Probability Theory and Stochastic Processes

LIST 6

Martingales

(1) Let $X_1, X_2, \ldots$ be a martingale with respect to the filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$. Show that:

(a) If $X_0 = E(X_1)$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$, then $X_0, X_1, X_2, \ldots$ is a martingale with respect to the filtration $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$.

(b) X_n is a martingale with respect to $\sigma(X_1, \ldots, X_n)$.

(2) Let $Y_1, Y_2, \ldots$ be independent random variables such that

$$P(Y_n = a_n) = \frac{1}{2n^2}$$
$$P(Y_n = 0) = 1 - \frac{1}{n^2}$$
$$P(Y_n = -a_n) = \frac{1}{2n^2}$$

where $a_1 = 2$, $a_n = 4\sum_{j=1}^{n-1} a_j$. Decide if X_n and $\sigma(Y_1, \ldots, Y_n)$ define a martingale when

(a) $X_n = \sum_{j=1}^{n} Y_j$.

(b) $X_n = \sum_{j=1}^{n} \frac{1}{2^j} Y_j$.

(c) $X_n = \sum_{j=1}^{n} Y_j^2$.

(3) Let $Y_1, Y_2, \ldots$ be a sequence of iid random variables such that $P(Y_n = 1) = p$ and $P(Y_n = -1) = 1 - p$. Let $S_n = \sum_{j=1}^{n} Y_j$. Decide if X_n and $\sigma(Y_1, \ldots, Y_n)$ define a martingale when

(a) $X_n = S_n$.

(b) $X_n = S_n^2 - n$.

(c) $X_n = (-1)^n \cos(\pi S_n)$.

(d) $X_n = \left(\frac{1-p}{p}\right)^{S_n}$.

(e) $X_n = S_n - (2p - 1)n$.

(4) Let $Y_1, Y_2, \ldots$ be a sequence of iid random variables with Poisson distribution and mean value λ. Consider also the sequence

$$X_n = X_{n-1} + Y_n - 1, \quad n \in \mathbb{N},$$

and $X_0 = 0$. Find the values of λ for which X_n is a martingale, sub-martingale or super-martingale, with respect to the filtration $\sigma(Y_1, \ldots, Y_n)$.

(5) Let X_n be a martingale with respect to the filtration $\mathcal{F}_n$ and τ is a stopping time. Determine $E(X_{\tau \wedge n})$.

(6) Let $Y_1, Y_2, \ldots$ be a sequence of iid random variables with distribution $P(Y_n = 1) = p$ and $P(Y_n = -1) = 1-p$ where $0 < p < 1$, and $X_n = \sum_{j=1}^{n} Y_j$. Compute $E(\tau)$ for the stopping time

$$\tau = \min\{n \geq 1 \colon X_n = 1\}$$

when

(a) $p \leq 1/2$. *Hint*: Use Wald's equation (when $p < 1/2$ try also using the optional stopping theorem for $Z_n = [(1 - p)/p]^{X_n}$).

(b) * $p > 1/2$. *Hint*: Use the optional stopping theorem for $Z_n = X_n - (2p - 1)n$. Look first at an application of the optional stopping theorem for $Z_{\tau \wedge n}$ in order to show that $E(\tau \wedge n)$ is bounded. Then take the same conclusion for $E(\tau)$.

(7) Let X_n be a martingale with respect to a filtration $\mathcal{F}_n$. Prove that $E(X_{n+j} | \mathcal{F}_n) = X_n$ for all $n, j \in \mathbb{N}$.