Financial Markets and Investments

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Answer directly on the exam sheet. **Duration: 1.5h**

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GROUP I (30 points)

1. Explain Lintner's definition of portfolio and its connection with shortselling restrictions. Consider an arbitrary number of $n > 3$ risky assets and a risk-free asset that can be used for both lending and borrowing. Sketch in the plan $(\sigma, \bar{R})$ the investment opportunity set under 3 different scenarios: (i) shortselling fully allowed, (ii) limited shortselling *a la*, Lintner, and (iii) shortselling forbidden. . . . . [15p]

   The standard definition of portfolio is simply a set of weights $(x_1, x_2, \cdots, x_n)$ that add up to 1, i.e. $\sum_{i=1}^n x_i = 1$. So feasible portfolios may include extreme short selling situations, with weights for some assets of as absurd $x_i = -500\%$ or more, which are never possible in real markets.

   To avoid such outcomes when optimizing to get the tangent portfolio – without totally forbidding short selling – Lintner proposed to redefined portfolio as a set of weights $(x_1, x_2, \cdots, x_n)$, whose absolute value add up to 1, i.e. $\sum_{i=1}^n |x_i| = 1$. Implicit in this definition is the idea that short selling does not come for "free", but instead at a cost. This can be interpreted as over colateralization of a short position which is also not realistic, but it has the advantage that the problem of finding out the tangent portfolio can still be found in terms of Z (just like for the standard portfolio case). In fact, the Lintner portfolio can be understood a combination of the unrestricted tangent portfolio with the deposit. Thus, if has the same Sharpe ratio as the tangent portfolio.

   When shortselling is forbidden it is almost sure that the restricted tangent has lower Sharpe ratio as it is tangent to the restricted envelop hyperbola which is almost surely in the interior of the unrestricted envelop hyperbola.

*** Sketch Here ***

2 Choose <u>ONE</u> of the following statements and discuss whether they are true or false. ..... [15p]

I. *To an investor who does not verify the Von-Neuman-Morgensten axioms, one should recommend safe portfolios according to criteria such as Roy, Kataoka or Telser.*
Comment: FALSE
If an investor does not satisfy the Von-Neuman-Morgenstern axioms, it means one cannot apply the principle of maximizing expected utility when choosing her optimal portfolio. However, this does not mean alternative portfolio proposals should necessarily be based upon safety criteria.
Portfolios based upon the safety criteria such as Roy, Kataoka or Telser should only be proposed to investors particularly worried about bad outcomes and for whom volatility is not a good measure of risk. The choice of the safety criteria to apply depends on the particular way investors express their worries, for instance, in terms of minimizing the probability of portfolio returns below a given level, choosing the portfolio with a higher quantile return or maximizing expected returns given a probability condition of returns below a given level is satisfied.

II. *If some analysts believe in a two-factor APT equilibrium model and others in the classical CAPM equilibrium model, they will never agree about equilibrium returns.*
Comment: FALSE
Financial analysts who consider the classical CAPM as valid believe equilibrium expected returns of individual assets are given by $\bar{R}_i^{\text{CAPM}} = R_F + \beta_i \left(\bar{R}_M - R_F\right)$ for all assets $i = 1, \cdots, N$ and where M is the so-called market portfolio and R_F the constant interest rate (both for deposit and lending) in that market.
On the other hand, financial analysts who consider an APT of two factors believe those two factors are able to capture the non-specific component of returns and that the process generating data depends linearly on the two factors. So, in equilibrium the return of individual assets will be give by $\bar{R}_i^{\text{APT}} = R_F + b_i 1 \left(\bar{I}_1 - R_F\right) + b_i 2 \left(\bar{I}_2 - R_F\right)$ for all assets $i = 1, \cdots, N$, where as before R_F is the constant interest rate (both for deposit and lending) in that market and $\bar{I}_j$ for $j = 1, 2$ are the expected value of two orthogonal indices.
So, the two types of models are different in spirit and are not likely to give the same equilibrium expected returns. Still, under very particular conditions they could lead to the same values.
Note that CAPM believers assume all indices/factors in the market are explained by the market portfolio M, thus we also have $\bar{I}_j = R_F + \beta_{I_j} \left(\bar{R}_M - R_F\right)$ for $j = 1, 2$.
And there would be agreement between the two equilibrium models if

$$\begin{aligned}
\bar{R}_i^{\text{CAPM}} &= \bar{R}_i^{\text{APT}} \\
R_F + \beta_i \left(\bar{R}_M - R_F\right) &= R_F + b_{i1} \left(\bar{I}_1 - R_F\right) + b_{i2} \left(\bar{I}_2 - R_F\right) \\
R_F + \beta_i \left(\bar{R}_M - R_F\right) &= R_F + b_{i1} \left[R_F + \beta_{I_1} \left(\bar{R}_M - R_F\right) - R_F\right] + b_{i2} \left[R_F + \beta_{I_2} \left(\bar{R}_M - R_F\right) - R_F\right] \\
\beta_i &= b_{i1}\beta_{I_1} + b_{i2}\beta_{I_2} \ .
\end{aligned}$$

That is, under very particular dependence relationships between the parameters of both models, there could be agreement.

GROUP II (20 points)

Characterize the risk profile of a *log–investor* with $U(W) = a + b\log(W)$, for $b > 0$. .... [10p]
Show that a portfolio that maximizes the geometric expected return is the optimal portfolio for a log–investor (for simplicity you may consider a discrete distribution of returns). .... [10p]

Proof.
For a log-investor we have

$$U(W) = a + b\log(W) \qquad \text{with} \qquad b > 0 \ ,$$

$$\begin{aligned}
U'(W) &= \frac{b}{W} > 0 & U''(W) &= -\frac{b}{W^2} < 0 \\
A(W) &= -\frac{U''(W)}{U(W)} = \frac{1}{W} & A'(W) &= -\frac{1}{W^2} > 0 \\
R(W) &= A(W) \times W = 1 & R'(W) &= 0
\end{aligned}$$

and we can conclude that this type of investors:

- prefer more to less ($U'(W) > 0$),
- they are risk averse ($U''(W) < 0$),
- they have decreasing absolute risk aversion ($A'(W) < 0$), i.e. as his wealth increases he invests more, at least in absolute terms, in risky assets;
- they have constant relative risk aversion ($R'(W) = 0$), i.e. as his wealth increases he keeps the same percentage of wealth invested in risky assets.

Whenever returns have follow a discrete distribution we have a finite set of scenarios with associated returns $R_1, R_2, \cdots, R_n$ for some finite n.

The geometric average of such returns, denoted $\bar{R}_g$, is defined as

$$1 + \bar{R}_g = (1 + R_1)^{p_1}(1 + R_2)^{p_2} \cdots (1 + R_n)^{p_n} \ .$$

The maximizations below are all equivalent

$$\begin{aligned}
&\max \quad (1 + R_1)^{p_1}(1 + R_2)^{p_2} \cdots (1 + R_n)^{p_n} \\
&\max \quad \log\left((1 + R_1)^{p_1}(1 + R_2)^{p_2} \cdots (1 + R_n)^{p_n}\right) \\
&\max \quad \sum_{i=1}^{n} p_i \log\left((1 + R_i)\right) \\
&\max \quad \mathbb{E}\left[\log\left((1 + R)\right)\right] \\
&\max \quad \mathbb{E}\left[\log\left(W\right)\right] \ ,
\end{aligned}$$

where we are using the fact that $W = (1 + R)W_0$ and the initial investment is positive ($W_0 > 0$). Therefore, we can conclude maximizing geometric expected returns is equivalent to finding the optimal portfolio for log investors. ∎

GROUP III (50 points)

The efficient frontier in the market under analysis is given by

$$\bar{R}_p = 3\% + SR_T \sigma_p \ ,$$

where SR_T is the highest possible attainable Sharpe Ratio.

In addition, we know the portfolios under consideration are based upon 16 risky assets and that the only combination of *just risky assets* that is efficient happen to be the *homogeneous* portfolio with expected return of 12% and a volatility of 15%.

1. What can you conclude about: (i) the existence or not a a riskless asset, (ii) de possibility of borrowing to invest in risky assets, (iii) the composition of the tangent portfolio, (iv) the value of SR_T. Explain. .... [10p]
   Answer:
   (i) Since the we can fully eliminate risk, setting $\sigma_p = 0$, then we know the riskless asset exist and we have $R_f = 3\%$.
   (ii) Because de EF is a straight line we know it is possible to both lend and borrow at the same rake R_f.
   (iii) Given we have a straighline as EF, the tangent portfolio is the *only* combination of risky assets that is efficient. In this case it happens to be the homogeneous portfolio of 16 risky assets, so its composition is $1/16 = 6.25\%$ in each of the risky assets.
   (iv) The Sharpe ratio of the tangent portfolio is given by

$$SR_T = \frac{\bar{R}_T - R_f}{\sigma_T} = \frac{\bar{R}_H - R_f}{\sigma_H} = \frac{12\% - 3\%}{15\%} = 0.6 \ .$$

2. Show that, for an average volatility of risky assets of 17.7%, the market implied average correlation is 0.7. ....[7.5p]
   Solution:
   We know that with $n = 16$ the homogeneous portfolio has a volatility of 15%. Using

$$\begin{aligned} \sigma_H^2 &= \frac{1}{n}\bar{\sigma}_i^2 + \frac{n-1}{n}\bar{\sigma}_{ij} \\ (15\%)^2 &= \frac{1}{16}(17.7\%)^2 + \frac{n-1}{n}\bar{\rho}(17.7\%)(17.7\%) \qquad \Leftrightarrow \qquad \bar{\rho} = 0.7 \end{aligned}$$

3. Mr. Iseg has a risk profile well described by the indifference curves $\bar{R}_p = \sigma_p^2 + 0.3\sigma_p + K$, with $K \in \mathbb{R}$.

   (a) Show the optimal investment volatility for Mr. Iseg is 15%.........................[7.5p]
   Solution:
   The optimal portfolio (*) is the one where the slope of the efficient frontier (EF) equals the slope of the indifference curves (IC):

$$\frac{\partial \bar{R}_p|_{EF}}{\partial \sigma_p} = \frac{\partial \bar{R}_p|_{IC}}{\partial \sigma_p} \quad \Leftrightarrow \quad 0.6 = 2\sigma_p^* + 0.3 \quad \Leftrightarrow \quad \sigma_p^* = \frac{0.6 - 0.3}{2} = 15\% \ .$$

   (b) How should Mr. Iseg invest 250 000 euros, to reach that desired level of risk? ....[5p]
   Solution:

Given his risk level is exactly the volatility of the tangent portfolio, we know he should invest all his weath in the tangent portfolio. Since that is the homogeneous portfolio, then he has to invest 6.25% in each of the 16 risky assets.

(c) What would need to be the certain return, R_C, that would make Mr.Iseg indifferent to invest or not in his optimal portfolio? Explain your computations. .... [10p]

Solution:

To determine the expected utility associated with the investment in the homogeneous portfolio we compute K_H from the indifference curves

$$\begin{aligned} \bar{R}_H &= \sigma_H^2 + 0.3\sigma_H + K_H \\ 12\% &= (15\%)^2 + 0.3(15\%) + K_H \qquad \Leftrightarrow \qquad K_H = 0.075 \end{aligned}$$

To get the certain returnR_C we ned to find R_C with $\sigma_C = 0$ and $K_C = K_H = 0.075$. Using the indifference curves we get

$$R_C = \sigma_C^2 + 0.3\sigma_C + K_C \qquad \Leftrightarrow \qquad R_C = K_C = 0.075\ .$$

So Mr. Iseg would be indifferent between investing in the homogenous portfolio and the riskless asset only if the riskless asset would have a return of 7.5%.

4. In terms of (i) the market's efficient frontier and (ii) the optimal allocation for Mr. Iseg, what would change if:

(a) Shortselling is forbidden. .... [5p]

Solution:

Nothing would change since the tangent portfolio is the homogeneous portfolio and it does not require shortselling. As long as it is still possible to deposit or borrow at $R_f = 3\%$ the EF is the same, and so is Mr. Iseg optimal allocation.

(b) It is not possible to take a loan for investment in risky assets. .... [5p]

Solution:

In this case the efficient frontier would be the previously considered EF only for volatilities lower or equal than the tangent portfolio volatility, i.e. $\sigma_p \leq 15\%$. For higher values of volatility the efficient frontier follow the upper part of the outer hyperbola of the risky assets investment opportunity set. So the efficient frontier would change.

Still, for Mr. Iseg nothing changes since his optimal volatility is the tangent portfolio.