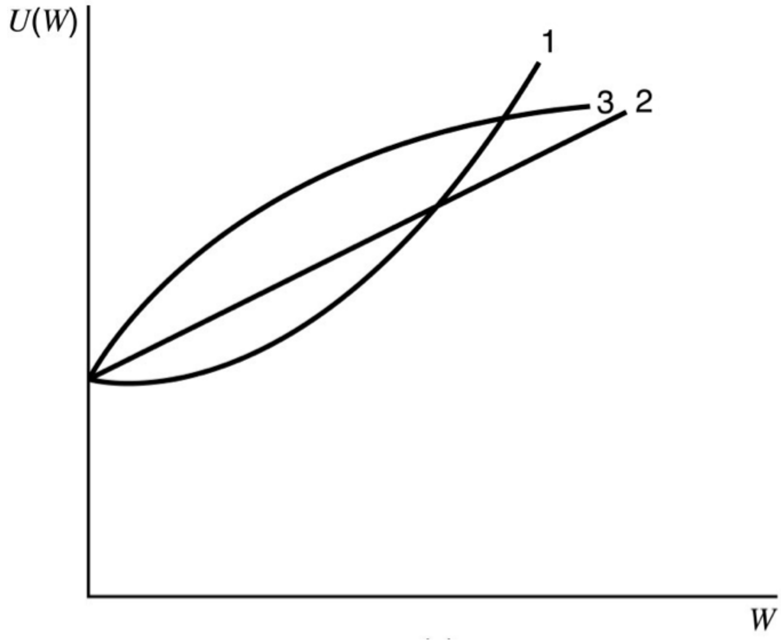


Figure 10: Exercise **??** – Shape of utility functions for risk (1) lovers, (2) neutral and (3) averse.

By definition of linear, concave and convex functions we have.

(a) For any a and b and $p \in [0, 1]$ if the utility function U is linear we have

$$U(pa + (1-p)b) = pU(a) + (1-p)U(b) \qquad \Leftrightarrow \qquad U\underbrace{(\mathbb{E}(W))}_{W_0} = \mathbb{E}(U(W)) \ ,$$

thus, we conclude that any risk neutral investor would be indifferent between entering or not a fair game.

(b) For any a and b and $p \in [0, 1]$ if the utility function U is concave we have

$$U(pa + (1-p)b)p \geq U(a) + (1-p)U(b) \qquad \Leftrightarrow \qquad U\underbrace{(\mathbb{E}(W))}_{W_0} \geq \mathbb{E}(U(W)) \ .$$

So, investors with concave utilities do not enter fair games.

(c) For any a and b and $p \in [0, 1]$ if the utility function U is convex we have

$$U(pa + (1-p)b)p \leq U(a) + (1-p)U(b) \qquad \Leftrightarrow \qquad U\underbrace{(\mathbb{E}(W))}_{W_0} \leq \mathbb{E}(U(W)) \ .$$

So, investors with convex utilities enter fair games.

Exercise 3.2.

(a) For the investor with utility $U(W) = -W^{-1/3}$ we compute the expected utility of both investments,

$$\mathbb{E}\left[U(W_A)\right] = 0.25U(4) + 0.5U(6) + 0.25U(8) = -0.5576$$
$$\mathbb{E}\left[U(W_B)\right] = \frac{1}{3}U(4) + \frac{1}{3}U(6.2) + \frac{1}{3}U(8) = -0.5581$$

and conclude that investor 1, $A \succ B$.

(b) For $U(W) = -W^{-0.1}$ we get,

$$\mathbb{E}\left[U(W_A)\right] = 0.25U(4) + 0.5U(6) + 0.25U(8) = -0.8386$$
$$\mathbb{E}\left[U(W_B)\right] = \frac{1}{3}U(4) + \frac{1}{3}U(6.2) + \frac{1}{3}U(8) = -0.8387$$

and conclude that also investor 2, $A \succ B$.

(c) Both investors have power utility, thus

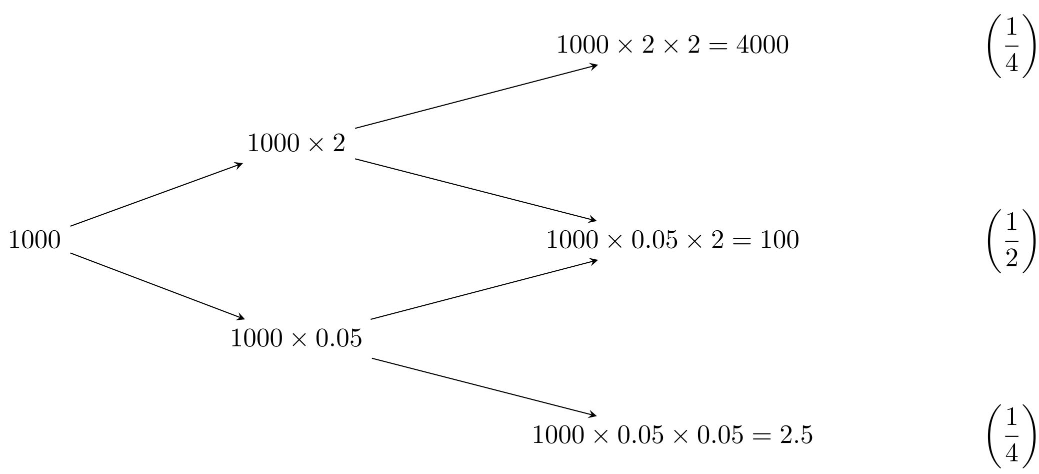
$$U(W) = -W^{-\alpha} \qquad \text{for} \alpha > 0$$
$$U'(W) = \alpha W^{-\alpha-1} > 0$$
$$U''(W) = -\alpha(\alpha+1)W^{-\alpha-2} < 0$$
$$ARA(W) = -\frac{U''(W)}{U'(W)} = \frac{\alpha(\alpha+1)W^{-\alpha-2}}{\alpha W^{-\alpha-1}} = \frac{1+\alpha}{W} \qquad \Rightarrow \qquad ARA'(W) = -\frac{1+\alpha}{W^2} < 0$$
$$RRA(W) = \frac{1+\alpha}{W}W = 1+\alpha \qquad \Rightarrow \qquad RRA'(W) = 0$$

they prefer more to less and they are risk averse with decreasing absolute risk aversion and constant relative risk aversion.

So, they always keep the same proportion of wealth invested in risky assets. Despite their similarities in terms of profiles, investor 1 has $\alpha = 1/3 = 0.3(3)$ while investor 2 has $\alpha = 0.1$, so their coefficients of RRAs are of 1.3(3) and 1.1, respectively, and we can conclude investor 1 has a higher degree of risk aversion than investir 2.

Exercise 3.3.

Since the coin is tossed twice the game can be summarised by the scheme below.

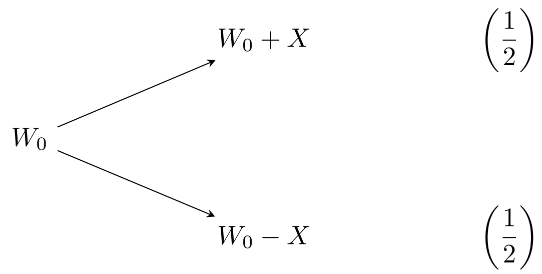


For log utility we have $U(W) = \log(W)$, and we have

$$\mathbb{E}\left[U\left(Game\right)\right] = \frac{1}{4}\underbrace{U(4000)}_{8.295} + \frac{1}{2}\underbrace{U(100)}_{4.6051} + \frac{1}{4}\underbrace{U(2.5)}_{0.916} = 4.6051$$

Since $U(100) = 4.60651$, we know the certainty equivalent of the game is $C = 100$ and, thus, the investor would be willing to pay up to 900 to avoid the situation.

Exercise 3.4.



(a) (i) For $W_0 = 1000$ and $X = 250$, the expected utility of the game, and the associated certainty equivalent, for each of the investor are:

$$\mathbb{E}\left[U(Game)\right] = \frac{1}{2}U(W_0 + X) + \frac{1}{2}U(W_0 - X)$$

$$U(W)) = \ln(W) \qquad \mathbb{E}\left[U(Game)\right] = \frac{1}{2}\log(1250) + \frac{1}{2}\log(750)$$

$$= \frac{1}{2}(7.13) + \frac{1}{2}(6.62) = 6.875$$

$$\ln(C) = 6.875 \quad \Rightarrow \quad C = 967.78$$

$$V(W)) = 1 - e^{-0.001W} \qquad \mathbb{E}\left[U(Game)\right] = \frac{1}{2}\left(1 - e^{-0.001\times 1250}\right) + \frac{1}{2}\left(1 - e^{-0.001\times 750}\right)$$

$$= \frac{1}{2}(0.7135) + \frac{1}{2}(0.5276) = 0.62055$$

$$1 - e^{-0.001C} = 0.62055 \quad \Rightarrow \quad C = 969.03$$

Investor 1 is willing to pay $1000 - 967.78 = 32.22$ and investor 2 is willing to pay $1000 - 969.03 = 30.97$.

(ii) The expected utility of the game is

$$\max_X \qquad \mathbb{E}\left[U(Game)\right] = \frac{1}{2}U(W_0 + X) + \frac{1}{2}U(W_0 - X)$$

the value X that maximizes expected utility is given by the first-order-condition (F.O.C)

$$\frac{1}{2}U'(W_0 + X) - \frac{1}{2}U'(W_0 - X) = 0$$

For both investors we get

$$U(W) = \ln(W)$$

$$U'(W) = \frac{1}{W}: \quad \frac{1}{2}\frac{1}{W_0 + X} - \frac{1}{2}\frac{1}{W_0 - X} = 0 \quad \Leftrightarrow \quad X = 0$$

$$V(W) = 1 - e^{-0.001W}$$

$$U'(W) = 0.001e^{-0.001W}: \quad \frac{0.001}{2}e^{-0.001(W_0+X)} - \frac{0.001}{2}e^{-0.001(W_0-X)} = 0 \quad \Leftrightarrow \quad X = 0$$

Which is not surprising as risk averse investors would rather not enter fair games (no matter the W_0 or X).

(b) The optimal $X = 0$ does not change. The amount investors are willing to pay to avoid the game, however, does depend on the initial wealth

$$
\begin{aligned}
U(W)) = \ln(W) \qquad \mathbb{E}[U(Game)] &= \frac{1}{2}\log(100250) + \frac{1}{2}\log(99750) \\
&= \frac{1}{2}(11.5154) + \frac{1}{2}(11.5104) = 11.5129 \\
\ln(C) &= 11.5129 \quad \Rightarrow \quad C = 99997.45 \\
V(W)) = 1 - e^{-0.001W} \qquad \mathbb{E}[U(Game)] &= \frac{1}{2}\left(1 - e^{-0.001\times 100250}\right) + \frac{1}{2}\left(1 - e^{-0.001\times 99750}\right) \\
&= \frac{1}{2}(0.9999) + \frac{1}{2}(0.9999) = 0.9999 \\
1 - e^{-0.001C} &= 0.9999 \quad \Rightarrow \quad C = 99999.99
\end{aligned}
$$

As the wealth increases the curvature of both utility functions decrease and so they are willing to pay less to avoid the game.

Exercise 3.5.

(a) Starting from an initial wealth of $W_0 = 50$, the final outcome may be $W = 25$ or $W = 75$, with equal probability.

Given the utility function, we have

$$
\begin{aligned}
\text{If he enters the game :} \qquad \mathbb{E}[U(Game)] &= \frac{1}{2}U(25) + \frac{1}{2}U(75) \\
&= \frac{1}{2}\left[25 - 0.005(25)^2 + 75 - 0.005(75)^2\right] \\
&= 34.375 \\
\text{If he does not enter the game :} \qquad U(50) &= 50 - 0.005(50)^2 = 37.5
\end{aligned}
$$

So, he chooses not to play the game.

(b) To be indifferent between playing the same or not we need the expected utility of the game to be the same as the utility of not playing the game. Let us assign a probability p to the outcome 75 and $(1-p)$ to 25. We, thus have

$$
\begin{aligned}
p\left[25 - 0.005(25)^2\right] + (1-p)\left[75 - 0.005(75)^2\right] &= 37.5 \\
46.875\,p + 21.875(1-p) &= 37.5 \\
p &= 62.5\%
\end{aligned}
$$

(c) The certainty equivalent of the game is the fixed amount that would make the investor indifferent between playing the game or nor.

In this case we have

$$
\begin{aligned}
U(C) &= \mathbb{E}[U(Game)] \\
C - 0.005C^2 &= 34.375 \\
C &= \frac{-1 \pm \sqrt{1 - 4\times(-0.005)\times(-34.375)}}{2\times(-0.005)} = \frac{1 \pm 0.5590}{0.01} \\
&\Rightarrow C = 44.1
\end{aligned}
$$

Exercise 3.6.

From the ranking of the projects, $X \succ Y \succ Z$, we know $\mathbb{E}(U_X) > \mathbb{E}(U_Y)$ and $\mathbb{E}(U_Y) > \mathbb{E}(U_Z)$. Using a second order Taylor approximation of the RTF we also have

$$\mathbb{E}(U) = f(\sigma, \bar{R}) \approx \bar{R} - \frac{1}{2} RRA(W_0)(\bar{R}^2 + \sigma^2) \ .$$

For each project we get

$$f_X(30\%, 20\%) \approx 0.2 - \frac{1}{2} RRA(W_0)(0.3^2 + 0.2^2) = 0.2 - 0.065 RRA(W_0)$$
$$f_Y(35\%, 15\%) \approx 0.15 - \frac{1}{2} RRA(W_0)(0.15^2 + 0.35^2) = 0.15 - 0.0725 RRA(W_0)$$
$$f_Z(5\%, 8\%) \approx 0.08 - half RRA(W_0)(0.08^2 + 0.05^2) = 0.08 - 0.00445 RRA(W_0) \ .$$

and it musty hold

$$\begin{cases} f_X(30\%, 20\%) > f_Y(35\%, 15\%) \\ f_Y(35\%, 15\% > f_Z(5\%, 8\%) \end{cases} \Leftrightarrow \begin{cases} 0.2 - 0.065\ RRA(W_0) > 0.15 - 0.0725\ RRA(W_0) \\ 0.15 - 0.0725\ RRA(W_0) > 0.08 - 0.00445\ RRA(W_0) \end{cases}$$

Solving the system we get $1.06 > RRA(W_0) > -6.67$, so any investor with $RRA(W_0)$ within that range would have the suggested ranking of projects. In particular for risk neutral investors, with $RRA(W_0) = 0$, we also get $X \succ Y \succ Z$.

Exercise 3.7.

(a) The preferred investment will be the one with the highest level of expected utility. Thus, we have to calculate the utility in each state of economy for the three investments. Given the utility function $U(W) = 20W - 0.5 * W^2$ we get,

For investment A:

$$U(5) = 20 * 5 - 0.5 \times 5^2 = 87.5$$
$$U(6) = 20 * 6 - 0.5 \times 6^2 = 102$$
$$U(9) = 20 * 9 - 0.5 \times 9^2 = 139.5$$

For investment B:

$$U(4) = 20 * 4 - 0.5 \times 4^2 = 72$$
$$U(7) = 20 * 7 - 0.5 \times 7^2 = 115.5$$
$$U(10) = 20 * 10 - 0.5 \times 10^2 = 150$$

For investment C:

$$U(1) = 20 * 1 - 0.5 \times 1^2 = 19.5$$
$$U(9) = 20 * 9 - 0.5 \times 9^2 = 139.5$$
$$U(18) = 20 * 18 - 0.5 \times 18^2 = 198$$

Therefore, the expected utility for each investment is

$$\mathbb{E}\left[U(W_A)\right] = 87.5 \times 1/3 + 102 \times 1/3 + 139.5 \times 1/3 = 109.67$$
$$\mathbb{E}\left[U(W_B)\right] = 72 \times 1/4 + 115.5 \times 1/2 + 150 \times 1/4 = 113.25$$
$$\mathbb{E}\left[U(W_C)\right] = 19.5 \times 1/5 + 139.5 \times 3/5 + 198 \times 1/5 = 127.20$$

So, Investment C is preferred because it has the highest level of expected utility.

(b) As before, the preferred investment will be the one with the highest level of expected utility, so that we have to calculate the utility in each state of economy for the three investments, now considering the new utility function $U(W) = -\frac{1}{\sqrt{W}}$.

For investment A:

$$\begin{aligned} U(5) &= -\frac{1}{\sqrt{5}} = -0.4472 \\ U(6) &= -\frac{1}{\sqrt{6}} = -0.4082 \\ U(9) &= -\frac{1}{\sqrt{9}} = -0.3333 \end{aligned}$$

For investment B:

$$\begin{aligned} U(4) &= -\frac{1}{\sqrt{4}} = -0.5 \\ U(7) &= -\frac{1}{\sqrt{7}} = -0.3750 \\ U(10) &= -\frac{1}{\sqrt{10}} = -0.3162 \end{aligned}$$

For investment C:

$$\begin{aligned} U(1) &= -\frac{1}{\sqrt{1}} = -1 \\ U(9) &= -\frac{1}{\sqrt{9}} = -0.3333 \\ U(18) &= -\frac{1}{\sqrt{18}} = -0.2351 \end{aligned}$$

Therefore, the expected utility for each investment is

$$\begin{aligned} \mathbb{E}\left[U(W_A)\right] &= -0.4472 \times 1/3 - 0.4082 \times 1/3 - 0.3333 \times 1/3 = -0.3963 \\ \mathbb{E}\left[U(W_B)\right] &= -0.5 \times 1/4 - 0.3780 \times 1/2 - 0.3162 \times 1/4 = -0.3930 \\ \mathbb{E}\left[U(W_C)\right] &= -1 \times 1/5 - 0.3333 \times 3/5 - 0.2357 \times 1/5 = -0.4471 \end{aligned}$$

With this new utility function, Investment B is preferred because it has the highest level of expected utility.

(c) For investments A and B be indifferent, using the first utility function, their expected utility must equal. Therefore, what must be the probability π associated to payoffs 4 and 10 of investment B to have such equality?

$$A \sim B \Longleftrightarrow \mathbb{E}\left[U(W_A)\right] = \mathbb{E}\left[U(W_B)\right]$$

Thus

$$\mathbb{E}\left[U(W_A)\right] = \mathbb{E}\left[U(W_B)\right]$$

$$109.67 = 72 \times \pi + 115.5 \times (1 - 2\pi) + 150 \times \pi$$

$$\pi = 0.648$$

Since we must have $0 \leq \pi \leq 0.5$, otherwise the new probabilities would not be between 0 and 1, this means investor 1 will never be indifferent between investments A and B. He always prefer B to A .

(d) For investments B and C be indifferent, using the second utility function, their expected utility must be the same. In part c we vary the probability associated to certain payoffs, now we allow for a change in the lowest payoff of these two investments, which is 1 for Investment C. So,

$$B \sim C \Longleftrightarrow \mathbb{E}\left[U(W_B)\right] = \mathbb{E}\left[U(W_C)\right]$$

Thus

$$\mathbb{E}\left[U(W_B)\right] = \mathbb{E}\left[U(W_C)\right]$$

$$-0.3963 = U(x) \times 1/5 - 0.3333 \times 3/5 - 0.2357 \times 1/5$$

$$U(x) = -0.7456$$

Since $U(x) = -\frac{1}{\sqrt{x}}$ we finally have

$$U(x) = -\frac{1}{\sqrt{x}}$$

$$-0.7456 = -\frac{1}{\sqrt{x}}$$

$$x = 1.7987$$

Exercise 3.8. (a) To analise the investor behaviour towards risk we need to study its utility function and its economics proprieties, which is done taking the first and the second derivative. With the utility function $U(W) = -w^{-1/2}$ and assuming $W > 0$, we have

$$U'(W) = \frac{1}{2}W^{-3/2}$$

Since $W > 0$ it comes $U'(W) > 0$, which means the investor prefers more to less. This attribute is known as nonsatiation. The second derivative is

$$U''(W) = -\frac{3}{4}W^{-5/2}$$

Which smaller than 0, so that the investor shows risk aversion.

(b) Absolute aversion is calculated by taking the first derivative of a measure of absolute aversion that is

$$ARA(W) = -\frac{U''(W)}{U'(W)}$$

Therefore,

$$ARA(W) = -\frac{U''(W)}{U'(W)} = \frac{\frac{3}{4}W^{-5/2}}{\frac{1}{2}W^{-3/2}} = \frac{3}{2}W^{-1}$$

And,

$$ARA'(W) = -\frac{3}{2}W^{-2}$$

Since $ARA'(W) < 0$, the investor exhibits decreasing absolute risk aversion. In practical terms, this means the investor increases the amount of money invested in risky assets when her wealth increases.

Relative aversion is a similar to absolute aversion, but its calculated in proportional terms. So, we need to take the first derivative of a measure of relative risk aversion that is

$$RRA(W) = -\frac{WU''(W)}{U'(W)}$$

Therefore,

$$RRA(W) = -\frac{WU''(W)}{U'(W)} = \frac{\frac{3}{4}W^{-5/2}W}{\frac{1}{2}W^{-3/2}} = \frac{3}{2}$$

And,

$$RRA'(W) = 0$$

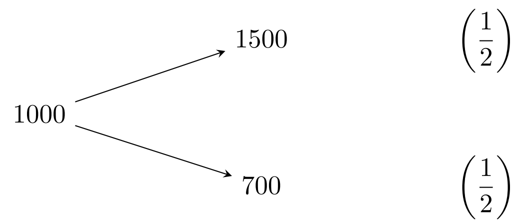
Since $RRA'(W) = 0$, the investor exhibits constant relative risk aversion. In practical terms, this means the percentage invested in risky assets remains constant when her wealth increases.

Exercise 3.9.

(a) Since $U(W) = ae^{-bW}$, we have $U'(W) = -abe^{-bW}$ and $U''(W) = ab^2e^{-bW}$. To have a risk averse investor we need $U''(W) = ab^2e^{-bW} < 0$. Since, $e^{-bW} > 0$ and b^2 is positive, then a must be negative ($a < 0$). On the other hand, to respect the nonsatiation assumption we need $U'(W) = -abe^{-bW} > 0$. Again $e^{-bW} > 0$. Because $a < 0$ we have $-a > 0$, which implies a positive b.

(b) (i) If the investor decides not to do the risky investment, he keep the 1000 and has an utility of $\mathbb{E}[U(Invest)] = ae^{-b1000}$.

If he decides do do the risky investment he faces



and his expected utility from the investment is

$$\mathbb{E}[U(Invest)] = \frac{1}{2}ae^{-b1500} + \frac{1}{2}ae^{-b700} = ae^{-b1000}\frac{e^{-b500} + e^{+b300}}{2}$$

To compare the utility of not investing with the expected utility of the investment we need to compare 1 with $\frac{e^{-b500}+e^{+b300}}{2}$, which does not depend on a, but only on b. The investor chooses the risky investment when

$$\begin{aligned}
\mathbb{E}[U(Invest)] &> U(1000)\\
ae^{-b1000}\frac{e^{-b500} + e^{+b300}}{2} &> ae^{-b1000}\\
e^{-b500} + e^{+b300} &> 2\ .
\end{aligned}$$

(ii)

$$\begin{aligned}
U(C) &= \mathbb{E}[U(Invest)]\\
e^{-Cb} &= \frac{1}{2}e^{-b1500} + \frac{1}{2}e^{-b700}\\
-Cb &= \ln\left(\frac{1}{2}e^{-b1500} + \frac{1}{2}e^{-b700}\right)\\
C &= -\frac{\ln\left(\frac{1}{2}e^{-b1500} + \frac{1}{2}e^{-b700}\right)}{b}
\end{aligned}$$

The certainty equivalent of a risky investment is the certain (fixed) amount that makes the investor indifferent between keeping that fixed amount or entering the risky investment. It can also be interpreted as the maximum amount the investor would be willing to “pay” to enter the risky investment.

(iii) For $b = 0.01$ we have $C = -\frac{\ln\left(\frac{1}{2}e^{-b1500}+\frac{1}{2}e^{-b700}\right)}{0.01} = 769.28$. Since it is less than 1000 we can conclude that in this case the investor will not do the risky investment.

Exercise 3.10.

(a) See Figure **??**.

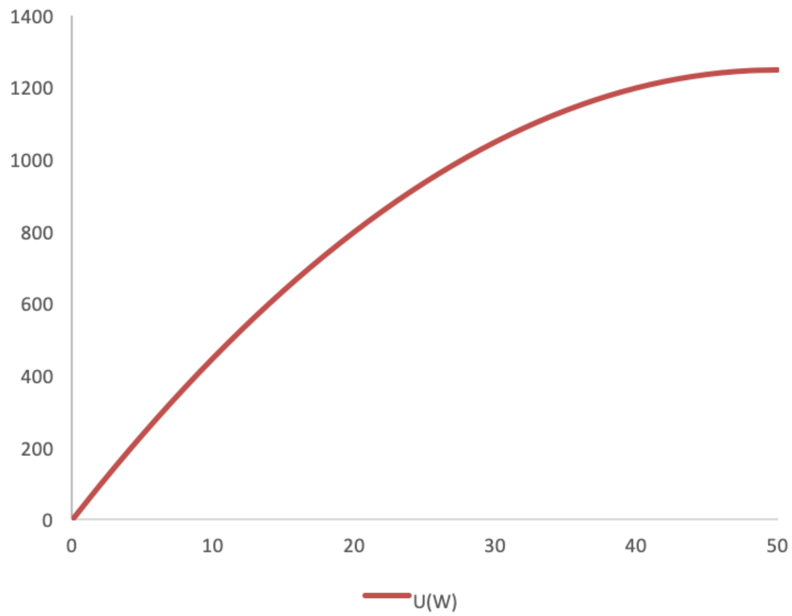


Figure 11: Exercise **??** – Utility function for relevant wealth levels ($W < 50$).

(b) To describe this investor behaviour towards risk we need to study the following properties

- Nonsaciation
- Risk attitude (risk aversion)
- Absolute risk aversion
- Relative risk aversion

The investor respects the nonsaciation assumption if $U'(W) > 0$. Since

$$U'(W) = 50 - W$$

This propriety is respected if and only if $W < 50$.

To study the second property we take te second derivative

$$U''(W) = -1 < 0$$

Consequently, the investor shows risk aversion for the feasible values for wealth ($W \in]0, 50[$). Geometrically, in the allowed domain, the function is increasing and concave, being a parable turned down (see Figure **??**).

About absolute risk aversion we known

$$ARA(W) = -\frac{U''(W)}{U'(W)} = (50 - W)^{-1} \qquad ARA'(W) = (50 - W)^{-2} > 0$$

Thus, this investor exhibits an increasing absolute risk aversion, i.e. when her wealth increases she will invest a small amount of money in risky assets.

About relative risk aversion we have

$$RRA(W) = -\frac{WU''(W)}{U'(W)} = W\,(50 - W)^{-1} \qquad RRA'(W) = \frac{50}{(50 - W)^2} > 0$$

Thus, this investor exhibits an increasing relative risk aversion, i.e. when her wealth increases she will invest a small percentage of her wealth in risky assets.

(c) This investor will chose the project with higher expected utility. Thus for investment X, we have for each state of economy

$$U(10) = 50W - \frac{1}{2}W^2 = 50 \times 10 - \frac{1}{2} \times 10^2 = 450$$

$$U(40) = 50W - \frac{1}{2}W^2 = 50 \times 40 - \frac{1}{2} \times 40^2 = 1,200$$

$$U(25) = 50W - \frac{1}{2}W^2 = 50 \times 25 - \frac{1}{2} \times 25^2 = 937.5$$

For investment Y,

$$U(20) = 50W - \frac{1}{2}W^2 = 50 \times 20 - \frac{1}{2} \times 20^2 = 800$$

$$U(40) = 50W - \frac{1}{2}W^2 = 50 \times 40 - \frac{1}{2} \times 40^2 = 1,200$$

$$U(45) = 50W - \frac{1}{2}W^2 = 50 \times 45 - \frac{1}{2} \times 45^2 = 1,237.5$$

Thus, expected utilities are

$$\mathbb{E}\left[U\left(W_X\right)\right] = \sum_{i=1}^{3} P_i U(W_{X_i}) = 0.1 \times 450 + 0.2 \times 1,200 + 0.7 \times 937,5 = 941.25$$

$$\mathbb{E}\left[U\left(W_Y\right)\right] = \sum_{i=1}^{3} P_i U(W_{y_i}) = 0.05 \times 800 + 0.9 \times 1,237.5 + 0.05 \times 1200 = 1,181.88$$

As $\mathbb{E}\left[U\left(W_Y\right)\right] > \mathbb{E}\left[U\left(W_X\right)\right]$, we have $Y \succ X$, i.e. investor's choice should be project Y.

(d) The risk premium π is the amount the investor is willing to pay to insure against risk, such that this is a measure of absolute risk aversion. The risk premium is calculated as $\pi = \mathbb{E}\left[W\right] - c$ where c is the certain equivalent. The certain equivalent is the amount received with certainty that has the same utility than a lottery

$$U\left(c\right) = \mathbb{E}\left[U\left(W\right)\right] \tag{9}$$

Thus, for Investment X, we have $\pi_X = \mathbb{E}\left[W_X\right] - c_X$, where

$$\mathbb{E}\left[W_X\right] = \sum_{i=1}^{3} P_i W_{X_i} = 0.1 \times 10 + 0.2 \times 40 + 0.7 \times 25 = 26.5$$

To find c_X we need to use (??)

$$U(c_X) = \mathbb{E}[U(W_X)]$$

$$50c_X - \frac{1}{2}c_X^2 = 941.25$$

$$c_X = 74.85 \vee c_X = 25.15$$

Since c_X must be in the range of possible values for W_X we have $c_X = 25.15$. Finally, the risk premium is

$$\pi_X = \mathbb{E}[W_X] - c_X = 26.5 - 25.15 = 1.35$$

Similarly for Investment Y, we have $\pi_Y = \mathbb{E}[W_Y] - c_Y$, where

$$\mathbb{E}[W_Y] = \sum_{i=1}^{3} P_i W_{Y_i} = 0.05 \times 20 + 0.9 \times 40 + 0.05 \times 45 = 39.25$$

To find c_Y we use again (??)

$$U(c_Y) = \mathbb{E}[U(W_Y)]$$

$$50c_Y - \frac{1}{2}c_Y^2 = 1181.88$$

$$c_Y = 38.33 \vee c_Y = 61.67$$

Since c_Y must be in the range of possible values for W_Y we have $c_Y = 38.33$. Finally, the risk premium is

$$\pi_Y = \mathbb{E}[W_Y] - c_Y = 39.25 - 38.33 = 0.92$$

As expected the risk premium for investment X is higher due its higher risk level.

Exercise 3.11.

(a) To discover the investor's attitudes towards risk we can draw her utility function. To do so we need as many points as we can. From the data in the problem we already have two points $\{(R, U) : (0\%, 0)\ (10\%, 10)\}$.

We also have data on three investment projects and their certain equivalents, $C_X = 10\%$, $C_Y = 20\%$ and $C_Z = 30\%$, that can give us another three points.

Thus, for project X

$$U(C_X) = \mathbb{E}[U(R_X)]$$

$$U(10\%) = 0.5U(0\%) + 0.5U(30\%)$$

$$5 = 0.5U(30\%)$$

$$U(30\%) = 10$$

For project Y we have

$$U(C_Y) = \mathbb{E}[U(R_Y)]$$

$$U(20\%) = 0.4U(10\%) + 0.6U(30\%)$$

$$U(20\%) = 0.4 \times 5 + 0.6 \times 10$$

$$U(20\%) = 8$$

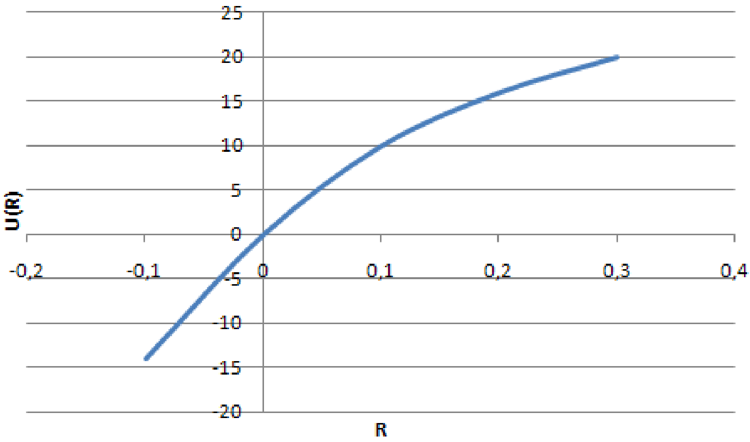


Figure 12: Exercise **??** - Utility Function

Finally, for project Z we have

$$U(C_Z) = \mathbb{E}[U(R_Z)]$$
$$U(10\%) = 0.2U(-10\%) + 0.8U(20\%)$$
$$5 = 0.2U(-10\%) + 0.8 \times 8$$
$$U(-10\%) = -7$$

With five points we can draw the utility function (see Figure **??**) and observe the function is increasing and concave, therefore for equal increases in return the marginal utility is decreasing. Thus, the investor is risk averse.

(b) The risk premium associated with each of the projects is given by $\pi = E(R) - C$, where C is the certainty equivalent. We thus have

$$\begin{aligned} \mathbb{E}(R_X) = 15\% &\implies \pi_X = 15\% - 10\% = 5\% \\ \mathbb{E}(R_Y) = 22\% &\implies \pi_X = 22\% - 20\% = 2\% \\ \mathbb{E}(R_Z) = 14\% &\implies \pi_X = 14\% - 10\% = 4\% \end{aligned}$$

(c) The previous answer is based on the expected utility theorem and the utility function proprieties. The expected utility theorem states the rational rules to order different investment projects and basically it claims that the decision criterion is the maximization of the expected utility.

(d) To rank the three projects we need to compute their expected utilities. Using the results from (a) we get

$$\mathbb{E}(U(R_X)) = 5 \qquad \mathbb{E}(U(R_Y)) = 8 \qquad \mathbb{E}(U(R_Z)) = 5 \ ,$$

so the investor prefers project Y to the other two projects and is indifferent between X and Z, i.e. $Y \succ X \sim Z$

(e) We know consider a game that pays 30% with probability h and 0% with probability $(1 - h)$. We need to find the probability level h that makes the investor indifferent between each project and this game.

$$\begin{aligned} h_X U(30\%) + (1-h_X)U(0\%) &= E(U(R_X)) \\ 10h_X &= 5 \\ h_X &= 0.5 \end{aligned}$$

$$\begin{aligned} h_Y U(30\%) + (1-h_Y)U(0\%) &= E(U(R_Y)) \\ 10h_Y &= 8 \\ h_Y &= 0.8 \end{aligned}$$

$$\begin{aligned} h_Z U(30\%) + (1-h_Z)U(0\%) = E(U(R_Z)) \\ 10h_Z &= 5 \\ h_Z &= 0.5 \end{aligned}$$

From the above we get the exact same ranking as before: $Y \succ X \sim Z$.

Exercise 3.12.

(a) To find the absolute and relative risk aversion coefficients we first need to take the first and second derivative of the utility function

$$U'(W) = \frac{4}{W} > 0 \wedge U''(W) = -\frac{4}{W^2} < 0$$

Thus, she respects the nonsatiation assumption and is risk averse. About absolute and relative risk aversion we know

$$ARA(W) = -\frac{U''(W)}{U'(W)} = -\frac{-\frac{4}{W^2}}{\frac{4}{W}} = \frac{1}{W} \Rightarrow ARA'(W) = -\frac{1}{W^2} < 0, \ \forall\, W > 0$$

$$RRA(W) = -\frac{WU''(W)}{U'(W)} = -W\frac{-\frac{4}{W^2}}{\frac{4}{W}} = 1 \Rightarrow RRA'(W) = 0$$

Therefore, the investor exhibits decreasing absolute risk aversion and constant relative risk aversion, i.e. as her wealth increases she always keeps the same proportion invested in risky assets.

(b) We consider three projects X, Y, Z with only two possible outcomes, 201 and 1, and for each of them we know $\mathbb{E}(W_X) = 101$, $\mathbb{E}(W_Y) = 61$ and $\mathbb{E}(W_Z) = 71$.

(i) Let us consider p_X to be the real probability of the outcome 201 in project X and $(1 - p_X)$ to be the real probability of the outcome 1. Likewise use p_Y and p_Z when dealing with the other two projects. Then we have

$$\begin{array}{rcrcl} \mathbb{E}(W_X) = 101 & \Leftrightarrow & 201p_X + (1-p_X) = 101 & \Leftrightarrow & p_X = 0.5 \\ \mathbb{E}(W_Y) = 61 & \Leftrightarrow & 201p_Y + (1-p_Y) = 61 & \Leftrightarrow & p_Y = 0.3 \\ \mathbb{E}(W_Z) = 71 & \Leftrightarrow & 201p_Z + (1-p_Z) = 71 & \Leftrightarrow & p_Z = 0.35 \end{array}$$

(ii) Using the probabilities from (i) we can determine the expected utility associated with each project. We have,

$$\begin{aligned} \mathbb{E}[U(W_X)] &= (1-p_X)U(1) + p_X U(201) = 0.5 \times 2 + 0.5 \times 23.2132 = 12.6066 \\ \mathbb{E}[U(W_Y)] &= (1-p_Y)U(1) + p_Y U(201) = 0.7 \times 2 + 0.3 \times 23.2132 = 8.3640 \\ \mathbb{E}[U(W_Z)] &= (1-p_Z)U(1) + p_Z U(201) = 0.65 \times 2 + 0.35 \times 23.2132 = 9.4246 \end{aligned}$$

and the ranking is $X \succ Z \succ Y$.

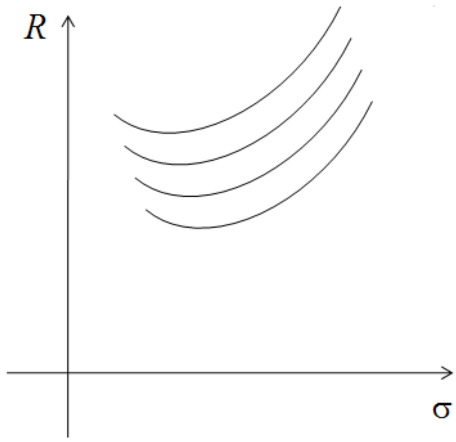


Figure 13: Exercise **??** - Indifference Curves

(iii) The certainty equivalent of project X, C_X is the certain amount that gives the investor the same utility as the expected utility of project X. Likewise for C_Y and C_Z for projects Y and Z, respectively. The risk premia is defined as $\pi_X = E(W_X) - C_X$ and likelwise for π_Y, π_Z.

$$U(C_X) = \mathbb{E}\left[U(W_X)\right] \Leftrightarrow 2 + 4\ln(C_X) = 12.6066 \Leftrightarrow C_X = e^{\frac{12.6066-2}{4}} = 14.1774$$
$$U(C_Y) = \mathbb{E}\left[U(W_Y)\right] \Leftrightarrow 2 + 4\ln(C_Y) = 8.3640 \Leftrightarrow C_Y = e^{\frac{8.3640-2}{4}} = 4.9086$$
$$U(C_Z) = \mathbb{E}\left[U(W_Z)\right] \Leftrightarrow 2 + 4\ln(C_Z) = 9.4246 \Leftrightarrow C_Z = e^{\frac{9.4246-2}{4}} = 6.3991$$

therefore, $\pi_X = 101 - 14.1774 = 86.8225$, $\pi_Y = 61 - 4.9086 = 56.0914$ and $\pi_Z = 71 - 6.3991 = 64.6009$.

(c) Since the new utility function is a linear transformation of the original function

$$V(W) = 2U(W) - 4 = 2\left(2 + 4\ln W\right) - 4 = 4 + 8\ln -4 = 8\ln W$$

and taking into account that the new information on expected payoffs is irrelevant because what matters are expected utilities, the three projects are now ordered exactly in the same way: $X \succ Y \succ Z$.

Exercise 3.13.

(a) To study her risk profile we need to take the first and the second derivative of the utility function $W - 6W^2$ with $W < 1/12$. So,

$$U'(W) = 1 - 12W > 0 \text{ for } W < 1/12, \qquad \text{and} \qquad U''(W) = -12 < 0\ .$$

Thus, the investor prefers more to less, as long as $W < 1/12$, and his risk averse. The indifference curves are plotted in Figure **??**.

(b) Absolute and relative risk aversion are as follows

$$ARA(W) = -\frac{U''(W)}{U'(W)} = \frac{12}{1-12W} \qquad ARA'(W) = \frac{144}{(1-12W)^2} > 0$$

$$RRA(W) = -\frac{WU''(W)}{U'(W)} = \frac{12W}{1-12W} \qquad RRA'(W) = \frac{12}{(1-12W)^2} > 0$$

Therefore, the investor exhibits increasing absolute and relative risk aversion, i.e. as her wealth increases she reduces the amount and the proportion invested in risky assets.

(c) While absolute risk aversion measures the variation in the amount invested in risky assets as a function of wealth, the relative risk aversion measures the change in the proportion invested in risky assets provoked by a variation in wealth.

Exercise 3.14.

(a) The risk tolerance function (RTF) $f(\sigma, \bar{R})$ is nothing but the mean-variance representation of the expected value of the utility function $U(W)$.

Utility functions are defined in terms of final wealth, while RTF are defined in terms of returns, but we can always write $W = W_0(1 + R)$. For some utility functions we may not get a closed-form expression for $f(\sigma, \bar{R})$, that only happens in special cases or whenever returns follow a distribution for which $\bar{R}$ and σ are sufficient statistics.

Indifference curves are level curves of the RTF, i.e., curves along which the expected utility is constant $f(\sigma, \bar{R}) = K$.

(b) For $\bar{R} = \exp(0.7\sigma) + K$ we have

$$\left(\frac{\partial \bar{R}}{\partial \sigma}\right)_{IC} = 0.7 \exp(0.7\sigma) > 0$$

It is only possible to keep the same K level of expected utility is higher risk levels are associated with higher expected returns, so we can conclude the investor is risk-averse.

(c) If the efficient frontier is given by $\bar{R} = 0.05 + 0.8\sigma$, then to find the investor optimal we must find the point where the slopes of the indifference curves and the efficient frontier are the same.

$$\left(\frac{\partial \bar{R}}{\partial \sigma}\right)_{IC} = \left(\frac{\partial \bar{R}}{\partial \sigma}\right)_{EF}$$

$$0.7 \exp(0.7\sigma^*) = 0.8$$

$$\sigma^* = \frac{\log\left(\frac{0.8}{0.7}\right)}{0.7} = 0.1907$$

Exercise 3.15. Solved during lectures.

Exercise 3.16.

(a) For a two assets portfolio the risk is

$$\sigma_P^2 = X_A^2\sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2X_A(1 - X_A)\sigma_{AB}$$

In this case we know $\sigma_{AB} = 0$ and we pretend $\sigma_P^2 = (9.22\%)^2$. Thus,

$$\sigma_P^2 = X_A^2\sigma_A^2 + (1 - X_A)^2 \sigma_B^2$$

$$0.0085 = (10\%)^2 X_A^2 + (20\%)^2 (1 - X_A)^2$$

$$P_1 : \ X_A = 0.9 \wedge X_B = 0.1 \vee P_2 \ X_A = 0.7 \wedge X_B = 0.3$$

Since,

$$\bar{R}_{P_1} = 0.9 \times 8\% + 0.1 \times 12\% = 8.4\%$$

$$\bar{R}_{P_2} = 0.7 \times 8\% + 0.3 \times 12\% = 9.2\%$$

Only P_2 is efficient. Therefore, $\{(X_A, X_B); (0.7, 0.3)\}$ and $R_P = 9.2\%$.

(b) For a two assets portfolio the return is

$$\bar{R}_P = X_A \bar{R}_A + (1 - X_A)\,\bar{R}_B$$

In this case we want to find a portfolio with a return of 11%, so

$$\bar{R}_P = X_A \bar{R}_A + (1 - X_A)\,\bar{R}_B$$

$$11\% = 8\% X_A + 12\%\,(1 - X_A)$$

$$X_A = 0.25 \wedge X_B = 0.75$$

Consequently, the portfolio's variance is

$$\begin{aligned}\sigma_P^2 &= X_A^2 \sigma_A^2 + (1 - X_A)^2\,\sigma_B^2 \\ &= 0.25^2 \times (10\%)^2 + 0.75^2 \times (20\%)^2 \\ &= 0.023125\end{aligned}$$

and its risk is $\sigma_P = 15.21\%$.

(c) To find the tangent portfolio between the capital market line and the efficient frontier of risky assets we have to solve the following system of simultaneous equations in order to Z_i, $\forall i \geqslant 0$

$$\begin{cases}\bar{R}_1 - R_F = Z_1\sigma_1^2 + Z_2\sigma_{12} + Z_3\sigma_{13} + \cdots + Z_N\sigma_{1N} \\ \bar{R}_2 - R_F = Z_1\sigma_{21} + Z_2\sigma_2^2 + Z_3\sigma_{23} + \cdots + Z_N\sigma_{2N} \\ \bar{R}_3 - R_F = Z_1\sigma_{31} + Z_2\sigma_{32} + Z_3\sigma_3^2 + \cdots + Z_N\sigma_{3N} \\ \vdots \\ \bar{R}_N - R_F = Z_1\sigma_{NN}^2 + Z_2\sigma_{N2} + Z_3\sigma_{N3} + \cdots + Z_N\sigma_N^2\end{cases}$$

which can be written using matricial notation

$$\left(\bar{R} - R_F 1\right) = \Sigma Z \qquad \Leftrightarrow \qquad \begin{pmatrix} 8\% - 4\% \\ \bar{1}2\% - 4\% \\ \bar{1}5\% - 4\% \end{pmatrix} = \begin{pmatrix} 0,01 & 0 & 0 \\ 0 & 0,04 & -0,03 \\ 0 & -0,03 & 0,0625 \end{pmatrix} Z$$

where we have used $\sigma_{BC} = \rho_{BC}\sigma_B\sigma_C = -0.6 \times 20\% \times 25\% = -0.03$.
Solving the above equation we get

$$Z = \Sigma^{-1}\left(\bar{R} - R_F 1\right)$$

where Σ^{-1} is the inverse covariance matrix, $\bar{R}$ is a column vector with the securities returns, R_F is a scalar and 1 is a column vector of 1s. Applying this last equation

$$Z = \Sigma^{-1}(R - R_F 1) = \begin{pmatrix} 100.0000 & 0.0000 & 0.0000 \\ 0.0000 & 39.0625 & 18.7500 \\ 0.0000 & 18.7500 & 25.0000 \end{pmatrix} \begin{pmatrix} 8\% - 4\% \\ 12\% - 4\% \\ 15\% - 4\% \end{pmatrix} = \begin{pmatrix} 4 \\ 5.1875 \\ 4.25 \end{pmatrix}$$

The Zs are proportional to the optimum amount to invest in each security. Then the optimum proportions to invest in stock k is X_k, where

$$X_k = \frac{Z_k}{\sum\limits_{i=1}^{N} Z_i}$$

Thus,

$$\begin{pmatrix} X_A \\ X_B \\ X_C \end{pmatrix} = \begin{pmatrix} 4/13.4375 \\ 5.1875/13.4375 \\ 4.25/13.4375 \end{pmatrix} = \begin{pmatrix} 29.77\% \\ 38.60\% \\ 31.63\% \end{pmatrix}$$

(d) The tangency portfolio's return is

$$\bar{R}_T = \sum_{i=1}^{3} X_i \bar{R}_i = 0.2977 \times 8\% + 0.386 \times 12\% + 0.3163 \times 15\% = 11.76\%$$

Since securities A and B and A and C are not correlated, the risk calculation is simplified

$$\begin{aligned}\sigma_T^2 &= \sigma_A^2 X_A^2 + \sigma_B^2 X_B^2 + \sigma_C^2 X_C^2 + 2X_B X_C \sigma_{BC} \\ &= 0.01 \times 0.2977^2 + 0.04 \times 0.3860^2 + 0.0625 \times 0.3163^2 + 2 \times 0.3860 \times 0.3163 \times (-0.03) \\ &= 0.005773\end{aligned}$$

Thus, the portfolio's risk is $\sigma_T = 7.60\%$.

The efficient frontier is given by the line:

$$\bar{R}_P = R_F + \frac{\bar{R}_T - R_F}{\sigma_T}\sigma_P = 4\% + \frac{11.76\% - 4\%}{7.59\%}\sigma_P = 4\% + 1.022\sigma_P$$

(e) (i) The indifference curves are give by $\bar{R} = 0.5\sigma^2 + 0.965\sigma + 0.01K$, and we have,

$$\frac{\partial \bar{R}^{IC}}{\partial \sigma} = \sigma + 0.965 > 0, \quad \text{for all } \sigma > 0.$$

Since the indifference curves are upward slopping in the space $(\sigma, \bar{R})$, we can conclude the investor is risk averse.

(ii) The investment decision criterion is to maximize the investor's expected utility subject to the efficient frontier. In this case we are given indifference curves, of each K level of expected utility. So we just need do equal the slopes of the indifference curves to the slope of the efficient frontier to find the optimal portfolio's risk. Let us denote the optimal portfolio with the letter P. Therefore,

$$\frac{\partial \bar{R}^{EF}}{\partial \sigma} = \frac{\partial \bar{R}^{IC}}{\partial \sigma}$$

$$1.022 = \sigma_P + 0.965$$

$$\sigma_P = 5.7\%$$

Remember that this optimal portfolio is compose by risk free and portfolio T, so that its risk is $\sigma_P = X_T \sigma_T$. Therefore, the weight of portfolio T in the optimal portfolio is

$$X_T = \frac{\sigma_P}{\sigma_T} = \frac{5.7\%}{7.60\%} = 0.75$$

And, of course, $X_F = 1 - X_T = 1 - 0.75 = 0.25$. Thus, she must invest 75% in portfolio T, which corresponds to

$$0.75X_T = 0.75\begin{pmatrix} 0.2977 \\ 0.3860 \\ 0.3163 \end{pmatrix} \Longrightarrow \begin{cases} X_A = 0.2233 \\ X_B = 0.2895 \\ X_C = 0.2372 \end{cases}$$

and 25% in the risk free asset. Therefore, she will invest

$$\text{Investment} = 400,000\begin{pmatrix} 0.2233 \\ 0.2895 \\ 0.2372 \\ 0.25 \end{pmatrix} \Longrightarrow \begin{cases} X_A = 89,302 \\ X_B = 115,814 \\ X_C = 94,884 \\ X_F = 100,000 \end{cases}$$

(iii) From the indifference curves $\bar{R} = 0.5\sigma^2 + 0.965\sigma + 0.01K$ we know K is the fixed expected utility level , for the three portfolios under analysis we have

$$
\begin{aligned}
&\bar{R}_T = 0.5\sigma_T^2 + 0.965\sigma_T + 0.01K_T \\
&11.76\% = 0.5(7.60\%)^2 + 0.965(7.60\%) + 0.01K_T \qquad \Longrightarrow K_T = 4,137
\end{aligned}
$$

$$
\begin{aligned}
&\bar{R}_O = 0.5\sigma_O^2 + 0.965\sigma_O + 0.01K_O \\
&9.82\% = 0.5(5.70\%)^2 + 0.965(5.70\%) + 0.01K_O \qquad \Longrightarrow K_O = 4,156
\end{aligned}
$$

$$
\begin{aligned}
&\bar{R}_F = 0.5\sigma_F^2 + 0.965\sigma_F + 0.01K_F \\
&4\% = 0.5(0\%)^2 + 0.965(0\%) + 0.01K_F \qquad \Longrightarrow K_F = 4
\end{aligned}
$$

from what we can conclude the investor preferences are $O \succ T \succ F$.

(f) (i) The RTF is nothing but the expected value of the utility function, with domain in the space $(\sigma, \bar{R})$. For the log utility we have

$$
\begin{aligned}
\mathbb{E}\left(U(W)\right) &= \mathbb{E}\left(\ln(W)\right) \\
&= \mathbb{E}\left(\ln(W_0(1+R))\right) \\
&= \ln(W_0) + \mathbb{E}\left(\ln(1+R)\right)
\end{aligned}
$$

and, for a general distribution of R, the last expectation cannot be written in terms of $\sigma = Var(R)$ and $\bar{R} = \mathbb{E}(R)$.

(ii) Using a second-order Taylor approximation around W_0 we get

$$
\begin{aligned}
U(W) &\approx U(W_0) + (W - W_0)U'(W_0) + \frac{1}{2}(W - W_0)^2 U''(W_0) \\
\ln(W) &\approx \ln(W_0) + \frac{W - W_0}{W_0} - \frac{1}{2}\frac{(W - W_0)^2}{W_0^2} \\
\ln(W) &\approx \ln(W_0) + R - \frac{1}{2}(R^2)
\end{aligned}
$$

where we used $U'(W) = 1/W$ and $U''(W) = -1/W^2$ and $W = W_0(1 + R)$.
The approximation to the RTF is thus

$$
\begin{aligned}
f(\sigma, \bar{R}) &\approx \mathbb{E}\left[\ln(W_0) + R - \frac{1}{2}(R^2)\right] \\
&\approx \ln(W_0) + \bar{R} - \frac{1}{2}\mathbb{E}(R^2) \\
&\approx \ln(W_0) + \bar{R} - \frac{1}{2}(\sigma^2 + \bar{R}^2)
\end{aligned}
$$

(iii) Recall the efficient frontier is

$$\bar{R}_P = 4\% + 1.022\sigma_P$$

The optimum to the log investor is to maximize the approximation to his RTF which is equivalent to

$$
\begin{aligned}
\max_{P} \quad & \bar{R}_P - \frac{1}{2}(\sigma_P^2 + \bar{R}_p^2) \\
\text{s.t.} \quad & \bar{R}_P = 4\% + 1.022\sigma_P
\end{aligned}
$$

Using the restriction in the objective function we get

$$\max_{\sigma_P} \quad (4\% + 1.022\sigma_P) - \frac{1}{2}(\sigma_P^2 + (4\% + 1.022\sigma_P)^2)$$

From the FCO we get

$$\begin{aligned} 1.022 - \frac{1}{2}\left(2\sigma_P^* + 2(4\% + 1.022\sigma_P^*)1.022\right) &= 0 \\ 1.022(1 - 0.04) - (1 + (1.022)^2)\sigma_P^* &= 0 \\ \sigma_P^* &= 0.4799 \end{aligned}$$

So, the log-investor has an optimal risk level of 47.99% and thus he should invest

$$x = \frac{47.99\%}{7.60\%} = 631.57\% \implies x_F = -531.57\% \ ,$$

assuming he faces no limits on borrowing, the optimal is to borrow 531.57% to invest 631.57% in the tangent portfolio.

(iv) Indifference curves are curves of fixed expected utility, i.e. fixed levels of the RTF, $f(\sigma, \bar{R}) = K$. Using the Taylor approximation in (ii) we have

$$\ln(W_0) + \bar{R} - \frac{1}{2}(\sigma^2 + \bar{R}^2) = K$$

Solving w.r.t. $\bar{R}$ would give us a quadratic form, so in this case it is easier to solve w.r.t. σ^2. We get

$$IC: \qquad \sigma^2 = 2\left(\ln(W_0) - K\right) + 2\bar{R} - \bar{R}^2$$

(v) Now we need to re-write the efficient frontier also w.r.t. σ^2, so we can compare its slope with the slope of the IC above.

$$EF: \qquad \bar{R} = 0.04 + 1.022\sigma \qquad \implies \qquad \sigma^2 = \left(\frac{\bar{R} - 0.04}{1.022}\right)^2$$

The two curves will have the same slope at

$$\begin{aligned} \left(\frac{\partial \sigma^2}{\partial \bar{R}}\right)_{IC} &= \left(\frac{\partial \sigma^2}{\partial \bar{R}}\right)_{EF} \\ 2 - 2\bar{R}^* &= 2\frac{\bar{R}^* - 0.04}{1.022}\frac{1}{1.022} \\ \bar{R}^* &= \frac{(1.022)^2 + 0.04}{1 + (1.022)^2} = 53\% \end{aligned}$$

An expected return of 53% is only possible if we leverage a lot to invest in T, concretely

$$53\% = (1 - x)4\% + x * 11.76\% \qquad \implies x = 631,57\%.$$

As expected we get exactly the same optimum as in (iii).

(g) Any investor who is risk neutral, cares only about maximising the expected return of investments. In the market situation of the exercise, when we can both lend and borrow at the same rate R_F without limits, it is always possible to borrow a bit more to increase the expected return. Without loss of generality – as the investor is indifferent between all investments with the same $\bar{R}$, we can focus on the efficient frontier to show the optimal risk level is $\sigma^*_{neutral} = +\infty$.

To see this note that

$$\max_{P} \quad \bar{R}_P \qquad \Leftrightarrow \qquad \max_{\sigma_p} \quad 4\% + 1.022\sigma_p \qquad \implies \qquad \sigma^*_{neutral} = +\infty$$

s.t. EF

(h) In the case of the risk lover we can focus on efficient portfolios, because for any fixed risk level, those are the ones that maximize expected return and a risk lover likes both risk and expected return. His optimum can be understood as, first maximize risk and then for the maximal risk maximize expected return. Or, maximize risk along the efficient frontier.

Recall the efficient frontier can be written both in terms of $\bar{R}_P = 0.04 + 1.022\sigma_P$ or $\sigma_P = \dfrac{\bar{R}_P - 0.04}{1.022}$.

Formally we can write

$$\max_{P} \quad \sigma_P \quad \Leftrightarrow \quad \max_{\bar{R}_P} \quad \frac{\bar{R}_P - 4\%}{1.022} \quad \Longrightarrow \quad \bar{R}^*_{lover} = +\infty$$
$$\text{s.t. EF}$$

3.2 Alternatives Techniques

Exercise 3.17.

(a) The geometric mean is given by

$$\bar{R}^G_j = \prod_{i=1}^{N} \left(1 + \bar{R}_{ij}\right)^{P_{ij}} - 1$$

Therefore, the geometric mean returns of the outcomes shown in Exercise **??** (assuming an initial investment of 100) are:

$$\bar{R}^G_A = \prod_{i=1}^{3} \left(1 + \bar{R}_{iA}\right)^{P_{iA}} - 1 = 1.05^{1/3} \times 1.06^{1/3} \times 1.09^{1/3} - 1 = 0.0665$$

$$\bar{R}^G_B = \prod_{i=1}^{3} \left(1 + \bar{R}_{iB}\right)^{P_{iB}} - 1 = 1.04^{1/4} \times 1.07^{1/2} \times 1.10^{1/4} - 1 = 0.0698$$

$$\bar{R}^G_C = \prod_{i=1}^{3} \left(1 + \bar{R}_{iC}\right)^{P_{iC}} - 1 = 1.01^{1/5} \times 1.09^{3/5} \times 1.18^{1/5} - 1 = 0.0907$$

Thus $C \succ B \succ A$.

(b) The idea of maximizing the geometric mean return to chose the optimal portfolio is supported by two main arguments:

1. has the highest return probability of reaching, or exceeding, any given wealth level in the shortest possible time; and
2. has the highest probability of exceeding any given wealth level over any given period of time.

Exercise 3.18.

(a) To use the stochastic dominance criterion we need to calculate the accumulated probability (first order stochastic dominance - FOSD) and the sum of accumulated probabilities (second order stochastic dominance - SOSD). Table **??** exhibits the accumulated and sum of accumulates probability.

Thus, using the accumulated probability we cannot find any FOSD. However, when we consider the sum of accumulated probability, the SOSD allows us to rank the projects, such that $C \succ B \succ A$.

	Accumulated Probability			Sum of Accumulated Probability		
Return	A	B	C	A	B	C
4%	0.2	0.0	0.0	0.2	0.0	0.0
5%	0.2	0.1	0.0	0.4	0.1	0.0
6%	0.5	0.4	0.4	0.9	0.5	0.4
7%	0.5	0.6	0.7	1.4	1.1	1.1
8%	0.9	0.9	0.9	2.3	2.0	2.0
9%	0.9	1.0	0.9	3.2	3.0	2.9
10%	1.0	1.0	1.0	4.2	4.0	3.9

Table 5: Exercise **??** - FOSD and SOSD

(b) Any risk averse investor would choose the same ranking as above. So any utility function with $U'(.) > 0$ and $U''(.) < 0$ would do. Log, negative exponencial, etc.

(c) Roy's safety first criterion is to minimize $Prob\,(R_P < R_L)$. Then,

$$Prob\,(R_A < 5\%) = 0.2; \quad Prob\,(R_B < 5\%) = 0.0; \quad Prob\,(R_C < 5\%) = 0.0$$

Therefore, under this decision criterion, investments B and C are preferable than investment A, but to the investor investments B and C are indifferent, $B \backsim C \succ A$.

(c) Kataoka's safety first criterion is to maximize R_L subject to $Prob\,(R_P < R_L) \leqslant \alpha$. For $\alpha = 10\%$, maximum R_L for each of the three possible investments is

$$I_A: \; R_L = 4\%; \quad I_B: \; R_L = 6\%; \quad I_C: \; R_L = 6\%$$

As before B and C are preferable to A, but B and C are indifferent, $B \backsim C \succ A$.

(d) Telser's safety first criterion is maximize $\bar{R}_P$ subject to $Prob\,(R_P \leqslant R_L) \leqslant \alpha$. In this problem, the restriction is $Prob\,(R_P \leqslant 0.5) \leqslant 0.1$, which excludes investment A, because $Prob\,(R_A \leqslant 0.5) = 0.2$ what does not respect the restriction. Investments B and C respect the restriction ($Prob\,(R_B \leqslant 0.5) = 0.1 \wedge Prob\,(R_C \leqslant 0.5) = 0.0$). However, these two investments are not indifferent as before. Actually, Telser's objective is to maximize $\bar{R}_P$, so that we must chose the investment with higher expected return. Thus,

$$\bar{R}_B = \sum_{i=1}^{5} P_{B_i} R_{B_i} = 0.1 \times 5 + 0.3 \times 6 + 0.2 \times 7 + 0.3 \times 8 + 0.1 \times 9 = 7$$

$$\bar{R}_C = \sum_{i=1}^{4} P_{C_i} R_{C_i} = 0.4 \times 6 + 0.3 \times 7 + 0.2 \times 8 + 0.1 \times 10 = 7.1$$

Then $\bar{R}_C > \bar{R}_B \Rightarrow C \succ B$.

(e) The geometric mean is given by

$$\bar{R}_j^G = \prod_{i=1}^{N} \left(1 + \bar{R}_{ij}\right)^{P_{ij}} - 1$$

Therefore, the geometric mean returns of the outcomes are:

$$\bar{R}_A^G = \prod_{i=1}^{4} \left(1 + \bar{R}_{A_i}\right)^{P_{A_i}} - 1 = 1.04^{0.2} \times 1.06^{0.3} \times 1.08^{0.4} \times 1.1^{0.1} - 1 = 0.0678$$

$$\bar{R}_B^G = \prod_{i=1}^{5} \left(1 + \bar{R}_{B_i}\right)^{P_{B_i}} - 1 = 1.05^{0.1} \times 1.06^{0.3} \times 1.07^{0.2} \times 1.08^{0.3} \times 1.09^{0.1} - 1 = 0.0699$$

$$\bar{R}_C^G = \prod_{i=1}^{4} \left(1 + \bar{R}_{C_i}\right)^{P_{C_i}} - 1 = 1.06^{0.4} \times 1.07^{0.3} \times 1.08^{0.2} \times 1.1^{0.1} - 1 = 0.0709$$

Thus $C \succ B \succ A$.

Exercise 3.19.

(a) The solution to this exercise is similar to that one of Exercise **??**. However, we now have a continuous distribution what makes the calculations considerably more nasty if done with bare hands and qualifies the exercise to be solved using Excel or a similar software. So you may want to ask your instructor the excel file with the solution. Nevertheless we present the charts with the FOSD and SOSD (see Figure **??**), from which we can conclude that none of these investments show FOSD or SOSD over the remaining ones.

(b) Recall that Roy's safety first criterion is to minimize $Prob\left(R_P < R_L\right)$. Therefore we want to calculate the following probabilities and rank them accordingly

$$\Pr\left(R_A < 5\%\right); \qquad \Pr\left(R_B < 5\%\right); \qquad \Pr\left(R_C < 5\%\right)$$

Since, the returns follow normal distributions that are not standardised, we need to standardise them. Recall that,

$$\frac{R_A - \bar{R}_A}{\sigma_A} = Z \backsim N\left(0, 1\right)$$

Then,

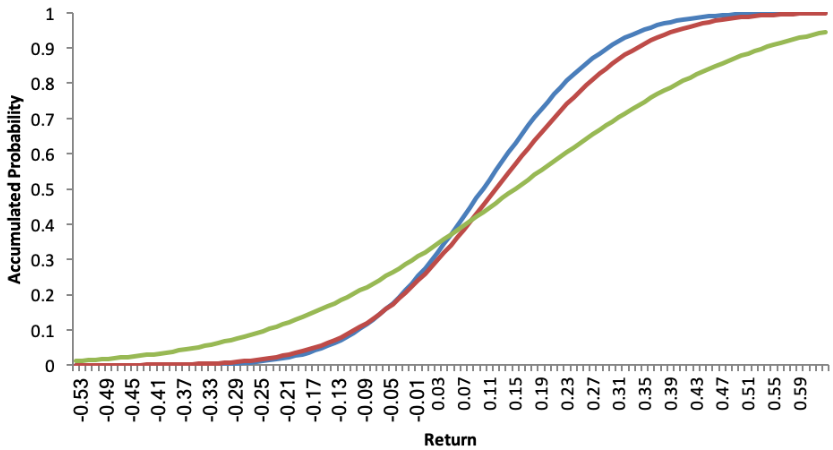
$$\Pr\left(R_A < 5\%\right) = \Pr\left(\frac{R_A - \bar{R}_A}{\sigma_A} < \frac{0.05 - 0.1}{0.15}\right) = \Pr\left(Z_A < -\frac{1}{3}\right) = N\left(-\frac{1}{3}\right) = 0.3694$$

$$\Pr\left(R_B < 5\%\right) = \Pr\left(\frac{R_B - \bar{R}_B}{\sigma_B} < \frac{0.05 - 0.12}{0.17}\right) = \Pr\left(Z_B < -0.41176\right) = N\left(-0.41176\right) = 0.3400$$

$$\Pr\left(R_C < 5\%\right) = \Pr\left(\frac{R_C - \bar{R}_C}{\sigma_C} < \frac{0.05 - 0.15}{0.30}\right) = \Pr\left(Z_C < -\frac{1}{3}\right) = N\left(-\frac{1}{3}\right) = 0.3694$$

Therefore, under this decision criterion, investments B is preferable than investment A and C, which are indifferent, $B \succ A \backsim C$.

(a)



(b)

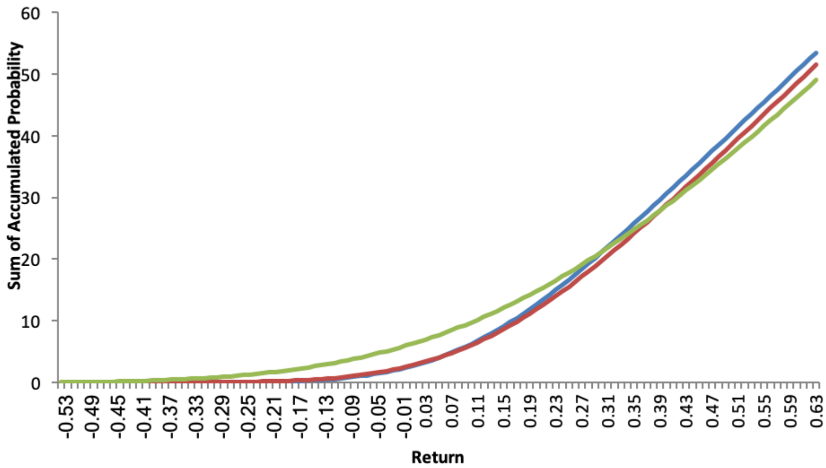


Figure 14: Exercise **??** – first (a) and second-order (b) stochastic dominance graphs.

(c) Kataoka's safety first criterion is to maximize R_L subject to $Prob\,(R_P < R_L) \leqslant \alpha$. For $\alpha = 10\%$, maximum R_L for each of the three possible investments is:

– Investment A

$$Prob\,(R_A \leqslant R_{L_A}) \leqslant \alpha$$

$$Prob\left(Z_A \leqslant \frac{R_{L_A} - \bar{R}_A}{\sigma_A}\right) \leqslant \alpha$$

$$Prob\left(Z_A \leqslant \frac{R_{L_A} - 0.1}{0.15}\right) \leqslant 0.1$$

$$\frac{R_{L_A} - 0.1}{0.15} \geqslant -1.282$$

$$R_{L_A} \geqslant -0.0923$$

$$R_{L_A} = -0.0922$$

- Investment B

$$Prob\,(R_B \leqslant R_{L_B}) \leqslant \alpha$$

$$Prob\left(Z_B \leqslant \frac{R_{L_B} - \bar{R}_B}{\sigma_B}\right) \leqslant \alpha$$

$$Prob\left(Z_B \leqslant \frac{R_{L_B} - 0.12}{0.17}\right) \leqslant 0.1$$

$$\frac{R_{L_B} - 0.12}{0.17} \geqslant -1.282$$

$$R_{L_B} \geqslant -0.0979$$

$$R_{L_B} = -0.0978$$

- Investment C

$$Prob\,(R_C \leqslant R_{L_C}) \leqslant \alpha$$

$$Prob\left(Z_C \leqslant \frac{R_{L_C} - \bar{R}_C}{\sigma_C}\right) \leqslant \alpha$$

$$Prob\left(Z_C \leqslant \frac{R_{L_C} - 0.15}{0.30}\right) \leqslant 0.1$$

$$\frac{R_{L_C} - 0.15}{0.30} \geqslant -1.282$$

$$R_{L_C} \geqslant -0.2346$$

$$R_{L_C} = -0.2345$$

Thus, A is preferable to B that is preferable to C, $A \succ B \succ C$.

(d) Telser's safety first criterion is maximize $\bar{R}_P$ subject to $Prob\,(R_P \leqslant R_L) \leqslant \alpha$. In this problem, the restriction is $Prob\,(R_P \leqslant 0.5) \leqslant 0.1$, which excludes the three investments, since

$$Prob\,(R_A \leqslant 0.5) = 0.3694 \nleqslant 0.1$$

$$Prob\,(R_B \leqslant 0.5) = 0.3400 \nleqslant 0.1$$

$$Prob\,(R_C \leqslant 0.5) = 0.3694 \nleqslant 0.1$$

(e) The Value at Risk is given by $\bar{R}_i - Z_\alpha \sigma_i$. Since we set $\alpha = 0.025$ we have $Z_{0.025} = 1.96$. Therefore,

$$VaR_A = \bar{R}_A - 1.96\sigma_A = 0.1 - 1.96 \times 0.15 = -0.196$$

$$VaR_B = \bar{R}_B - 1.96\sigma_B = 0.12 - 1.96 \times 0.17 = -0.2139$$

$$VaR_C = \bar{R}_C - 1.96\sigma_C = 0.15 - 1.96 \times 0.30 = -0.4392$$

Thus, A is preferable to B that is preferable to C, $A \succ B \succ C$.

4 Equilibrium in Financial Markets

4.1 CAPM

Exercise 4.1.

(a) Using the single-index model, the risk of a security i is given by $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{e_i}^2$, where the first term is the systematic risk and the second term is the specific risk. Using in the expression the values given in the problem

$$\sigma_A^2 = \beta_A^2 \sigma_m^2 + \sigma_{e_A}^2 = 1.5^2 + 0.5^2 + 0.05 = 0.6125$$

Therefore the risk is $\sigma_A = 0.783$

(b) If the specific risk is null, then $\sigma_{eC}^2 = 0$. Security's C variance is $\sigma_C^2 = 0.75$. Thus, using the single-index model the β of C is

$$\begin{aligned} \sigma_C^2 &= \beta_C^2 \sigma_m^2 + \sigma_{eC}^2 \\ 0.75 &= \beta_C^2 \times 0.25 + 0 \\ \beta_C &= 1.73205 \end{aligned}$$

(c) From CAPM we know the return of a security is $R_A = R_f + \beta\,(R_m - R_f)$. From the data we know $R_A = 20\%$ and security B is risk-free ($\beta = 0$), so that the risk-free interest rate is 10%. Thus,

$$\begin{aligned} \bar{R}_A &= R_f + \beta\left(\bar{R}_m - R_f\right) \\ 0.2 &= 0.1 + 1.5(\bar{R}_m - 0.1) \\ \bar{R}_m &= \frac{0.25}{1.5} \\ &= 0.1667 \end{aligned}$$

(d) These assumptions are those of CAPM. See your notes or the textbook.

Exercise 4.2.

(a) From CAPM we know the return of a security is $R_A = R_f + \beta\,(R_m - R_f)$ and its β is $\beta = \frac{\sigma_{i,m}}{\sigma_m^2}$. Since the market risk is 0.1, its variance is $\sigma_m^2 = 0.01$. The covariance between asset's i return and the market return is given by $\sigma_{i,m} = \sigma_i \sigma_m \rho_{i,m}$. Finally, $\rho_{i,m} = 1$, since security i is perfectly correlated with the market. So, using the given data, $\sigma_{i,m} = 0.2 \times 0.1 \times 1 = 0.02$. Thus,

$$\beta = \frac{\sigma_{i,m}}{\sigma_m^2} = \frac{0.02}{0.01} = 2$$

and

$$\begin{aligned} R_i &= R_f + \beta\,(R_m - R_f) \\ &= 0.05 + 2\,(0.1 - 0.05) \\ &= 0.15 \end{aligned}$$

(b) The request line is given by the single-index model $R_i = \alpha_i + \beta_i \bar{R}_m$. We know β_i and $\bar{R}_m$. To draw the line we need to find α_i, which is given by the expression $\alpha_i = R_i - \beta_i \bar{R}_m$. In this case, $\alpha_i = 0.15 - 2 \times 0.1 = -0.05$. The line is represented in Figure **??**.

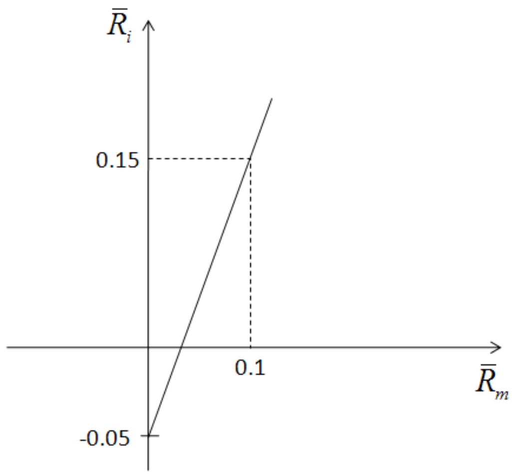


Figure 15: Exercise **??** - Characteristic line

Exercise 4.3.

(a) Using CAPM to calculate the expected return

$$\bar{R}_X = R_f + \beta_X \left(\bar{R}_m - R_f\right) = 0.07 + \beta\,(0.09 - 0.07)$$

β_X can be found using $\beta_X = \frac{\sigma_{Xm}}{\sigma_m^2}$. Thus

$$\beta_X = \frac{\sigma_{Xm}}{\sigma_m^2} = \frac{0.02}{0.025} = 0.8$$

Finally,

$$\bar{R}_X = 0.07 + 0.8\,(0.09 - 0.07) = 0.086$$

(b) If $\bar{R}_m = 0.12$ then the expected return is

$$\bar{R}_X = R_f + \beta_X \left(\bar{R}_m - R_f\right) = 0.07 + 0.08\,(0.12 - 0.07) = 0.11$$

Since the CAPM's expected return is lower than the market expected return, the price is underpriced.

Exercise 4.4. To know the return of each portfolio to look for an arbitrage opportunity we need to find each portfolio β, which is the weighted average of each security's β, and each portfolio's expected return. Thus

$$\begin{aligned}\beta_1 &= x_{1A}\beta_A + x_{1B}\beta_B + x_{1C}\beta_C \\ &= -0.5 \times 1.5 + 0 \times 1 + 1.5 \times 0.5 \\ &= 0\end{aligned}$$

$$\begin{aligned}\beta_2 &= x_{2A}\beta_A + x_{2B}\beta_B + x_{2C}\beta_C \\ &= 0 \times 1.5 - 1 \times 1 + 2 \times 0.5 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\bar{R}_1 &= x_{1A}\bar{R}_A + x_{1B}\bar{R}_B + x_{1C}\bar{R}_C \\ &= -0.5 \times 0.12 + 0 \times 0.1 + 1.5 \times 0.05 \\ &= 0.015\end{aligned}$$

$$\begin{aligned}\bar{R}_2 &= x_{2A}\bar{R}_A + x_{2B}\bar{R}_B + x_{2C}\bar{R}_C \\ &= 0 \times 0.12 - 1 \times 0.1 + 2 \times 0.05 \\ &= 0\end{aligned}$$

Therefore, we have two risk-free portfolios with different expected returns, implying an arbitrage opportunity. So, without investing a single penny we can short-sale portfolio 2 and buy portfolio 1, earning an arbitrage profit of 1.5%.

Exercise 4.5.

(a) To fill the table given in the exercise we need to find β_m, β_c, $\bar{R_A}$ and $\bar{R_B}$. By definition, $\beta_m = 1$. Since security C is risk-free, its β is null and $R_f = 0.02$. Thus the expected return of securities A and B is

$$\begin{aligned}\bar{R}_A &= R_f + \beta_A\left(\bar{R}_m - R_f\right) \\ &= 0.02 + 0.08 \times 0.5 \\ &= 0.06\end{aligned}$$

$$\begin{aligned}\bar{R}_B &= R_f + \beta_B\left(\bar{R}_m - R_f\right) \\ &= 0.02 + 0.08 \times (-0.1) \\ &= 0.012\end{aligned}$$

(b) Accordingly to the single-index model total risk is

$$\sigma_i^2 = \underbrace{\beta_i^2\sigma_m^2}_{\text{Systematic Variance}} + \underbrace{\sigma_{e_i}^2}_{\text{Specific Variance}}$$

Thus, for security A the systematic variance is $\beta_A^2\sigma_m^2 = 0.5^2 \times 0.04^2 = 0.0004$ and the specific variance is $\sigma_{e_A}^2 = \sigma_A^2 - \beta_A^2\sigma_m^2 = 0.12^2 - 0.0004 = 0.014$. Thus, systematic risk é $\sqrt{0.0004} = 0.02$ and specific risk is $\sqrt{0.014} = 0.1183$. For security B the systematic risk is $\beta_B^2\sigma_m^2 = (-0.1)^2 \times 0.04^2 = 0.000016$ and the specific risk is $\sigma_{e_B}^2 = \sigma_B^2 - \beta_B^2\sigma_m^2 = 0.12^2 - 0.000016 = 0.014384$. Thus, systematic risk é $\sqrt{0.000016} = 0.004$ and specific risk is $\sqrt{0.014384} = 0.1199$.

(c) If CAPM holds any investor has always incentives to compose a portfolio with a risk-free asset and the market portfolio. By holding the market portfolio, well diversified by definition, the investor will eliminate the portfolio's specific risk. If CAPM holds, expectations are homogeneous meaning that all investors share the same expectations, which should imply a very low level of trading. If, for some reason the expected return in the market for a given security is the predict by CAPM, it should means the security is not rewarding properly its systematic risk, therefore, it is not an equilibrium return and we have an arbitrage opportunity. In this case, expectations are temporarily heterogenous, until the market adjust to its equilibrium on the security market line.

Exercise 4.6.

(a) The equation for the security market line is $\bar{R}_i = R_f + \beta_i \left(\bar{R}_m - R_f\right)$. Thus, from the data in the problem we have:

$$\begin{cases} \bar{R}_1 = R_f + \beta_1 \left(\bar{R}_m - R_f\right) \\ \bar{R}_2 = R_f + \beta_2 \left(\bar{R}_m - R_f\right) \end{cases} \Leftrightarrow \begin{cases} 0.06 = R_f + 0.5 \left(\bar{R}_m - R_f\right) \\ 0.12 = R_f + 1.5 \left(\bar{R}_m - R_f\right) \end{cases}$$

Solving in order to $\bar{R}_m$ and R_f,

$$\begin{cases} \bar{R}_m = 0.09 \\ R_f = 0.03 \end{cases}$$

Finally, the the security market line is

$$\bar{R}_i = 0.03 + 0.06\beta_i$$

(b) Using the above security market line, an asset with a beta of 2 would have an expected return of:

$$\bar{R}_i = 0.03 + 0.06\beta_i \qquad = 0.03 + 0.06 \times 2 = 0.15$$

(c) To exploit an arbitrage strategy we need to find a portfolio with asset 1 and asset 2 that replicates the risk ($\beta_p = 1.2$) of the given asset, but with a different return. since the β of a portfolio is the weighted average of each security β and the weights of asset 1 and asset 2 must sum 1, it comes

$$\begin{cases} x_1 + x_2 = 1 \\ \beta_p = x_1\beta_1 + x_2\beta_2 \end{cases} \Leftrightarrow \begin{cases} x_2 = 1 - x_1 \\ 1.2 = 0.5x_1 + 1.5(1 - x_2) \end{cases} \Leftrightarrow \begin{cases} x_1 = 0.3 \\ x_2 = 0.7 \end{cases}$$

The return of this replication portfolio is $R_p = 0.3 \times 0.06 + 0.7 \times 0.12 = 0.102$. Therefore, we have an arbitrage opportunity that can be exploited by short-selling the replication portfolio and buying asset 3, making an arbitrage profit of $0.15 - 0.102 = 0.048$.

Exercise 4.7.

Given the security market line in this problem, for the two stocks to be fairly priced their expected returns must be:

$$\bar{R}_X = 0.04 + 0.08 \times 0.5 = 0.08$$
$$\bar{R}_X = 0.04 + 0.08 \times 2 = 0.2$$

If the expected return on either stock is higher than its return given above, the stock is a good buy.

Exercise 4.8.

Given the security market line in this problem, the two funds' expected returns would be:

$$\bar{R}_A = 0.04 + 0.19 \times 0.8 = 0.192 > 0.1 \rightarrow \text{ bad performance}$$

$$\bar{R}_B = 0.04 + 0.19 \times 1.2 = 0.268 > 0.15 \rightarrow \text{ bad performance}$$

Comparing the above returns to the funds' actual returns, we see that both funds performed poorly, since their actual returns were below those expected given their beta risk.

Exercise 4.9. Part (a) and Part (b) can be answered simultaneously.
The security market line is:

$$\bar{R}_i = R_f + \beta\left(\bar{R}_m - R_f\right)$$

Substituting the given values for assets 1 and 2 gives two equations with two unknowns and solving simultaneously gives:

$$\begin{cases} 0.094 = R_f + 0.8\left(\bar{R}_m - R_f\right) \\ 0.134 = R_f + 1.3\left(\bar{R}_m - R_f\right) \end{cases} \Leftrightarrow \begin{cases} \bar{R}_f = 0.03 \\ \bar{R}_m = 0.11 \end{cases}$$

Exercise 4.10. [OBS: this exercise is out of place, it should be in the APT subsection]

A general equilibrium relationship for security returns must imply absence of arbitrage. In this case we consider systematic risk to be concerned with market risk and interest rate risk. So it would be interesting to find an expression that explain returns with two risk factors: market risk; and interest rate risk. To do so, we need to create an arbitrage portfolio as follows:

$$\sum_i X_i^{ARB} \times 1 = 0 \tag{10}$$

$$a_{ARB} = \sum_i X_i^{ARB} a_i = 0 \tag{11}$$

$$b_{ARB} = \sum_i X_i^{ARB} b_i = 0 \tag{12}$$

Since the above portfolio has zero net investment and zero risk with respect to the given two-factor model, by the force of arbitrage its expected return must also be zero:

$$\bar{R}_{ARB} = \sum_i X_i^{ARB} \bar{R}_i = 0 \tag{13}$$

From a theorem of linear algebra, since the above orthogonality conditions (**??**), (**??**) and (**??**) with respect to the X_i^{ARB} result in orthogonality condition (**??**) with respect to the X_i^{ARB}, $\bar{R}_i$ can be expressed as a linear combination of 1, a_i and b_i:

$$\bar{R}_i = \lambda_0 \times 1 + \lambda_1 a_i + \lambda_2 b_i \tag{14}$$

We can create a zero-risk investment portfolio (without systematic risk) to find λ_0 as follows:

$$\sum_i X_i^Z = 1$$

$$a_Z = \sum_i X_i^Z a_i = 0$$

$$b_Z = \sum_i X_i^Z b_i = 0$$

Substituting the above equations into equation (**??**) gives:

$$\bar{R}_Z = \sum_i X_i^Z \bar{R}_i = \lambda_0 \sum_i X_i^Z + \lambda_1 \sum_i X_i^Z a_i + \lambda_2 \sum_i X_i^Z b_i = \lambda_0$$

Then, we can create a strictly market-risk investment portfolio to find λ_1 as follows:

$$\sum_i X_i^M = 1$$

$$a_M = \sum_i X_i^M a_i = 1$$

$$b_M = \sum_i X_i^M b_i = 0$$

Substituting the above equations into equation (??) gives:

$$\bar{R}_M = \sum_i X_i^M \bar{R}_i = \lambda_0 \sum_i X_i^M + \lambda_1 \sum_i X_i^M a_i + \lambda_2 \sum_i X_i^M b_i = \lambda_0 + \lambda_1$$

or

$$\lambda_1 = \bar{R}_M - \lambda_0 = \bar{R}_M - \bar{R}_Z$$

Finally, we can create a strictly interest rate-risk investment portfolio to find λ_2 as follows:

$$\sum_i X_i^C = 1$$

$$a_C = \sum_i X_i^C a_i = 0$$

$$b_C = \sum_i X_i^C b_i = 1$$

Substituting the above equations into equation (??) gives:

$$\bar{R}_C = \sum_i X_i^C \bar{R}_i = \lambda_0 \sum_i X_i^C + \lambda_1 \sum_i X_i^C a_i + \lambda_2 \sum_i X_i^C b_i = \lambda_0 + \lambda_2$$

or

$$\lambda_2 = \bar{R}_C - \lambda_0 = \bar{R}_C - \bar{R}_Z$$

Substituting the derived values for λ_0, λ_1 and λ_2 into equation (??), we have:

$$\bar{R}_i = \bar{R}_Z + \left(\bar{R}_M - \bar{R}_Z\right) \times a_i + \left(\bar{R}_C - \bar{R}_Z\right) \times b_i$$

Exercise 4.11.

(a) In the graph (see Figure ??) , the efficient frontier with riskless lending but no riskless borrowing is the ray extending from R_F to the tangent portfolio L and then along the minimum-variance curve through the market portfolio M and out toward infinity (assuming unlimited short sales). All investors who wish to lend will hold tangent portfolio L in some combination with the riskless asset, since no other portfolio offers a higher slope. Furthermore, unless all investors lend or invest solely in portfolio L, the market portfolio M will be along the minimum-variance curve to the right of portfolio L, since the market portfolio is a wealth-weighted average of all the efficient risky-asset portfolios held by investors, and no rational investor would hold a risky-asset portfolio along the curve to the left of L.

The expected return on a zero-beta asset is the intercept of a line tangent to the market portfolio, and the zero-beta portfolio on the minimum-variance frontier must be below the global minimum variance portfolio of risky assets by the geometry of the graph. Furthermore, by the geometry of the graph, since the risk-free lending rate is the intercept of the line tangent to portfolio L, and since L is to the left of M on the minimum-variance curve, the risk-free lending rate must be below the expected return on a zero-beta asset.

(b) The zero-beta security market line is the line in the graph (see Figure ??) extend from the expected return on a zero-beta asset through the market portfolio and out toward infinity (assuming unlimited short sales). The expected return-beta relationships of all risky securities risky-asset portfolios (including the market portfolio M and portfolio L) are described by that line. The other line from the risk-free lending rate to portfolio L only

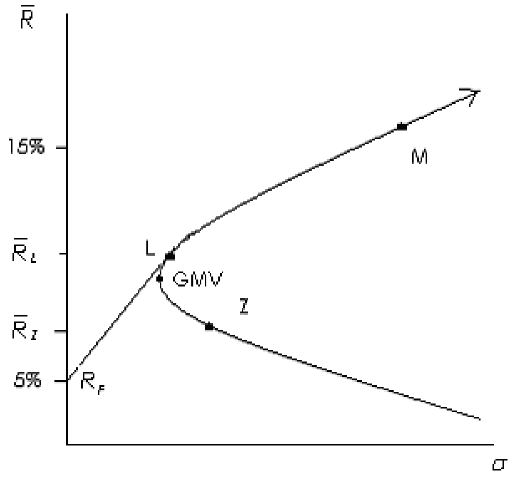


Figure 16: Exercise ?? - Efficient Frontier

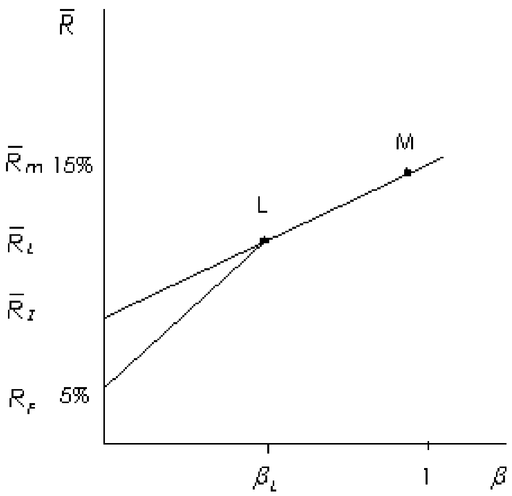


Figure 17: Exercise ?? - Zero-Beta Security Market Line

describes the expected return-beta relationships of combination portfolios of the risk-free asset and portfolio L; those combination portfolios are not described by the zero-beta security market line.

Exercise 4.12. If the post-tax form of the equilibrium pricing model holds, then:

$$\bar{R}_i = R_F + \left[\left(\bar{R}_m - R_F\right) - (\delta_m - R_F)\,\tau\right]\beta_i + (\delta_i - R_F)\,\tau$$

If the standard CAPM model holds, then:

$$\bar{R}_i = R_F + \left(\bar{R}_m - R_F\right)\beta_i$$

Let us assume that the post-tax model holds instead of the standard model, and $\delta_m = R_F$.

Then, for a stock with $(\delta_i - R_F)\,\tau > 0$, if you are right and use the post-tax model, you would correctly believe that the stock has a higher expected return than the stock's return expected by the other investor using the standard model.

Similarly, for a stock with $(\delta_i - R_F)\,\tau < 0$, you would correctly believe the stock has a lower expected return than the stock's return expected by the other investor using again the standard model.

Therefore, if you manage two securities, one with $(\delta_i - R_F)\,\tau > 0$ and the other with $(\delta_i - R_F)\,\tau < 0$, you can swap them with the other investor. Since you both have heterogenous expectations, each one of you will believe that are making an excess return.

Now consider a specific example using the following data for stocks A and B, the market portfolio and the riskless asset:

$$\beta_A = 1.0;\ \delta_A = 8\%;\ \beta_B = 1.0;\ \delta_B = 0\%;\ \bar{R}_M = 14\%;\ \delta_m = 4\%;\ R_F = 4\%;\ \tau = 0.25$$

If the post-tax model holds, then you would correctly believe that the equilibrium expected returns for the two stocks are:

$$\begin{cases} \bar{R}_A = 4 + ((14-4) - (4-4)\times 0.25)\times 1.0 + (8-4)\times 0.25 \\ \bar{R}_B = 4 + ((14-4) - (4-4)\times 0.25)\times 1.0 + (0-4)\times 0.25 \end{cases} \Leftrightarrow \begin{cases} \bar{R}_A = 15\% \\ \bar{R}_B = 13\% \end{cases}$$

While the other investor using the standard model would incorrectly believe that the stocks' equilibrium expected returns are:

$$\begin{cases} \bar{R}_A = 4 + (14-4)\times 1.0 \\ \bar{R}_B = 4 + (14-4)\times 1.0 \end{cases} \Leftrightarrow \begin{cases} \bar{R}_A = 14\% \\ \bar{R}_B = 14\% \end{cases}$$

You would tend to buy stock A and sell stock B short. Of course, residual risk puts a limit to the amount of unbalancing you would do. But by some unbalancing, you earn an excess return. At the same time the other investor using the standard model would be indifferent between the two stocks. If your tax factor was below the aggregate tax factor (τ lower than 0.25) then you should buy stock B from the other investor and sell that investor stock A. The fact that this will lead to higher after-tax cash flows for you is straightforward.

4.2 APT

Exercise 4.13.

(a) If APT's model holds, returns are generated by a multi-index model such that

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$$

Where,

λ_j is the risk premium associated to the risk factor I_j, $j = 1, 2$
b_{ik} is the sensitivity of security i to the risk factor I_j, $j = 1, 2$

To derive the equilibrium model we need to calculate λ_j. Since we know the expected returns for three portfolios X, Y and Z and the sensitivity of each to the risk factors, we can build a equation system with three equations and three variables:

$$\begin{cases} \bar{R}_X = \lambda_0 + \lambda_1 b_{X1} + \lambda_2 b_{X2} \\ \bar{R}_Y = \lambda_0 + \lambda_1 b_{Y1} + \lambda_2 b_{Y2} \\ \bar{R}_Z = \lambda_0 + \lambda_1 b_{Z1} + \lambda_2 b_{Z2} \end{cases} \Leftrightarrow \begin{cases} 0.16 = \lambda_0 + \lambda_1 1 + \lambda_2 0.7 \\ 0.14 = \lambda_0 + \lambda_1 0.6 + \lambda_2 1 \\ 0.11 = \lambda_0 + \lambda_1 0.5 + \lambda_2 1.5 \end{cases} \Leftrightarrow \begin{cases} \lambda_0 = 0.095929 \\ \lambda_1 = 0.0572816 \\ \lambda_2 = 0.009709 \end{cases}$$

Finally,

$$\bar{R}_i = 0.0959 + 0.0573 b_{i1} + 0.0097 b_{i2}$$

(b) If this portfolio does not respect the equilibrium conditions defined in part a, we will find an arbitrage opportunity. Thus, first we need to check the non arbitrage expected return for portfolio W:

$$\begin{aligned} \bar{R}^e_W &= 0.0959 + 0.0573 b_{i1} + 0.0097 b_{i2} \\ &= 0.0959 + 0.0573 \times 1 + 0.0097 \times 0 \\ &= 0.1532 \end{aligned}$$

Since, $\bar{R}^e_W = 0.1489 > \bar{R}_W = 0.13$, this portfolio W is not at equilibrium, allowing the existence of arbitrage opportunities. The low level of the market expected return implies that the current market price is too high, meaning portfolio W is overpriced. Thus, we would like to short sell it and buy a fairly priced portfolio that replicates W's cash flows and risk. The subsequent increase in W's supply will force its price to fall until reach a non arbitrage price, such that $\bar{R}^e_W = \bar{R}_W$.

(c) Recall that APT equilibrium model with a risk-free asset is

$$\bar{R}_i = R_F + b_{i1}\lambda_1 + b_{i2}\lambda_2 \tag{15}$$

and that if the CAPM is the equilibrium model, it holds for all securities, as well as all portfolios of securities. Assume the indexes can be represented by portfolios of securities. Then, if the CAPM holds, the equilibrium return on each λ_j is given by the CAPM or

$$\lambda_1 = \beta_{\lambda 1}\left(\bar{R}_m - R_F\right)$$

$$\lambda_2 = \beta_{\lambda 2}\left(\bar{R}_m - R_F\right)$$

Substituting into Equation (**??**) yields

$$\begin{aligned} \bar{R}_i &= R_F + b_{i1}\beta_{\lambda 1}\left(\bar{R}_m - R_F\right) + b_{i2}\beta_{\lambda 2}\left(\bar{R}_m - R_F\right) \\ &= R_F + \left(b_{i1}\beta_{\lambda 1} + b_{i2}\beta_{\lambda 2}\right)\left(\bar{R}_m - R_F\right) \end{aligned}$$

Defining β_i as $(b_{i1}\beta_{\lambda 1} + b_{i2}\beta_{\lambda 2})$ results in the expected return of $\bar{R}_i$ being priced by the CAPM:

$$\bar{R}_i = R_F + \beta_i\left(\bar{R}_m - R_F\right)$$

Which is a solution with multiple factors fully consistent with CAPM.

Exercise 4.14.

(a) (i) As in the previous exercise, if APT's model holds, returns are generated by a multi-index model such that

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$$

Thus, to find the equation that holds with these three securities we should proceed as before

$$\begin{cases} \bar{R}_X = \lambda_0 + \lambda_1 b_{X1} + \lambda_2 b_{X2} \\ \bar{R}_Y = \lambda_0 + \lambda_1 b_{Y1} + \lambda_2 b_{Y2} \\ \bar{R}_Z = \lambda_0 + \lambda_1 b_{Z1} + \lambda_2 b_{Z2} \end{cases} \Leftrightarrow \begin{cases} 0.10 = \lambda_0 + \lambda_1 0.5 + \lambda_2 1 \\ 0.12 = \lambda_0 + \lambda_1 1 + \lambda_2 1.5 \\ 0.11 = \lambda_0 + \lambda_1 0.5 + \lambda_2 2 \end{cases} \Leftrightarrow \begin{cases} \lambda_0 = 0.0675 \\ \lambda_1 = 0.0015 \\ \lambda_2 = 0.025 \end{cases}$$

Finally,

$$\bar{R}_i = 0.0675 + 0.015 b_{i1} + 0.025 b_{i2}$$

(ii) The risk-free rate is given by λ_0, thus $R_F = 0,0675$.

(b) Security D will be at equilibrium if its equilibrium expected return rate equals its market expected return rate. Thus, we first need to compute the equilibrium expected return using our APT model,

$$\begin{aligned} \bar{R}^e_D &= 0.0675 + 0.015 b_{i1} + 0.025 b_{i2} \\ &= 0.0675 + 0.015 \times 2 + 0.025 \times 0.5 \\ &= 0.1075 \end{aligned}$$

Since, $\bar{R}^e_D = 0.1075 < \bar{R}_D = 0.12$, this portfolio D is not at equilibrium, allowing the existence of arbitrage opportunities. The high level of market expected return implies that the current market price is too low, meaning portfolio D is underpriced. Thus, we would like to buy it and short sell a fairly priced portfolio that replicates D's cash flows and risk. The subsequent increase in D's demand will force its price to increase until reach a non arbitrage price, such that $\bar{R}^e_D = \bar{R}_D$.

(c) As long as we can manage to find the right proportions to invest in each security, it should be possible to build the replication portfolio with securities A, B and C. This new portfolio sensitivity to factor 1 and 2 must equal the sensitivity of security D to these same risk factors. Since, the portfolio sensitivity is given by the weighted average of each security sensitivity and the proportions invested in the three securities must sum 1, it comes

$$\begin{cases} b_{D1} = x_A b_{A1} + x_B b_{B1} + x_C b_{C1} \\ b_{D2} = x_A b_{A2} + x_B b_{B2} + x_C b_{C2} \\ x_A + x_B + x_C = 1 \end{cases} \Leftrightarrow \begin{cases} 2 = x_A 0.5 + x_B 1 - x_C 0.5 \\ 0.5 = x_A 1 + x_B 1.5 + x_C 2 \\ x_A + x_B + x_C = 1 \end{cases} \Leftrightarrow \begin{cases} x_A = 1 \\ x_B = 1 \\ x_C = -1 \end{cases}$$

Exercise 4.15.

(a) To create an arbitrage opportunity, it must be possible to make a profit without investment and risk, which means

$$\begin{cases} \sum_{1=1}^{3} x_i = 0 \\ \sum_{1=1}^{3} x_i b_{i,1} = 0 \end{cases}$$

A possible portfolio that respects these conditions is

$$\begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_3 = 1 \end{cases}$$

Its expected return is $\bar{R}_p = \sum_{i=1}^{3} x_i \bar{R}_i = 1 \times 12 - 2 \times 15 + 1 \times 40 = 22$.

(b) The equilibrium relationship associated to the arbitrage pricing model is

$$\begin{cases} 0.10 = \lambda_0 + \lambda_1 \times 1 \\ 0.20 = \lambda_0 + \lambda_1 \times 3 \end{cases} \Leftrightarrow \begin{cases} \lambda_0 = 0 \\ \lambda_1 = 0.1 \end{cases}$$

Therefore, the APT line is

$$\bar{R}_i = 0 + 0.1 b_{i1} = 0.1 b_{i1}$$

Thus, the missing value is $\bar{R}_3 = 0.1 b_{i1} = 0.1 \times 3 = 0.3$

If we compare the expected returns with the equilibrium returns we can conclude

- Since $\bar{R}_1 = 12\% > \bar{R}_1^e = 10\%$, if you buy it you will get a return higher than what you would receive in equilibrium because Security 1 is underpriced. Therefore you should buy it
- Since $\bar{R}_2 = 15\% < \bar{R}_2^e = 20\%$, if you buy it you will get a return lower than what you would receive in equilibrium because Security 1 is overpriced. Therefore you should (short) sell it
- Since $\bar{R}_3 = 40\% > \bar{R}_3^e = 30\%$, if you buy it you will get a return higher than what you would receive in equilibrium because Security 1 is underpriced. Therefore you should buy it

(c) Without transaction costs, a linear relationship between βs and returns implies that any point outside this line represents an arbitrage opportunity and a abnormal return. However, if we consider transaction costs, the expected return in equilibrium must be corrected, falling by the amount they assume. If transaction costs were not constant, the relationship between βs and returns will not be linear at all. But, if the abnormal return and the transaction costs occur at the same time, they may cancel or at least be lower than transactions costs, reaching a new equilibrium outside the original line, since one cannot earn abnormal returns. Thus, transaction costs may imply a non linear relationship, which still respects the law of one price and the non arbitragem assumption.

Exercise 4.16.

(a) From the relationship between CAPM and APT we know that $\lambda_j = \left(\bar{R}_m - R_F\right) \beta_{\lambda j}$. Thus, to have consistency between CAPM and the data we need to observe

$$\begin{cases} \lambda_1 = \left(\bar{R}_m - R_F\right) \beta_{\lambda 1} \\ \lambda_2 = \left(\bar{R}_m - R_F\right) \beta_{\lambda 2} \end{cases} \Leftrightarrow \begin{cases} \beta_{\lambda 1} = \dfrac{\lambda_1}{\left(\bar{R}_m - R_F\right)} \\ \beta_{\lambda 2} = \dfrac{\lambda_2}{\left(\bar{R}_m - R_F\right)} \end{cases}$$

From the data in the problem we know $\bar{R}_m - R_F = 0,04$, so we have to calculate λ_1, λ_2 and λ_3, using the previously used equilibrium condition $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$. Thus,

$$\begin{cases} \bar{R}_A = \lambda_0 + \lambda_1 b_{A1} + \lambda_2 b_{A2} \\ \bar{R}_B = \lambda_0 + \lambda_1 b_{B1} + \lambda_2 b_{B2} \\ \bar{R}_C = \lambda_0 + \lambda_1 b_{C1} + \lambda_2 b_{C2} \end{cases} \Leftrightarrow \begin{cases} 0.12 = \lambda_0 + \lambda_1 1 + \lambda_2 0.5 \\ 0.134 = \lambda_0 + \lambda_1 3 + \lambda_2 0.2 \\ 0.12 = \lambda_0 + \lambda_1 3 - \lambda_2 0.5 \end{cases} \Leftrightarrow \begin{cases} \lambda_0 = 0.1 \\ \lambda_1 = 0.01 \\ \lambda_2 = 0.02 \end{cases}$$

Finally,

$$\begin{cases} \beta_{\lambda 1} = \dfrac{\lambda_1}{(\bar{R}_m - R_F)} \\ \beta_{\lambda 2} = \dfrac{\lambda_2}{(\bar{R}_m - R_F)} \end{cases} \Leftrightarrow \begin{cases} \beta_{\lambda 1} = \dfrac{0.01}{0.04} = 0.25 \\ \beta_{\lambda 2} = \dfrac{0.02}{0.04} = 0.5 \end{cases}$$

(b) Again, from the relationship between CAPM and APT, the β of each portfolio is given by $\beta_i = (b_{i1}\beta_{\lambda 1} + b_{i2}\beta_{\lambda 2})$. Thus

$$\begin{cases} \beta_A = (b_{A1}\beta_{\lambda 1} + b_{A2}\beta_{\lambda 2}) \\ \beta_B = (b_{B1}\beta_{\lambda 1} + b_{B2}\beta_{\lambda 2}) \\ \beta_C = (b_{C1}\beta_{\lambda 1} + b_{C2}\beta_{\lambda 2}) \end{cases} \Leftrightarrow \begin{cases} \beta_A = 1 \times 0.25 + 0.5 \times 0.5 \\ \beta_B = 3 \times 0.25 + 0.2 \times 0.5 \\ \beta_C = 3 \times 0.25 - 0.5 \times 0.5 \end{cases} \Leftrightarrow \begin{cases} \beta_A = 0.5 \\ \beta_B = 0.85 \\ \beta_C = 0.5 \end{cases}$$

(c) Since $\lambda_0 = R_F$ and $\lambda_0 = 0.1$, then $R_F = 0.1$

Exercise 4.17.

(a) If the APT assumptions hold then, in equilibrium, all securities are in the same plane $b_1/b_2/\bar{R}$. Thus, we can use deduce the equilibrium condition $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$ solving the equation system, as before

$$\begin{cases} \bar{R}_X = \lambda_0 + \lambda_1 b_{X1} + \lambda_2 b_{X2} \\ \bar{R}_Y = \lambda_0 + \lambda_1 b_{Y1} + \lambda_2 b_{Y2} \\ \bar{R}_Z = \lambda_0 + \lambda_1 b_{Z1} + \lambda_2 b_{Z2} \end{cases} \Leftrightarrow \begin{cases} 0.19 = \lambda_0 + \lambda_1 1 + \lambda_2 0.5 \\ 0.14 = \lambda_0 + \lambda_1 1.4 + \lambda_2 0 \\ 0.08 = \lambda_0 + \lambda_1 3 - \lambda_2 1 \end{cases} \Leftrightarrow \begin{cases} \lambda_0 = 0.07 \\ \lambda_1 = 0.05 \\ \lambda_2 = 0.14 \end{cases}$$

Thus,

$$\begin{aligned} R_F &= \lambda_0 = 0.07 \\ \bar{R}_{I_1} &= \lambda_1 + R_F = 0.05 + 0.07 = 0.12 \\ \bar{R}_{I_2} &= \lambda_2 + R_F = 0.14 + 0.07 = 0.21 \end{aligned}$$

(b) The expected return for portfolio W at equilibrium is given by $\bar{R}^e_W = 0.07 + 0.05b_{i1} + 0.14b_{i2} = 0.07 + 0.05 \times 1 + 0.14 \times 0 = 0.12$. Since $\mathbb{E}[R_W] = 0.13 > \bar{R}^e_w = 0.12$ we know the security is underpriced being an interesting investment to make (we should buy). Portfolio's W risk is similar to the risk of factor I_1 $(b_1 = 1 \wedge b_2 = 0)$, so that a possible arbitrage strategy is to short sell the index factor (assuming you could do so) and buy portfolio W.

An alternative is to form a new portfolio P using portfolios A, B and C, such that $b_1 = 1 \wedge b_2 = 0$:

$$\begin{cases} b_{P1} = b_1 x + b_1 y + b_1 z \\ b_{P2} = b_2 x + b_2 y + b_2 z \\ x + y + z = 1 \end{cases} \Leftrightarrow \begin{cases} 1 = 1x + 1.4y + 3z \\ 0 = 0.5x - z \\ \end{cases} \Leftrightarrow \begin{cases} x = -1 \\ y = 2.5 \\ z = -0.5 \end{cases}$$

To compose Portfolio P you would short sell X and Z to buy Y, in the proportions just computed.

(c) To evaluate the fund's performance, we need to compare the equilibrium expected return with the actual return. The equilibrium expected return is calculated as $\bar{R}^e_W = 0.07 +$

$0.05b_{i1} + 0.14b_{i2} = 0.07 + 0.05 \times 1.2 + 0.14 \times 0.2 = 0.158$. Now, to find the actual return we can use the Sharpe's Ratio (SR), defined as

$$SR = \frac{\bar{R}_{Fund} - R_F}{\sigma_{Fund}}$$

$$\bar{R}_{Fund} = SR \times \sigma_{Fund} + R_F \qquad (16)$$

σ_{Fund} is not given, but if this portfolio is fully diversified it only faces systematic risk, such that the correlation between the two factors is null and, therefore,

$$\begin{aligned}\sigma^2_{Fund} &= b^2_{1Fund}\sigma^2_{I_1} + b^2_{2Fund}\sigma^2_{I_2} \\ &= (1.2)^2(0.1)^2 + (0.2)^2(0.25)^2 \\ &= 0.0169\end{aligned}$$

And $\sigma_{Fund} = 0.13$. Applying it in (**??**), it comes

$$\bar{R}_{Fund} = 0.75 \times 0.13 + 0.07 = 0.1675$$

Thus, the fund has achieved a performance higher than what was expected under equilibrium.

(d) It is possible since the indexes' returns I_1 and I_2 can be explain by CAPM. In that case, APT and CAPM are equivalents, as shown in a previous exercise. In this case $\bar{R}_{I_1} = R_F + \beta_{I_1}\left(\bar{R}_m - R_F\right)$ and $\lambda_1 = \bar{R}_{I_1} - R_F$ such that

$$\beta_{I_1} = \frac{\lambda_1}{\bar{R}_{I_1} - R_F} = \frac{0.05}{0.15 - 0.07} = 0.625$$

and

$$\beta_{I_2} = \frac{\lambda_2}{\bar{R}_{I_2} - R_F} = \frac{0.14}{0.15 - 0.07} = 1.75$$

To calculate the portfolios' βs we know that, in general, $\beta_i = b_{i1}\beta_{\lambda_1} + b_{i2}\beta_{\lambda_2}$, then

$$\beta_X = 1 \times 0.625 + 0.5 \times 1.75 = 1,5$$

$$\beta_Y = 1.4 \times 0.625 + 0 \times 1.75 = 0.875$$

$$\beta_Z = 3 \times 0.625 - 1 \times 1.75 = 0.125$$

Exercise 4.18.

(a) CAPM and APT pretend to explain expected returns, although through with quite different assumptions. CAPM is a general equilibrium model with very strong assumptions like homogeneous expectations, while APT only assumes the absence of arbitrage. APT is also a must more general model than CAPM in the sense it allows returns to be explained by a set of variables that can help to better explain returns. Nevertheless, under certain circunstancies (APT's risk factors being explained by CAPM's market portfolio) the two models are equivalent. From an empirical point of view, both models face a major drawback. CAPM's market portfolio is impossible to capture since it englobes all possible and imaginable assets, including non tradable assets like our home. APT can be used with all kind of variables, however what are the relevant variables no one really knows and eventually we may not have databases for them.

(b) If APT holds, then the two indexes returns are also explained by APT $\left(\bar{R}_i = 0.07 + 0.03b_{1i} + 0.05b_{2i}\right)$, but with one singularity: each index only shows sensitivity to one risk factor b. Thus

$$\{I_1 \therefore b_1 = 1 \wedge b_2 = 0\}$$

$$\{I_2 \therefore b_1 = 0 \wedge b_2 = 1\}$$

and

$$\bar{R}_{I_1} = 0.07 + 0.03 \times 1 + 0.05 \times 0 = 0.1$$

$$\bar{R}_{I_2} = 0.07 + 0.03 \times 0 + 0.05 \times 1 = 0.12$$

Finally, it should be straight forward to you that $R_F = 0.07$.

(c) If CAPM holds, then the equilibrium condition is given by $\bar{R}_i = R_F + \beta_i \left(\bar{R}_m - R_F\right)$, where $\beta_i \left(\bar{R}_m - R_F\right)$ captures the systematic risk premium appropriate to security i. The model does not reward specific risk because we assume fully diversified portfolios. Thus, it must apply to the securities described in the problem. Using the data given it comes

$$\begin{cases} 0.304 = R_F + 1.8\left(\bar{R}_m - R_F\right) \\ 0.135 = R_F + 0.5\left(\bar{R}_m - R_F\right) \end{cases} \Leftrightarrow \begin{cases} R_F = 0.07 \\ R_m = 0.2 \end{cases}$$

(d) Yes. CAPM and APT are equivalent if the indexes' returns were explained by CAPM. In this case,

$$\begin{cases} \bar{R}_{I_1} = R_F + \beta_{I_1}\left(\bar{R}_m - R_F\right) \\ \bar{R}_{I_2} = R_F + \beta_{I_2}\left(\bar{R}_m - R_F\right) \end{cases} \Leftrightarrow \begin{cases} \beta_{I_1} = \dfrac{\bar{R}_{I_1}}{\bar{R}_m - R_F} \\ \beta_{I_2} = \dfrac{\bar{R}_{I_2}}{\bar{R}_m - R_F} \end{cases} \Leftrightarrow \begin{cases} \beta_{I_1} = \dfrac{0.10 - 0.07}{0.20 - 0.07} = 0.23 \\ \beta_{I_2} = \dfrac{0.12 - 0.07}{0.20 - 0.07} = 0.385 \end{cases}$$

and the indexes I_1 and I_2 βs are given by the expression $\beta_i = b_{i1}\beta_{I_1} + b_{i2}\beta_{I_2}$. Thus,

$$\beta_i = b_{i1}\beta_{I_1} + b_{i2}\beta_{I_2} = 0.23b_{i1} + 0.385b_{i2}$$

5 Portfolio Management

Exercise 5.1.

(a) Volatility is not always judged as a good risk measure since it considers both systematic and unsystematic risk. Actually, unsystematic or specific risk can be fully diversified, therefore the systematic risk should be the only one rewarded, what explains why measures of systematic risk are more often judged as better risk measures.

(b) Using standard deviation as the measure for variability, the reward-to-variability ratio for a fund is the fund's excess return (average return over the riskless rate) divided by the standard deviation of return, i.e., the fund's Sharpe ratio. E.g., for fund A we have:

$$\frac{\bar{R}_A - R_F}{\sigma_A} = \frac{14-3}{6} = 1.833$$

See Table **??** for the remaining funds' Sharpe ratios.

(c) A fund's differential return, using standard deviation as the measure of risk, is the fund's average return minus the return on a naïve portfolio, consisting of the market portfolio and the riskless asset, with the same standard deviation of return as the fund's. E.g., for fund A we have:

$$\bar{R}_A - \left(R_F + \frac{\bar{R}_m - R_F}{\sigma_m} \times \sigma_A\right) = 14 - \left(3 + \frac{13-3}{5} \times 6\right) = -1\%$$

See Table **??** for the remaining funds' differential returns based on standard deviation.

(d) A fund's differential return, using beta as the measure of risk, is the fund's average return minus the return on a naïve portfolio, consisting of the market portfolio and the riskless asset, with the same beta as the fund's. This measure is often called "Jensen's alpha". E.g., for fund A we have:

$$\bar{R}_A - \left(R_F - \left(\bar{R}_m - R_F\right) \times \beta_A\right) = 14 - (3 + (13-3) \times 1.5) = -4\%$$

See Table **??** for the remaining funds' Jensen alphas.

(e) Treynor's ratio is quite similar to Sharpe's Ratio, but considering β as the appropriate risk measure. E.g., for fund A we have:

$$\frac{\bar{R}_A - R_F}{\beta_A} = \frac{14-3}{1.5} = 7.833$$

(f) This differential return measure is similar to Jensen's Alpha, except that the riskless rate is replaced with the average return on a zero-beta asset. E.g., for fund A we have:

$$\bar{R}_A - \left(\bar{R}_Z - \left(\bar{R}_m - \bar{R}_Z\right) \times \beta_A\right) = 14 - (4 + (13-4) \times 1.5) = -3.5\%$$

(g) Fund B shows a better performance than Fund A when considering Sharpe's Ratio. To invert this the following relationship should hold

$$\frac{\bar{R}_A - R_F}{\sigma_A} \geqslant \frac{\bar{R}_B - R_F}{\sigma_B} \Leftrightarrow 1.833 \geqslant \frac{\bar{R}_B - R_F}{\sigma_B} = \frac{\bar{R}_B - 3}{4} \Leftrightarrow \bar{R} \geqslant 10.33$$

So, for the ranking to be reversed, Fund B's average return would have to be lower than 10.33%.

Fund	Sharpe Ratio	Treynor Ratio	Differential Return (sigma)	Jensen's Alpha	Differential Return (Beta and $\bar{R}_Z$)
A	1.833	7.333	-1%	-4%	-3.5%
B	2.250	18.000	2%	4%	3.5%
C	1.625	13.000	-3%	3%	3.0%
D	1.063	14.000	-5%	2%	1.5%
E	1.700	8.500	-3%	-3%	-2.0%

Table 6: Exercise **??** - Answers (b to f)

Exercise 5.2. To compute Sharpe's ratio (SR), defined as the fund's excess return (average return over the riskless rate) divided by the standard deviation of return, we need to know the funds' volatility, which we can calculate using the single index model. Thus

$$\sigma_A = \sqrt{\beta_A^2 \sigma_m^2 + \sigma_{e_A} = 1,3^2 \times 0.3^2 + 0.003} = 0.3938$$

$$\sigma_B = \sqrt{\beta_B^2 \sigma_m^2 + \sigma_{e_B} = 0.9^2 \times 0.3^2 + 0.04} = 0.336$$

Therefore,

$$SR_A = \frac{\bar{R}_A - R_F}{\sigma_A} = \frac{0.15 - 0.05}{0.3938} = 0.2539$$

$$SR_B = \frac{\bar{R}_B - R_F}{\sigma_B} = \frac{0.09 - 0.05}{0.336} = 0.119$$

6 Miscellaneous

Exercise 6.1.

(a) (i) Since in this country it is possible to both deposit and lend at the same interest rate $R_F = 4\%$, we know the efficient frontier in the plan risk/expected return is a straight line passing trough the risk free asset and the so-called tangent portfolio (that is the only portfolio composed only of risky investments that is efficient). Thus, the efficient frontier in this country is given by

$$\bar{R}_p = R_F + \frac{\bar{R}_T - R_F}{\sigma_T}\sigma_p \quad \Leftrightarrow \quad \bar{R}_p = 4\% + \frac{4}{3}\sigma_p$$

where p is an efficient portfolio.
To check whether A is efficient or not we must see if it is on the straight line above

$$\bar{R}_A = 4\% + \frac{4}{3}\sigma_A \quad \Leftrightarrow \quad 8\% = 4\% + \frac{4}{3}3\% \quad \Leftrightarrow \quad 8\% = 8\%$$

and we can conclude portfolio A belong to the efficient frontier and, thus, is an efficient portfolio. The optimal portfolio for a *super averse* investor is the portfolio that maximizes the risk tolerance function $f(\sigma, \bar{R}) = 2.272\bar{R} - \bar{R}^2 - \sigma^2$ subject to the restriction it must be an efficient portfolio so, $\bar{R}_p = 4\% + \frac{4}{3}\sigma_p$. To get the optimal portfolio we need to

$$\max f(\sigma, \bar{R}) = 2.272\bar{R}_p - \bar{R}^2 - \sigma_p^2 \quad s.t \quad \bar{R}_p = 4\% + \frac{4}{3}\sigma_p$$

which is equivalent to the following unrestricted problem

$$\max \tilde{f}(\sigma) = 2.272\left(4\% + \frac{4}{3}\sigma_p\right) - \left(4\% + \frac{4}{3}\sigma_p\right)^2 - \sigma_p^2$$

The FOC of the problem is

$$\frac{\partial \tilde{f}}{\partial \sigma} = 0 \quad \Leftrightarrow \quad 2.272 \times \frac{4}{3} - 2\left(4\% + \frac{4}{3}\sigma_p\right)\frac{4}{3} - 2\sigma_p = 0 \quad \Leftrightarrow \quad \sigma_O = 0.96\% \ .$$

The expected return of the optimal portfolio O is then given by

$$\bar{R}_O = 4\% + \frac{4}{3}0.96\% = 5.297\%.$$

To obtain the optimal portfolio's composition we must rely on the fact the optimal portfolio is efficient and any efficient portfolio is a combination of the risk free asset with the tangent portfolio. Thus

$$\bar{R}_O = x_F R_F + (1 - x_f)\,\bar{R}_T \quad \Leftrightarrow \quad 5.297\% = 4\% x_F + 12\%\,(1 - x_F) \quad \Leftrightarrow \quad x_F = 84\% \ .$$

So, the optimal portfolio for a *super averse* investor requires depositing 84% of the initial amount and investing the remaining 16% in the tangent portfolio T.

(iii) If the simply averse invest 120% in the tangent portfolio that means they are leveraging themselves and taking a loan equivalent to 20% of their initial amount. Thus, they are shortselling the risk free asset, i.e. $x_F = -20\%$. Their expected return is

$$\bar{R}_{simply} = 4\% \times (-20\%) + 12\% \times 120\% = 13.6\% \ .$$

Since this point must also be on the efficient frontier we also have optimal risk level must satisfy

$$\bar{R}_{simply} = 4\% + \frac{4}{3}\sigma_{simply} \quad \Leftrightarrow \quad 13.6\% = 4\% + \frac{4}{3}\sigma_{simply} \quad \Leftrightarrow \quad \sigma_{simply} = 7.2\% \ .$$

(iv) Total amount deposited = 1 million super averse $\times$ 1000 euros $\times$ 84% = 840 000 euros
Total amount of loans = 4 million simply averse $\times$ 2000 euros $\times$ 20% = 1 600 000 euros Since $1600000 \neq 840000$ we conclude the market is *not* in equilibrium.

(b) (i) We now know two efficient portfolios: T and B both belonging to the hyperbola that by the envelop theorem is the frontier of the investment opportunity set of combinations of risky assets. By the Merton theorem we also know two portfolios are enough to derive the entire frontier, so the minimum variance portfolio MV is also a combination of T and B.
The variance of any combination of T and B is given by

$$\sigma^2 = x_T\sigma_T^2 + (1-x_T)^2\sigma_B^2 + 2x_T(1-x_T)\sigma_T\sigma_B\rho_{TB} \ .$$

The minimum variance portfolio minimizes is the only with the lowest possible risk, so it is it is the one that

$$\begin{aligned} \min \quad & x_T\sigma_T^2 + (1-x_T)^2\sigma_B^2 + 2x_T(1-x_T)\sigma_T\sigma_B\rho_{TB} \\ \Leftrightarrow & \\ \min \quad & x_T(6\%)^2 + (1-x_T)^2(12\%)^2 + 2x_T(1-x_T)(6\%)(12\%)0.6 \end{aligned}$$

From the FOC we get

$$\begin{aligned} \frac{\partial\sigma^2}{\partial x_T} = 0 \quad & \Leftrightarrow \quad (6\%)^2 - 2(12\%)^2(1-x_T) + 2(6\%)(12\%)0.6 - 4x_T(6\%)(12\%)0.6 = 0 \\ & \Leftrightarrow \quad x_T = 107.69\% \ , \end{aligned}$$

and the minimum variance portfolio involves shortselling portfolio B ($x_B = -7.69\%$) to invest more than 100% in portfolio T ($x_T = 107.69\%$).

(ii) See slides from classes for the sketch.
In this case the efficient frontier has three branches: (i) a segment of a straight line from the deposit rate to the first tangent portfolios; (ii) a portion of the envelope hyperbola (between the two tangent portfolios) and (iii) another segment of a line for risk levels higher than the risk of the second tangent portfolio (the tangent obtained using the active interest rate).

(iii) If the optimal risk levels do not change, then we know $\sigma^*_{super} = 0.96\%$ (from the exercise) and $\sigma^*_{super} = 7.2\%$ (from point a(iii)).
For the super averse investor nothing changes since their optimal risk level is below the risk of portfolio T and the deposit rate did not change. So they still invest 84% in the risk free asset and 16% in portfolio T.
For the simply averse investors we only know their optimal risk level is higher than σ_T, but we do not know whether it is bellow risk level of the tangent portfolio when we take the intersection with the yy-axis to be 7%, the portfolio usually denoted by T'.
We thus need first to determine portfolio T'. This portfolio must also be a combination of T and B and is the portfolio that

$$\max_{x_T,x_B} \frac{\bar{x}_T\bar{R}_T + x_B\bar{R}_B - 7\%}{\sqrt{x_T^2\sigma_T^2 + x_B^2\sigma_B^2 + 2x_Tx_B\sigma_T\sigma_B\rho_{TB}}} \quad s.t. \quad x_T + x_B = 1.$$

The first order conditions are equivalent to the following system of linear equations in z_T, Z_B and we know the z's are proportional to the x's,

$$\begin{cases} \bar{R}_T - 7\% = \sigma_T^2 z_T + \sigma_{TB}z_B \\ \bar{R}_B - 7\% = \sigma_{TB}z_T + \sigma_B^2 z_B \end{cases} \Leftrightarrow \begin{cases} \bar{1}2\% - 7\% = (6\%)^2 z_T + (6\%)(12\%)0.6z_B \\ \bar{1}5\% - 7\% = (6\%)(12\%)0.6z_T + (12\%)^2 z_B \end{cases} \Leftrightarrow$$

$$\begin{cases} \bar{z}_T = 11.28472 \\ \bar{z}_B = 2.170139 \end{cases} \Leftrightarrow \begin{cases} \bar{x}_T = 83.87\% \\ \bar{x}_B = 16.13\% \end{cases}$$

Portfolio T' requires investing 83.87% in portfolio T and 16.13% in portfolio B. The expected return and risk of T' are given by

$$\bar{R}_{T'} = 83.37\% \times 12\% + 16.13\% \times 15\% = 12.48\%$$
$$\sigma_{T'} = \sqrt{(83.37\%)^2 \times (6\%)^2 + (16.13\%)^2(12\%)^2 + 2(83.37\%)(16.13\%)(6\%)(12\%)0.6} = 6.38\%$$

Since the optimal risk level of the simply averse is higher than $\sigma_{T'}$, we know simply averse investors will take a loan to leverage themselves, even with the higher rate of 7% and invest more than 100% in T'.
The expected return is $\bar{R}_{simply} = 7\% + \frac{12.48\% - 7\%}{6.38\%} 7.2\% = 13.18\%$ and therefore we can see how much is the leverage:

$$13.18\% = 7\% x_F + 12.48\%(1 - x_F) \quad \Leftrightarrow \quad x_F = -12.77\% \quad \Rightarrow \quad x_{T'} = 112.77\% \ .$$

Simply averse investor take a loan to increase their capital by 12.77% and invest all their money in portfolio T'.

Exercise 6.2.

(a) Since the expression for the efficient frontier is a straight line we know

$$\bar{R}_p = R_F + \frac{\bar{R}_T - R_F}{\sigma_T} \sigma_p \ ,$$

which tells us that: (i) in this market there is a risk-free asset and that borrowing and lending is possible at the exact same rate $R_F = 3.5\%$, also (ii) since the slope of the efficient frontier equals the Sharpe ratio of the tangent portfolio we have $SR_T = \dfrac{\bar{R}_T - R_F}{\sigma_T} = 0.3436$

(b) (i) Mr. Silva has a quadratic utility function. For his particular function we have:

- $U'(W) = 50 - 2(0.01)W > 0$ for wealth levels that satisfy $W < \dfrac{50}{0.02} = 2500$. So, for a interval big enough around his initial wealth he prefers more to less.
- $U''(W) = -0.02 < 0$. From this we conclude Mr. Silva is risk averse.
- $ARA(W) = -\dfrac{U''(W)}{U'(W)} = \dfrac{0.02}{50 - 0.02W}$. Evaluating this function at the initial wealth $W_0 = 1000$ we get his absolute risk aversion coefficient before investment $ARA(1000) = \frac{0.02}{50 - 0.02 \times 1000} = \frac{0.02}{30}$. Taking the first derivative of the absolute risk aversion function we get $ARA'(W) = \frac{0.0004}{(50 - 0.02W)^2} > 0$ and we can conclude Mr. Silva has increasing absolute risk aversion, i.e. when his wealth increases he will decrease the amount of euros invested in risky assets.
- $RRA(W) = ARA(W)W = \dfrac{0.02W}{50 - 0.02W}$. Evaluating this function at the initial wealth $W_0 = 1000$ we get his relative risk aversion coefficient before investment $RRA(1000) = \frac{0.02 \times 1000}{50 - 0.02 \times 1000} = \frac{20}{30}$. Taking the first derivative of the relative risk aversion function we get $RRA'(W) = \frac{50}{(50 - 0.02W)^2} > 0$. Not surprisingly (given his increasing absolute risk aversion) Mr.Silva also has increasing relative risk aversion, i.e. when his wealth increases he keeps a smaller portion of his wealth in risky assets.

(ii) The risk tolerance function gives us for each pair of volatility and expected return, $(\sigma, \bar{R})$, the expected utility of an investor.

To derive Mr. Silva's risk tolerance function we need to compute the expected value of his utility function rewriting it in terms of returns, instead of wealth. Note that by definition of what wealth W and return R are, we get $W = W_0(1+R)$.

$$\begin{aligned}
f(\sigma,\bar{R}) &= \mathbb{E}[U(W)] = \mathbb{E}[U(W_0(1+R))] \\
&= \mathbb{E}\left[50W_0(1+R) - 0.01W_0^2(1+R)^2\right] \\
&= 50W_0(1+\mathbb{E}(R)) - 0.01W_0^2\mathbb{E}\left[(1+R)^2\right] \\
&= 50W_0(1+\mathbb{E}(R)) - 0.01W_0^2\left[1 + 2\bar{R} + \underbrace{\mathbb{E}(R^2)}_{\sigma^2+\bar{R}^2}\right]
\end{aligned}$$

Using $W_0 = 1000$ and simplifying we have Mr.Silva risk tolerance function

$$f(\sigma,\bar{R}) = 40000 + 30000\bar{R} - 10000\sigma^2 - 10000\bar{R}^2$$

(iii) To find Mr.Silva's optimal risk level we have to maximize his risk tolerance function, subject to the efficient frontier.

$$\max_{\sigma,\bar{R}} f(\sigma,\bar{R}) \quad s.t. \quad \bar{R} = 3.5\% + 0.3436\sigma$$

Including the restriction in the objective function we get

$$f(\sigma,\bar{R})|_{R=3.5\%+0.3436\sigma} = 40000+30000(3.5\%+0.3436\sigma)-10000\sigma^2-10000(3.5\%+0.3436\sigma)^2$$

This new restricted f function, depends only on σ. So to get its maximum we need to take its first derivative w.r.t. σ and set it to zero

$$\begin{aligned}
\frac{\partial f}{\partial \sigma^*} &= 0 \\
30000 \times 0.03436 - 20000\sigma^* - 20000(0.035 + 0.03436\sigma^*)0.3436 &= 0 \\
3 \times 0.3436 - 2\sigma^* - 2 \times 0.3436\left[0.035 + 0.3436\sigma^*\right] &= 0 \\
\sigma^* &= 23.13\%
\end{aligned}$$

(c) (i) We start by computing the inputs to mean-variance theory

$$\begin{aligned}
\bar{R}_1 &= 0.25(-5\%) + 0.5(0\%) + 0.25(50\%) = 11.25\% \\
\bar{R}_2 &= 0.25(10\%) + 0.5(-5\%) + 0.25(35\%) = 8.75\% \\
\sigma_1^2 &= 0.25(-5\% - 11.25\%)^2 + 0.5(0\% - 11.25\%)^2 + 0.25(50\% - 11.25\%)^2 = 0.05047 \\
&\quad \Rightarrow \quad \sigma_1 = 22.46\% \\
\sigma_2^2 &= 0.25(10\% - 8.75\%)^2 + 0.5(-5\% - 8.75\%)^2 + 0.25(35\% - 8.75\%)^2 = 0.02672 \\
&\quad \Rightarrow \quad \sigma_2 = 16.35\% \\
\sigma_{12} &= 0.25(-5\% - 11.25\%)(-5\% - 11.25\%) + 0.5(0\% - 11.25\%)(-5\% - 8.75\%) + \\
&\quad +0.25(50\% - 11.25\%)(35\% - 8.75\%) = 0.03265
\end{aligned}$$

From before we also know there is a risk-free asset with $R_F = 3.5\%$. The tangent portfolio is the one that maximizes the Sharpe ratio which is the same as solving a linear system of equations in z_1, z_2 which are proportional to the optimal weights

$$\begin{cases} \bar{R}_1 - R_f = \sigma_1^2 z_1 + \sigma_{12} z_2 \\ \bar{R}_2 - R_f = \sigma_{12} z_1 + \sigma_2^2 z_2 \end{cases} \Rightarrow \begin{cases} 11.25\% - 3.5\% = 0.05047 z_1 + 0.03265 z_2 \\ \bar{R}_2 - R_f = 0.03265 z_1 + 0.02672 z_2 \end{cases} \Leftrightarrow \begin{cases} z_1 = 1.263158 \\ z_2 = 0.421053 \end{cases}$$

Since z_1, z_2 are proportional to the tangent portfolio weights we can easily find them

$$x_1^T = \frac{z_1}{z_1 + z_2} = \frac{1.263158}{1.263158 + 0.421053} = 75\% \qquad x_2^T = \frac{z_2}{z_1 + z_2}\frac{0.421053}{1.263158 + 0.421053} = 25\%$$

The expected return as risk of the tangent portfolio are as follows

$$\begin{aligned}\bar{R}_T &= 0.75 \times 11.25\% + 0.25 \times 8.75\% = 10.625\% \\ \sigma_T^2 &= 0.75^2 \times 0.05047 + 0.25^2 \times 0.02672 + 2 \times 0.75 \times 0.25 \times 0.03265 = 0.0423 \\ \sigma_T &= 20.57\%\end{aligned}$$

An alternative to compute the tangent portfolio's volatility would be to use its expected return $\bar{R}_T$ and the equation for the efficient frontier

$$10.625\% = 3.5\% + 0.3436\sigma_T \Leftrightarrow \sigma_T = 20.57\% \, .$$

(ii) From before we know the optimal risk level of Mr. Silva is 23.13%. This is a point in the efficient frontier, so the optimal portfolio expected return is

$$\bar{R}^* = 3.5\% + 0.3436 \times 23.13\% = 11.51\% \, .$$

The optimal portfolio is a particular combination of the risk-free asset and the tangent portfolio. We find out the exact composition by solving

$$11.51\% = 3.5\% x_F + (1 - x_F)10.625\% \quad \Leftrightarrow \quad x_F = -12.45\% \quad \Rightarrow x_T = 112.45\% \, .$$

The optimal for Mr.Silva is to take a loan (of about 12.45% of his initial investment) to leverage a bit his position and invest 112.45% in the tangent portfolio.

(iii) Yes it would change since the current optimal portfolio involves taking a loan. Possibly at the new active rate he is no longer interested in taking a loan. His new optimum is most likely a combination of the tangent portfolio with a second portfolio belonging to the hyperbola that is the frontier of the investment opportunity set of risky assets.

Exercise 6.3.

(a) (i) *We are in a scenario were the correlation between the returns of any two assets is constant. So the tangent portfolio can be computed using a cut-off method.*

However, since shortselling is allowed, one can also simply use the general mean-variance theory. The inputs are:

$$\bar{R} = \begin{pmatrix} 8\% \\ 12\% \\ 15\% \end{pmatrix} \qquad V = \begin{pmatrix} 0.01 & 0.01 & 0.0125 \\ 0.01 & 0.04 & 0.025 \\ 0.0125 & 0.025 & 0.0625 \end{pmatrix}$$

where all covariances are obtained by multiplying each pair of individual assets volatility by the constant correlation of +0.5.

To find the tangent portfolio we need to solve the system $\left[\bar{R} - R_F\right] = VZ$

$$\begin{pmatrix} 5\% \\ 9\% \\ 12\% \end{pmatrix} = \begin{pmatrix} 0.01 & 0.01 & 0.0125 \\ 0.01 & 0.04 & 0.025 \\ 0.0125 & 0.025 & 0.0625 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \Leftrightarrow Z = V^{-1}\left[\bar{R} - R_F\right] = \begin{pmatrix} 2.85 \\ 0.95 \\ 0.95 \end{pmatrix} \Rightarrow X = \begin{pmatrix} 0.6 \\ 0.2 \\ 0.2 \end{pmatrix}$$

(ii) The expected return and risk of the tangent portfolio are:

$$\begin{aligned}\bar{R}_T &= X'\bar{R} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} 8\% \\ 12\% \\ 15\% \end{pmatrix} = 10.22\% \\ \sigma_T^2 &= X'VX = \begin{pmatrix} 0.6 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} 0.01 & 0.01 & 0.0125 \\ 0.01 & 0.04 & 0.025 \\ 0.0125 & 0.025 & 0.0625 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.2 \\ 0.2 \end{pmatrix} = 0.0151 \\ &\Rightarrow \quad \sigma_T = 12.323\%\end{aligned}$$

(iii) Since it is possible to deposit and borrow at the same rate $R_F = 3\%$, the efficient frontier is a straight line tangent to the investment opportunity set of risky assets. This line passes trough the risk-free point and the tangent portfolio, thus

$$\bar{R}_P = R_F + \frac{\bar{R}_T - R_F}{\sigma_T}\sigma_p \text{ , in our case, } \bar{R}_P = 3\% + \frac{10.22\% - 3\%}{12.323\%}\sigma_p \Leftrightarrow \bar{R}_P = 3\% + 0.586\sigma_p$$

(b) (i) The optimal risk level is attained at the point where the some indifference curve is tangent to the efficient frontier. I.e., they both have the same slope at that point

$$\left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{\text{EF}} = \left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{\text{IC}}$$

The efficient frontier is $\bar{R}_P = 3\% + 0.586\sigma_p$, and we have $\left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{\text{EF}} = 0.586$ Differentiating the indifference curves we get $\left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{\text{IC}} = 2\sigma_p + 0.415$

The optimal is thus $0.586 = 2\sigma_p^* + 0.415 \quad \Leftrightarrow \sigma_p^* = 8.55\%$.

(ii) Given the optimal risk level $\sigma_p^* = 8.55\%$ and the efficient frontier equation, we get the optimal expected return

$$\bar{R}^* = 3\% + 0.586 \times 8.55\% = 8\%$$

This is attainable by depositing part of the initial wealth and investing the remaining in the tangent portfolio

$$8\% = x_F 3\% + (1 - x_F)10.22\% \qquad \Leftrightarrow \qquad x_F = 30\% \quad \Rightarrow \quad x_T = 70\% \text{ .}$$

The optimal for this investor s to deposit 30% of his wealth and to invest the remaining 70% in the tangent portfolio.

(c) (i) Nothing changes. It is still possible to deposit and borrow at the same rate, which means portfolio T is the only combination of risky assets that is efficient. The exact same portfolio T is feasible because it does not involve shortselling.

(ii) The optimal portfolio remains the same, for the same reason, portfolio T is feasible even in a world with restrictions to shortsell.

(d) The ranking of assets according to Roy ranks higher assets with lower probability of undesirable returns. In this case those are returns lower than $R_L = 5\%$.

When returns follow normal distributions we know that

$$\min \Pr(\bar{R} \leq 5\%) \qquad \Leftrightarrow \qquad \max \frac{\bar{R} - 5\%}{\sigma}$$

and the ranking of the three assets is

$$C : \frac{15\% - 5\%}{25\%} = 0.4 \quad > \quad B : \frac{12\% - 5\%}{20\%} = 0.35 \quad > \quad A : \frac{8\% - 5\%}{10\%} = 0.35$$

The best, according to Roy, is C, than B, than A.