Probability Theory and Stochastic Processes

Solutions

List 1

1) a) yes b) yes c) yes

3) 2^n

4) yes, no

5) yes

6) a) Since $\sigma(\mathcal{A}_2)$ is a σ-algebra that contains $\mathcal{A}_2$, it also contains $\mathcal{A}_1$. Moreover, as $\sigma(\mathcal{A}_1)$ is the intersection of all σ-algebras containing $\mathcal{A}_1$, it must be inside $\sigma(\mathcal{A}_2)$.

b) This follows because $\sigma(\mathcal{A})$ is already a σ-algebra.

c) Recall that $\mathcal{B} = \sigma(\mathcal{I})$ where $\mathcal{I}$ is the collection of all intervals on the form $]a, b]$. From

$$[a, +\infty[= \left(\bigcup_{n\in\mathbb{N}} \left] -\infty, a - \frac{1}{n} \right] \right)^c,$$

it follows that $\mathcal{A} \subset \sigma(\mathcal{I})$. Furthermore,

$$\begin{aligned}]a, b] &=]a, +\infty[\cap]b, +\infty[^c \\ &= \left(\bigcup_{n\in\mathbb{N}} \left[a + \frac{1}{n}, +\infty \right[\right) \cap \left(\bigcup_{n\in\mathbb{N}} \left[b + \frac{1}{n}, +\infty \right[\right)^c \end{aligned}$$

So, $\mathcal{I} \subset \sigma(\mathcal{A})$. By (a), $\sigma(\mathcal{A}) = \mathcal{B}$.

7) see lecture notes

8) yes, no

List 2

1)

2)

3)

4) a)

b)

5)

6) a) yes

b) $f(a_1) + f(a_2) + f(a_3)$

7)

8)

9) a) Let $\psi = f - g$ so that ψ is $\mathcal{F}$-measurable and $\int_B \psi\, d\mu = 0$ for every $B \in \mathcal{F}$. Consider the measurable set

$$B = \left\{\psi^+ > 0\right\} \in \mathcal{F}$$

by writing $\psi^+ = \max\{\psi, 0\}$ which is also $\mathcal{F}$-measurable. Notice that $B = \{\psi = \psi^+ > 0\}$ and $\int_B \psi\, d\mu = 0$. In addition, take

$$B_n = \left\{\psi^+ \geq \frac{1}{n}\right\} \in \mathcal{F}$$

Hence, $B_n \uparrow B$. So, using the Markov inequality,

$$0 \leq \mu(B_n) \leq n \int_{B_n} \psi^+\, d\mu \leq n \int_B \psi^+\, d\mu = n \int_B \psi\, d\mu = 0.$$

Then, $\mu(B) = \lim \mu(B_n) = 0$ which means that $\psi^+ = 0$ μ-a.e. The same idea for ψ^- implies that $\psi = 0$ μ-a.e.

b) Let $\Omega = [0,1]$, $\mathcal{A} = \{\emptyset, \Omega\}$ and $\mathcal{F} = \mathcal{B}(\Omega)$. Take the Lebesgue measure m on Ω. Consider $h(x) = 1/2$ and $f(x) = x$. For any $A \in \mathcal{A}$, we have $\int_A h\, dm = \int_A f\, dm$. However, $f \neq h$ m-a.e.

10) a) 0

b) $\arctan \pi$

c) 0

d) 1/2

e) 1

List 3

1) a)

$$F(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{3}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

b)

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{3}x, & 0 \leq x < 1 \\ \frac{1}{3}, & 1 \leq x < 2 \\ \frac{2}{3}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

2) a) For each $x \in D$ we have $F(x^-) < r_x < F(x^+)$ for some choice of a rational number $r_x \in \mathbb{Q}$ (for example, take r_x to be the number with the smallest possible number of decimal places of $(F(x^-) + F(x^+)/2)$.

We can define a function $g\colon D \to \mathbb{Q}$ given by $g(x) = r_x$. Thus, for $x_1 < x_2$ both in D we have $g(x_1) < F(x_1^+) \leq F(x_2^-) < g(x_2)$, meaning that g is strictly increasing. Therefore, g is a bijection between D and $g(D) \subset \mathbb{Q}$ and D is countable.

b) $\alpha(\{x\}) = F(x) - F(x^-) = 0$

3)

4)

5) a) $\phi(t) = e^{ita}$

b) $(1 - p + pe^{it})^n$

c) $e^{\lambda(e^{it}-1)}$

d) $p/(1 - (1-p)e^{it})$

e) $pn/(1 - (1-p)e^{it})^n$

6) a)

$$\phi(t) = \frac{e^{itb} - e^{ita}}{(b-a)it}$$

b) $1/(1 - it)$

c) $1/(1 + t^2)$

d) $e^{-|t|}$

e) $e^{it\mu - \sigma^2 t^2/2}$

List 4

1) a) Notice that $f(x) \in B$ is equivalent to $x \in f^{-1}(B)$. So, for any $B_1, B_2 \in \mathcal{B}$,

$$P(f(X) \in B_1, g(Y) \in B_2) = P(X \in f^{-1}(B_1), Y \in g^{-1}(B_2)).$$

Since X, Y are independent we get that

$$P(X \in f^{-1}(B_1), Y \in g^{-1}(B_2)) = P(X \in f^{-1}(B_1))\, P(Y \in g^{-1}(B_2))$$

Thus, $f(X), g(Y)$ are also independent.

b) Since by a) $f(X), g(Y)$ are independent, then $E(f(X)g(Y)) = E(f(X))\, E(g(Y))$.

2) The characteristic function is $\phi(t) = e^{-|t|}$. So, for $S_n/n = (X_1 + \cdots + X_n)/n$ we have the characteristic function

$$\phi_n(t) = (\phi(t/n))^n = e^{-|t|}.$$

Therefore, for any n, S_n/n has the Cauchy distribution as well, and the limit distribution is not the normal distribution.

3) e^{-1}

4) 0

5) b) $\frac{1}{2}\mathcal{X}_C + \frac{3}{2}\mathcal{X}_{C^c}$

c) 1

List 5

4) a) Stationary distribution: $\alpha = (\alpha_1, \alpha_2, \dots)$ where

$$\alpha_1 = \frac{1-r}{2-r}, \qquad \alpha_i = \frac{1}{2^{i-1}(2-r)}, \quad i \geq 2$$

5) a) $S = R_+$, $\tau_1 = 4 = \tau_3$, $\tau_2 = 2$

b) $S = R_+$, $\tau_1 = \tau_2 = \tau_3 = \frac{1}{4}$

List 6

2) a) Yes.

b) Yes.

c) No. Just define a sub-martingale.

3) a) X_n is a martingale if $p = 1/2$, a super-martingale if $0 < p < 1/2$, and a sub-martingale if $1/2 < p < 1$.

b) X_n is a martingale if $p = 1/2$.

c) Yes.

d) Yes.

4. If $\lambda = 1$, X_n is a martingale. If $\lambda \geq 1$, X_n is a sub-martingale, and if $\lambda \leq 1$, X_n is a super-martingale.

5. $E(X_{\tau\wedge n}) = E(X_1)$

6)

$$E(\tau) = \begin{cases} +\infty, & p \leq \frac{1}{2} \\ \frac{1}{2p-1}, & p > \frac{1}{2}. \end{cases}$$