Answer directly on the exam sheet.   **Duration: 1.5h**

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GROUP I (30 points)

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1. State the assumptions underlying mean-variance theory (MVT) and explain why, even in the presence of risk neutral or risk loving investors, we can focus on the so-called efficient frontier. Finally, discuss equilibrium under MVT, if there would be only risk neutral and/or risk lover investors. .... [15p]

   Answer:

   The main assumptions of mean-variance theory (MVT) are: (i) that investors care only about expected returns and volatilities of portfolios, and (ii) that volatility – standard deviation of returns – is a good measure os risk.

   MVT focus of efficient portfolios because they are the ones that, for any given risk level, guarantee the highest possible expected return and the vast majority of investors in financial markets are risk averse, i.e they like expected returns and dislike volatility (risk).

   Nonetheless, even if we consider risk-neutral investors we can focus on the efficient frontier as they would be indifferent between any portfolio with the same level of expected return, so for any expected return level, we can always decide to pick the efficient one. For the risk lover investors we know they like both expected return and risk. Efficient frontiers have increasing expected returns for increasing levels of risk, and for each level of risk it guarantees the maximum possible expected return. So, the risk lovers problem can be understood choosing the efficient portfolio with the highest possible risk .

   In terms of equilibrium, if there would be only risk neutral and/or risk lover investors, equilibrium would not be possible. All risk-neutral investors would choose highest possible expected return portfolio. All risk loving investors would choose the highest possible volatility on the efficient frontier. Thus, all investors end up choosing exactly the same extreme portfolio, opting to leverage up as much as possible or taking extreme shortselling positions if leverage would not be possible. Either way this could not be possibly an equilibrium as to some investors to borrow, cash or assets, there must be other investor wishing to lend and that can only happen in the presence of risk-averse investors.

2 Choose <u>ONE</u> of the following statements and discuss whether they are true or false. .....[15p]

I. *Most return generating models are based upon unrealistic assumptions, thus, there is no sound ground for applying them in practice.*
Comment: FALSE
It is true that return generating models such as constant correlation models, single factor models or multi-factor models, rely on assumptions that may, or may not, hold in practice, thus their use may lead to the introduction of *model risk* in mean-variance optimisation. However, that does not mean they are useless as they also contribute to reduce another important risk associated with the application of mean-variance theory - *estimation risk*.
Indeed, without any model, the application of mean-variance theory requires estimation of all its inputs: all expected returns, $\bar{R}_i$, all volatilities σ_i and all possible correlations ρ_{ij}. This means that for n risky assets the number of parameters to estimate are

$$n + n + \frac{n(n-1)}{2} : \quad \bar{R}_i, \quad \sigma_i, \quad \rho_i j, \quad \forall \quad i = 1, \cdots, n \quad , \quad j \neq i, \quad j = 1, \cdots, n.$$

Note the number of expected returns and volatilities grows linearly with the number of risky assets, but the number of correlations grows in a quadratic way. In fact, the number of correlations to be estimated is huge and even small error on each estimate may lead to quite different conclusions in the end.
Using return generating models, the number of parameters to estimate tends to be much smaller and to grow either linearly with the number of assets – that is the case of constant correlation models and single factor models – or to be proportional to $n \times K$ for a relatively small K that stands for the number of factors in multi-factor models. The parameters associated with return generating models are not only less but also easier to estimate in a robust way, so using models help eliminating part of the estimation risk which may more the compensate the possible introduction of model risk.

II. *An investor worried with safety is indifferent between the optimal portfolios according to Roy, Kataoka or Telsser.*
Comment: FALSE.
Roy, Kataoka and Telser define alternative safety criteria so the optimal portfolios according to each of them may differ.
Roy criterium is appropriate for investors who are extremely averse to returns below a limit R_L and wish to minimize the probability of occurrence of that event.
Kataoka criterium should be used for investors that express their concerns in terms of the worst $\alpha\%$ outcomes/scenarios and choose portfolios that maximize the $\alpha\%$ quantile of the returns distribution.
Finally, Telser criterium should be applied whenever investors like to say both R_L and $\alpha\%$ requiring one should only consider portfolios that have a probability of returns lower or equal to R_L smaller than $\alpha\%$. In Telser's case if more than one portfolio satisfies the safety constraint one should then pick the one with maximum expected return, since the investor's concern about risk was already considered.

Show that if a portfolio stochastically dominates another in a first order sense, then, it is preferred by any investor who prefers more to less, independently of being risk averse, neutral or lover.

Clarify your notation and justify all steps of the proof.

Proof.

Portfolio F is preferred to portfolio G if the expected utility of distribution F is greater than the expected utility of G.

Both expected utilities are given by

$$\mathbb{E}\left[U(F)\right] = \int_a^b U(x)dF(x) \qquad \text{and} \qquad \mathbb{E}\left[U(G)\right] = \int_a^b U(x)dG(x) \ ,$$

where a and b are simply the smallest and largest values of F and G can take on.

Thus, for F to be preferred to G, we must have

$$\int_a^b U(x)dF(x) > \int_a^b U(X)dG(x) \qquad \Leftrightarrow \qquad \int_a^b U(x)dF(x) - \int_a^b U(x)dG(x) > 0 \ .$$

Using $d(uw) = udw + wdu$ and integrating both sides we get $\int_a^b udw = uw\big|_a^b - \int_a^b wdu$, the usual expression of integration by parts.

Defining $u = U(x)$ and $dw = d\left[F(x) - G(x)\right]$ and integrating by parts yields

$$\int_a^b U(x)d\left[F(x) - G(x)\right] = U(x)\left[F(x) - G(x)\right]\big|_a^b - \int_a^b U'(x)\left[F(x) - G(x)\right]dx \ .$$

$F(b) = G(b) = 1$ and $F(a) = G(a) = 0$ by definition. Thus, F dominates G if the last term is positive (i.e if the integral is negative).

If investors prefer more to less, we must have $U'(x) > 0$.

The integral adds up values of $U'(x)$ and $F(x) - G(x)$. For this integral to be negative no matter the pattern of $U'(x)$, i.e., no matter if investors are risk averse, lovers or neutral, $F(x)$ must be less or equal to $G(x)$ for all x. For it to have a value different than zero, the strict inequality must hold for some value.

This completes the proof. ∎

In a country *NearByTheSea* the efficient mean-variance frontier is given by

$$\begin{cases} \bar{R}_p = 0.03 + 1.2\sigma_p & \sigma_p \leq 0.10 \\ \sigma_p^2 = 5.56\bar{R}_p^2 - 1.50\bar{R}p + 0.11 & 0.10 < \sigma_p < 0.20 \\ \bar{R}_p = 0.114 + 0.48\sigma_p & 0.20 \leq \sigma_p \leq 0.35 \end{cases}$$

1. Based upon the above information:

   (a) Determine the expected returns of the efficient portfolios, $T1$ and $T2$ with 10% and 20% volatility, respectively. .... [5p]

   Solution:
   The expected returns of $T1$ can be computed using the first straightline (as it is the tangent between that line and the envelop hyperbola) $\bar{R}_{T1} = 0.03 + 1.2 \times 10\% = 15\%$.
   The same logic applies to $T2$ so we have $\bar{R}_{T2} = 0.114 + 0.48 \times 20\% = 21\%$.

   (b) Knowing the minimum variance portfolio can be seen as the combination of $T1$ and $T2$ where we invest 125% in $T1$, find out the implicit correlation between the returns of portfolios $T1$ and $T2$. .... [5p]

   Solution:
   If the minimum variance portfolio has $x_{T1}^{MV} = 125\%$

   $$x_{T1}^{MV} = \frac{\sigma_{T2}^2 - \sigma_{T1,T2}}{\sigma_{T1}^2 + \sigma_{T2}^2 - 2\sigma_{T1,T2}} \quad \Leftrightarrow \quad 1.25 = \frac{(0.20)^2 - \sigma_{T1,T2}}{(0.1)^2 + (0.2)^2 - 2\sigma_{T1,T2}} \quad \Leftrightarrow \quad \sigma_{T1,T2} = 0.015$$

   $$\Rightarrow \quad \rho_{T1,T2} = \frac{0.015}{0.1 \times 0.3} = 0.75 \ .$$

   (c) From the shape of the efficient frontier, what can you conclude about: (i) the existence or not of a riskless asset, (ii) passive and active interest rates, (iii) possible shortselling restrictions, (iv) possible borrowing limits .... [10p]

   Answer:
   (i) Since it is possible to have an investment with zero risk, the riskless asset exist.
   (ii) Because the EF has two different segments of lines, with different slopes, we conclude the passive and active interest rates must be different. From the equations we get $R_f^p = 3\%$ and $R_f^a = 11.4\%$.
   (iii) Because it is not possible to have more than 35% volatility, we conclude the envelop hyperbola must be bounded somehow, and have a maximum volatility point at a level less or equal than 35%. Otherwise there would always exist a combination of at least just risky assets for any volatility level (upper hyperbola is increasing for all σ_p values). Thus, there must exist shortselling restrictions.
   (iv) Because the maximum volatility feasible on the EF is 35% and the volatility of $T2$ is 20% we conclude the maximum vol portfolio is achieved by investing in $T2$, $x_{T2} = 35\%/20\% = 175\%$, which means borrowing is limited to 75% of the initial investment.

   (d) Sketch the shape of the investment opportunity set (IOS) set and of the efficient frontier (EF) in the mean-variance $(\sigma, \bar{R})$ space and describe how each efficient point could be achieved. .... [7.5p]

   Solution:
   From the above we can conclude the IOS is delimited by an envelop hyperbola and two

straight lines.

The efficient frontier has three segments. The first segment – part of a line for volatilities lower than 10% – can be understood as combinations of deposit with the first tangent portfolio $T1$. The second segment – part of the envelop hyperbola between $T1$ and $T2$ – can be understood as combinations of $T1$ and $T2$. Finally, the third segment – part of a second line, valid for volatilities between 20% and 35% – can be interpreted as leveraging up (borrowing) to invest more than 100% of the initial wealth in $T2$.
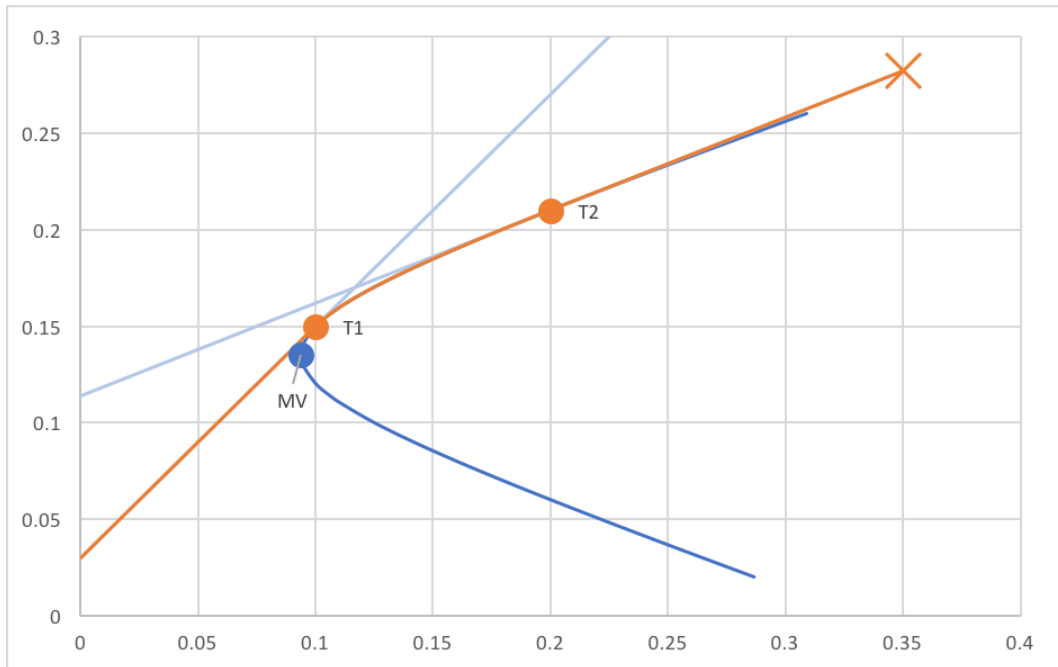
2. Consider Mr. Quelhas has an utility function $U(W) = -e^{bW}$, with $b < 0$.

(a) Evaluate Mr. Quelhas risk profile, interpreting your conclusions about his absolute and relative risk aversion. . . . . [7.5p]

Solution:

From Mr. Quelhas utility function we get

$$U(W) = -e^{bW} \qquad U'(W) = -be^{bW} > 0 \qquad U''(W) = -b^2 e^{bW} < 0 \quad \text{for} \quad b < 0,$$

and conclude (i) he prefers more to less – $U'(W) > 0$ – and he is risk averse – $U''(W) < 0$. (ii) Also, we can derive his absolute and relative risk aversion functions

$$ARA(W) = -\frac{U''(W)}{U'(W)} = -\frac{b^2 e^{bW}}{be^{bW}} = -b \qquad\qquad ARA'(W) = 0$$

$$RRA(W) = WA(W) = -bW \qquad\qquad RRA'(W) = -b > 0 \quad \text{for} \quad b < 0,$$

and conclude Mr. Quelhas has constant absolute risk aversion with $-b > 0$ his coefficient of absolute risk aversion. This means that if the money available to invest would increase he would not invest any additional amount in risky assets, nor would he disinvest. Consistently, with a constant absolute risk aversion, the investor has a increasing relative risk aversion, which means that for increasing amounts of money he decides to risk lower percentages of that money in risky assets.

(b) Take $W_0 = 1$ and use the second order Taylor approximation to the risk tolerance function of Mr. Quelhas to determine for which levels of the parameter b, his optimal volatility is the maximum allowed volatility $\sigma^* = 35\%$. Explain all steps of your solution. ...... [15p]

Solution:

The second order Taylor approximation of any RTF is given by

$$f(\sigma, \bar{R}) \approx \bar{R} - \frac{RRA(W_0)}{2}(\bar{R}^2 + \sigma^2) \ ,$$

where $RRA(\cdot)$ is the relative risk aversion function and W_0 is the initial wealth to be invested.

In the case of Mr. Quelhas we have $RRA(W) = -bW$, so

$$f(\sigma, \bar{R}) \approx \bar{R} + \frac{bW_0}{2}(\bar{R}^2 + \sigma^2) \ .$$

Also for optimal volatilities higher than $\sigma_{T2} = 20\%$, we know all efficient portfolios lie on the second line $\bar{R}_p = 0.114 + 0.48\sigma_p$.

Since Mr. Quelhas would always choose and efficient portfolio we can replace the relevante equation of the EF on the RTF. Using also $W_0 = 1$, we get RTF it in terms of σ alone,

$$f(\sigma) \approx (0.114 + 0.48\sigma) + \frac{b}{2}\left[(0.114 + 0.48\sigma)^2 + \sigma^2\right] \ .$$

To find the portfolio that maximizes the RTF we must set $f'(\sigma) = 0$. Since,

$$f'(\sigma) \approx 0.48 + \frac{b}{2}\left[2(0.114 + 0.48\sigma)0.48 + 2\sigma\right] \ ,$$

we get $\quad 0.48 + b\left[0.48(0.114 + 0.48\sigma^*)0.48 + \sigma^*\right] = 0 \quad \Leftrightarrow \quad \sigma^* = -\frac{0.48(1 + 0.114b)}{(0.48^2 + 1)b} \ .$

Given the efficient frontier has a maximum volatility of 35%, for any $\sigma^* \geq 35\%$ the optimal is the extreme point, thus,

$$-\frac{0.48(1 + 0.114\ b)}{(0.48^2 + 1)b} \geq 0.35$$

$$-0.48(1 + 0.114\ b) \leq 0.35\ b(1 + 0.48^2)$$

$$b \geq -\frac{0.48}{0.114 \times 0.48 + 0.35 \times (1 + 0.48^2)}$$

$$b \geq -0.989$$

Thus, the coefficient of absolute risk aversion $-b \leq 0.989$. For higher degrees of risk aversion the optimal volatility of Mr. Quelhas would be lower than 35%.