Universidade de Lisboa - ISEG

Mathematical Economics

FIRST EXAM

January 19, 2021

Solutions

PART I

(1) Consider the following correspondence $F : [0,2] \rightrightarrows [0,2]$,

$$F(x) = \begin{cases} [\sqrt{x}, 2 - \frac{x}{2}], & 0 \leq x \leq 1 \\ \{a\}, & 1 < x \leq 2 \end{cases}$$

where $a \in [0,2]$.

(a) State the Kakutani fixed point theorem.

Solution: Let $F \colon K \rightrightarrows K$ be a correspondence defined on a compact and convex subset $K \subset \mathbb{R}^n$. If F is upper hemicontinuous and $F(x) \neq \emptyset$ and convex for every $x \in K$, then F has a fixed point x^* in K, i.e., $x^* \in F(x^*)$.

(b) Determine the values of a such that F satisfies the assumptions of the Kakutani fixed point theorem.

Solution: The domain of F is the interval $[0,2]$, hence compact and convex. $F(x)$ is either a closed interval or a point, thus $F(x)$ is convex and non-empty for every $x \in [0,2]$. Finally, F is uhc for every $x \neq 1$. At $x = 1$, F is uhc iff it has the closed graph property, which is the case when $a \in [\sqrt{1}, 2 - \frac{1}{2}] = [1, \frac{3}{2}]$.

(c) Find the fixed points of F for those values of a found in (b). In case you did not solve (b), you may take $a = 2$.

Solution: The fixed points satisfy the inclusion $x \in F(x)$. By drawing the graph of F one concludes that F has fixed points $\{0, 1, a\}$.

(2) Consider the function $f : [0,1]^2 \to \mathbb{R}^2$ defined by

$$f(x,y) = \left(\frac{(x+y)^2}{4}, x \right).$$

(a) Verify that f satisfies the hypothesis of the Brouwer fixed point theorem.

Solution: The Brouwer fixed point theorem states that if K is compact and convex and $f\colon K \to K$ is continuous, then f has a fixed point in K. Here $K = [0,1]^2$ is a square, hence compact and convex. Morever, f being polynomial in each component it is continuous. It remains to show that $f(x,y) \in [0,1]^2$ for every $(x,y) \in [0,1]^2$. That follows since $0 \le \frac{(x+y)^2}{4} \le \frac{(1+1)^2}{4} = 1$ and $0 \le x \le 1$ for every $(x,y) \in [0,1]^2$.

(b) Find the fixed points of f.

Solution: The fixed points satisfy $f(x,y) = (x,y)$. Thus

$$\begin{cases} \frac{(x+y)^2}{4} = x \\ x = y \end{cases} \Leftrightarrow \begin{cases} x^2 = x \\ x = y \end{cases} \Leftrightarrow \begin{cases} x = 0 \vee x = 1 \\ x = y \end{cases}$$

So, f has fixed points $(0,0)$ and $(1,1)$.

PART II

(1) Find and classify the critical points of

$$f(x, y, z) = x^2 + y^2 + z^2 + xy + x - 2z$$

Solution: The derivative of f is

$$Df(x, y, z) = \begin{bmatrix} 2x + y + 1 & 2y + x & 2z - 2 \end{bmatrix}$$

The critical points satisfy $Df(x, y, z) = (0, 0, 0)$, that is

$$\begin{cases} 2x + y + 1 = 0 \\ 2y + x = 0 \\ 2z - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{2}{3} \\ y = \frac{1}{3} \\ z = 1 \end{cases}$$

The hessian of f is

$$D^2 f(x, y, z) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which is positive definite since the leading principal minors are $\Delta_1 = 2$, $\Delta_2 = 3$, $\Delta_3 = 6$. Whence, f is strictly convex and $(-2/3, 1/3, 1)$ is a minimizer.

(2) Solve the following problem:

$$\begin{aligned} &\text{maximize } x + y \\ &\text{subject to } 5x^2 - 6xy + 5y^2 \leq 4 \\ &x \geq 0 \end{aligned}$$

Explain carefully all the steps in your reasoning.

Solution: The problem has the Lagrangian

$$L(x, y, \lambda, \mu) = x + y + \lambda(4 - 5x^2 + 6xy - 5y^2) + \mu x$$

The associated Kuhn-Tucker conditions are

$$\begin{cases} 1+\lambda(6y-10x)+\mu=0 \\ 1+\lambda(6x-10y)=0 \\ \lambda(4-5x^2+6xy-5y^2)=0 \\ \mu x=0 \\ 5x^2-6xy+5y^2\leq 4 \\ x\geq 0 \end{cases}$$

The system has solutions

$$(1,1,\frac{1}{4},0) \quad \text{and} \quad (0,\pm\frac{2}{\sqrt{5}},\pm\frac{\sqrt{5}}{20},-\frac{8}{5}).$$

Since it is a maximization problem, only the first solution matters because all multipliers are non-negative. Moreover,

$$D^2L(x,y,\frac{1}{4},0)=\begin{bmatrix} -5/2 & 3/2 \\ 3/2 & -5/2 \end{bmatrix} < 0$$

since $\Delta_1=-5/2$ and $\Delta_2=8$. Hence, $L(x,y,\frac{1}{4},0)$ is a concave function and $(1,1)$ is the solution of the maximization problem.

PART III

(1) Consider the differential equation

$$x'(t) + 2t^2 x(t) = t^2$$

(a) Classify the differential equation and determine its general solution.

Solution: The ODE is 1st order, non-autonomous, linear and non-homogeneous. By the integrating factor method, the ODE has general solution

$$x(t) = \frac{1}{2} + Ce^{-\frac{2}{3}t^3}, \quad C \in \mathbb{R}$$

(b) Let $x(t)$ denote the particular solution when $x(0) = 1$. Find $\lim_{t\to+\infty} x(t)$.

Solution:

$$\lim_{t\to+\infty} x(t) = \frac{1}{2}.$$

(2) Consider the system of ODEs

$$\begin{cases} x' = -2y \\ y' = y - x \end{cases}$$

(a) Write the system in matrix form $X' = AX$.

Solution:

$$A = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}$$

(b) Solve the IVP assuming that $x(0) = 1$ and $y(0) = -1$.

Solution: The eigenvalues of A are $\lambda = 2$ or $\lambda = -1$ with eigenvectors $(-1, 1)$ and $(2, 1)$, respectively. . Hence, A has Jordan normal form of type I

$$J = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

and the matrix of change of variables

$$P = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}.$$

The exponential matrix of A is

$$\begin{aligned} e^{At} &= Pe^{Jt}P^{-1} \\ &= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2e^{-t} + e^{2t} & 2e^{-t} - 2e^{2t} \\ e^{-t} - e^{2t} & e^{-t} + 2e^{2t} \end{bmatrix} \end{aligned}$$

Finally, the solution to the IVP

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{At} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2e^{-t} + e^{2t} & 2e^{-t} - 2e^{2t} \\ e^{-t} - e^{2t} & e^{-t} + 2e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$$

(c) Sketch the phase portrait of the system.

Solution: A saddle with a stable axis along the eigenvector $(2, 1)$ and an unstable axis along the eigenvector $(-1, 1)$.

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Part:IV

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Solutions

Question 1: The problem

$$
\begin{aligned}
V(x_0) \equiv \max_{\{u_t\}_{t=0}^{T-1}} & \sum_{t=0}^{T-1} \ln(u_t) \\
& \text{subject to} \\
& x_{t+1} = (1+\mu)\, x_t - u_t, \text{ for } t \in \{0, 1, \ldots, T-1\} \\
& x_0 = 1 \\
& x_T = (1+\mu)^T
\end{aligned}
$$

(a) Hamiltonian function $H(u, x, \psi) = \ln(u) \; + \psi\left((1+\mu)\, x - u\right)$. Then $H_t = H(u_t, x_t, \psi_t)$. The first order conditions are

$$
\begin{aligned}
\frac{\partial H_t}{\partial u_t} = 0 \iff \frac{1}{u_t} &= \psi_t, \text{ for } \; t \in \{0, 1, \ldots, T-1\} \\
\psi_t = \frac{\partial H_{t+1}}{\partial x_{t+1}} &= (1+\mu)\, \psi_{t+1}, \text{ for } \; t \in \{0, 1, \ldots, T-1\} \\
x_{t+1} &= (1+\mu)\, x_t - u_t, \text{ for } \; t \in \{0, 1, \ldots, T-1\} \\
x_0 &= 1, \text{ for } \; t = 0 \\
x_T &= (1+\mu)^T, \text{ for } \; t = T
\end{aligned}
$$

(b) The first two equations yield $u_{t+1} = (1+\mu)\, u_t$ which has the general solution $u_t = u_0\,(1+\mu)^t$ with u_0 unknown. Substituting in the constraint we have $x_{t+1} = (1+\mu)\, x_t - u_0\,(1+\mu)^t$. Solving with $x_0 = 1$ yields $x_t = (1+\mu)^t - t\, u_0\,(1+\mu)^{t-1}$. The terminal condition is satisfied if

$$
x_t\Big|_{t=T} = (1+\mu)^T - T\, u_0\,(1+\mu)^{T-1} = (1+\mu)^T \iff u_0 = 0.
$$

The solution to the problem is,

$$
\begin{aligned}
u_t^* &= 0, \text{ for } \; t \in \{0, \ldots, T-1\} \\
x_t^* &= (1+\mu)^t, \text{ for } \; t \in \{0, \ldots, T\}
\end{aligned}
$$

Off course $V(x_0) = -\infty$

Question 2: The problem

$$V(k_0) \equiv \max_{c(\cdot)} \int_{t=0}^{\infty} \ln(c(t))\, e^{-(\rho-n)\,t}$$
$$\text{subject to}$$
$$\dot{k} = (A-n)\,k - c, \text{ for } t \in \mathbb{R}_+$$
$$k(0) = k_0$$
$$\lim_{t\to\infty} e^{-\rho\,t}\; k(t) \geq 0$$

(a) Hamiltonian function $H(c, k, q) = \ln(c) + q\,((A-n)\,k - c)$. Then $H(t) = H(c(t), a(t), q(t))$. The first order conditions are

$$\frac{\partial H(t)}{\partial c(t)} = 0 \iff \frac{1}{c(t)} = q(t), \text{ for } t \in \mathbb{R}_+$$
$$\dot{q} = (\rho - n)\,q - \frac{\partial H(t)}{\partial k(t)} = q\,(\rho - A), \text{ for } t \in \mathbb{R}_+$$
$$\dot{k} = (A-n)\,k - c, \text{ for } t \in \mathbb{R}_+$$
$$k(0) = k_0, \text{ for } t = 0$$
$$\lim_{t\to\infty} q(t)\,k(t)\,e^{-(\rho-n)t} = 0$$

(b) Define $z(t) \equiv \dfrac{c(t)}{k(t)}$. The f.o.c separate into a terminal value

$$\begin{cases} \dot{z} = 1 - (\rho - n)\,z, \text{ for} t \in \mathbb{R}_+ \\ \lim_{t\to\infty} z(t)\; e^{-(\rho-n)t} = 0 \end{cases}$$

and an initial value problem

$$\begin{cases} \dot{k} = \left(A - n - z(t)^{-1}\right) k, \text{ for} t \in \mathbb{R}_+ \\ k(0) = k_0. \end{cases}$$

The first has the solution, because $\rho > n > 0$ $z(t) = \dfrac{1}{\rho - n}$, for any $t \in \mathbb{R}_+$, then $c(t) = (\rho - n)\,k(t)$. Substituting in the second problem yields

$$\begin{cases} \dot{k} = \left(A - \rho\right) k, \text{ for} t \in \mathbb{R}_+ \\ k(0) = k_0. \end{cases}$$

Therefore, the solution to the problem is

$$k^*(t) = k_0\, e^{(A-\rho)\,t}$$
$$c^*(t) = (\rho - n)\; k_0\, e^{(A-\rho)\,t}.$$