Statistics I: Chapter 2: Random Variables

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Random variable

Random variable, informally, is a variable that takes on numerical values and has an outcome that is determined by an experiment.

Random Variable: Let S be a sample space with a probability measure. A random variable (or stochastic variable) X is a real-valued function defined over the elements of S.

$$\begin{aligned} X : & S \to \mathbb{R} \\ & s \to X(s) \end{aligned}$$

Important convention: Random variables are always expressed in capital letters. On the other hand, particular values assumed by the random variables are always expressed by lowercase letters.

Remark: Although a random variable is a function of s; usually we drop the argument, that is we write X; rather than $X(s)$.

Random variable

Remark:

- Once the random variable is defined, R is the space in which we work with;
- The fact that the definition of a random variable is limited to real-valued functions does not impose any restrictions;
- If the outcomes of an experiment are of the categorical type, we can arbitrarily make the descriptions real-valued by coding the categories, perhaps by representing them with the numbers.

Example (Coin Tossing)

One flips a coin and observes if a head or tail is obtained.

Sample Space: $$S = \{H, T\}$$

Random Variable:

$$X : S \to \{0, 1\} \text{ with } X(H) = 0 \text{ and } X(T) = 1.$$

Random variable

The definition of random variable does not rely explicitly on the concept of probability, it is introduced to make easier the computation of probabilities. Indeed, if $B \subset \mathbb{R}$, then

$$P(X \in B) = P(A), \quad \text{where} \quad A = \{s \in S : X(s) \in B\}$$

Is now clear that:

$$P(X \in B) = 1 - P(X \notin B).$$

In particular,

$$P(X \leq x) = 1 - P(X > x);$$
$$P(X < x) = 1 - P(X \geq x)$$

Cumulative distribution function

Let X be a random variable. The **cumulative distribution function** F_X is a real function of real variable given by:

$$F_X(x) = P(X \leq x) = P(X \in (-\infty, x])$$

Properties of CDFs:

1) $0 \leq F_X(x) \leq 1$;
2) $F_X(x)$ is non-decreasing: $\forall \Delta_x > 0 : F_X(x) \leq F_X(x + \Delta_x)$.
3) $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to +\infty} F_X(x) = 1$.
4) $P(a < X \leq b) = F_X(b) - F_X(a)$, for $b > a$
5) $\lim_{x \to a^+} F_X(x) = F_X(a)$; therefore X is **right continuous**
6) $P(X = a) = F_X(a) - \lim_{x \to a^-} F_X(x)$ for any real finite number.

Cumulative distribution function

Example (Coin Tossing)

One flips a coin and observes if a head or tail is obtained.

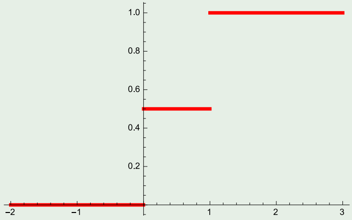
Sample Space: $S = \{H, T\}$

Random Variable:
$X : S \to \{0, 1\}$ with $X(H) = 0$ and $X(T) = 1$.

X counts the number of tails obtained.

It is easy to see that:
$P(X = 0) = 1/2$,
$P(X = 1) = 1/2$. Since we have $F_X(x) = P(X \leq x)$, then

$$F_X(x) = P(X \leq x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$



Cumulative distribution function

Example (Dice Casting)

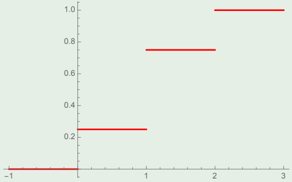
One flips a coin twice and counts the number of tails obtained.

Sample Space: $S = \{(H,T),(H,H),(T,H),(T,T)\}$

Random Variable:
$X : S \to \{0,1,2\}$ with
$X((H,T)) = 1$, $X((H,H)) = 0$,
$X((T,H)) = 1, X((T,T)) = 2$.

It is easy to see that:
$P(X = s) = 1/4$, for $s = 0,2$ and $P(X = 1) = 1/2$. Since we have $F_X(x) = P(X \leq x)$, then

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \leq x < 1 \\ 3/4, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$



Cumulative distribution function

Further properties:

- $P(X < b) = F_X(b) - P(X = b)$
- $P(X > a) = 1 - F_X(a)$
- $P(X \geq a) = 1 - F_X(a) + P(X = a)$
- $P(a < X < b) = F_X(b) - F_X(a) - P(X = b)$
- $P(a \leq X < b) = F_X(b) - F_X(a) - P(X = b) + P(X = a)$
- $P(a \leq X \leq b) = F_X(b) - F_X(a) + P(X = a)$

Prove the previous properties!

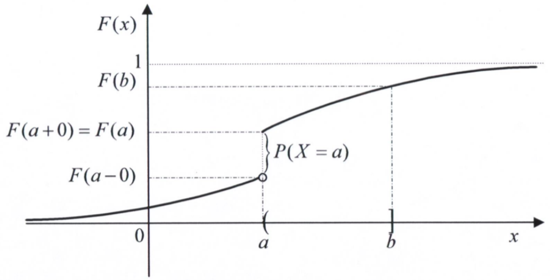
Proof: To prove that $P(X \geq a) = 1 - F_X(a) + P(X = a)$, one notes that:

$$\begin{aligned} P(X \geq a) &= 1 - P(X < a) = 1 - P(X \leq a) + P(X = a) \\ &= 1 - F_X(a) + P(X = a) \end{aligned}$$

Cumulative distribution function

The set of discontinuities of the cumulative distribution function D_X is given by $D_X = \{x \in \mathbb{R} : P(X = x) > 0\}$. Note that by property 6 this the same as

$$D_X = \left\{ a \in \mathbb{R} : F_X(a) - \lim_{x \to a^-} F_X(x) > 0 \right\}.$$



Types of random variables

Discrete Random Variable: X is a discrete random variable if

$$D_X \neq \emptyset \quad \text{and} \quad \sum_{x \in D_x} P(X = x) = 1.$$

Continuous Random Variable: X is a continuous random variable if $D_X = \emptyset$ and there is a non-negative function f such that

$$F_X(x) = \int_0^x f(s)ds.$$

Mixed Random Variable: X is a mixed random variable if

$$D_X \neq \emptyset, \quad \sum_{x \in D_x} P(X = x) < 1 \quad \text{and}$$

$$\exists \lambda \in (0,1) \text{ tal que } F_X(x) = \lambda F_{X_1}(x) + (1-\lambda) F_{X_2}(x)$$

where X_1 is a discrete random variable and X_2 is a continuous random variable.

Discrete random variables

X is a **discrete random variable** if

$$D_X \neq \emptyset \quad \text{and} \quad \sum_{x \in D_x} P(X = x) = 1.$$

Additionally, the function $f_X : \mathbb{R} \to [0, 1]$ defined by

$$f_X(x) = \begin{cases} P(X = x), & x \in D_X \\ 0, & x \in D_X \end{cases}.$$

is called the **probability function**.

Theorem: A function can serve as the probability function of a discrete random variable X if and only if its values, $f_X(x)$, satisfy the conditions

- $0 \leq f_X(x_j) \leq 1$, $j = 1, 2, 3, \ldots$
- $\sum_{j=1}^{\infty} f_X(x_j) = 1$.

Discrete Random Variables

For discrete random variables, the *cumulative distribution function* is given by :

$$F_X(x) = P(X \leq x) = \sum_{x_j \leq x} f_X(x_j).$$

Generally,

$$P(X \in B) = \sum_{x_j \in B \cap D_X} f_X(x_j).$$

Theorem: If the range of a random variable X consists of the values $x_1 < x_2 < \cdots < x_n$, then

$$f_X(x_1) = F_X(x_1), \quad \text{and} \quad f_X(x_i) = F_X(x_i) - F_X(x_{i-1}),$$

for all $i = 2, 3, \cdots n.$

Discrete Random Variables

Example

Check whether the function given by $f(x) = \frac{x+2}{25}$, for $x = 1, 2, 3, 4, 5$ can serve as the probability function of a discrete random variable X. Compute the cumulative distribution function of X.

Continuous Random Variables

X is a **continuous random variable** if $D_X = \emptyset$ and there is a function $f_X : \mathbb{R} \to \mathbb{R}_0^+$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(s)ds.$$

Additionally, f_X is called the **probability density function**.

Remark:

- Continuity of F_X is necessary, but not sufficient to guarantee that X is a continuous random variable;
- Note that $P(X \in D_X) = P(X \in \emptyset) = 0$;
- The function f_X provides information on how likely the outcomes of the random variable are.

Probability density function

Theorem. A function can serve as a probability density function of a continuous random variable X if its values, $f_X(x)$, satisfy the conditions:

- $f_X(x) \geq 0$ for $-\infty < x < +\infty$;
- $\int_{-\infty}^{+\infty} f_X(x)dx = 1$.

Example (Uniform Distribution)

Let X be a continuous random variable with a probability density function f_X given by

$$f_X(x) = \begin{cases} 1/5, & x \in [3, a] \\ 0, & x \in \mathbb{R} \setminus [3, a] \end{cases}$$

Find the value of the parameter a.

According to the previous theorem, we know that

$$f_X(x) \geq 0, \text{ for } -\infty < x < +\infty$$

$$\int_{-\infty}^{+\infty} f_X(x)dx = 1$$

From the second condition, we get that $\frac{a}{5} - \frac{3}{5} = 1 \Leftrightarrow a = 8$.

Probability density function

Theorem. If $f_X(x)$ and $F_X(x)$ are the values of the probability density and the distribution function of X at x, then

$$P(a \leq X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(t)dt$$

for any real constants a and with $a \leq b$, and

$$f_X(x) = \frac{dF_X(x)}{dx}, \quad \text{almost everywhere.}$$

Remarks:

- At the points x where there is no derivative of the CDF, F_X, it is agreed that $f_X(x) = 0$. In fact, it does not matter the value that we give to $f_X(x)$ as it does not affect the computation of F_X.
- The probability density function is not a probability and therefore it can assume values bigger than one.
- If X is a continuous random variable

$$P(X = a) = \int_a^a f_X(t)dt = 0.$$

Probability density function

Example (Triangle Distribution)

Consider the continuous random variable X with a probability density function f_X and cumulative distribution function given by

$$f_X(x) = \begin{cases} 0, & x < 0 \\ 4x, & 0 \leq x \leq \frac{1}{2} \\ 4 - 4x, & \frac{1}{2} \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

Cumulative density function:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 2x^2, & 0 \leq x < \frac{1}{2} \\ -1 + 4x - 2x^2, & \frac{1}{2} \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

Is this function F_X differentiable?

Probability density function

Theorem: If X is a **continuous random variable** and a and b are real constants with $a \leq b$, then

$$\begin{aligned} P(a &\leq X \leq b) = P(a \leq X < b) \\ &= P(a < X \leq b) \\ &= P(a < X < b) \end{aligned}$$

Proof: To prove the previous theorem one needs notice that:

$$\begin{aligned} P(a \leq X \leq b) =& P(a < X < b) + P(X = a) + P(X = b) \\ =& P(a < X \leq b) + P(X = a) \\ =& P(a \leq X < b) + P(X = b) \end{aligned}$$

Additionally, for $c = a$ or $c = b$ we have

$$P(X = c) = P(c \leq X \leq c) = \int_c^c f_X(t)dt = 0$$

Remark: The previous inequalities are not necessarily true for **discrete random variables**.

Mixed random variable

Mixed Random Variable: X is a mixed random variable if

$$D_X \neq \emptyset, \quad \sum_{x \in D_x} P(X = x) < 1 \quad \text{and}$$

$$\exists \lambda \in (0,1) \text{ tal que } F_X(x) = \lambda F_{X_1}(x) + (1-\lambda) F_{X_2}(x)$$

where X_1 is a discrete r.v. and X_2 is a continuous r.v..

Example

A company has received 1 million € to invest in a new business. With probability $\frac{1}{2}$, the firm does nothing but with probability $\frac{1}{2}$ the money is invested. If it does not invest the money, 1 million € is kept. Otherwise, the firm gets back a random amount uniformly distributed between 0 and 3 million €.

Let X be the following random variable:

$$X = \text{“Amount received by the company in millions”}$$

What type of random variable is X?

Mixed random variable

Example

$$S = [0,3] \quad \text{and} \quad X = \begin{cases} 1, & \text{with probability } \frac{1}{2} \text{ (Scenario 1)} \\ [0,3], & \text{with probability } \frac{1}{2} \text{ (Scenario 2)} \end{cases}$$

- X is not a discrete r.v. because it takes values in a continuous set;
- X is not a continuous random variable because $P(X=1) = 1/2$ (For continuous random variables the probability to take one single point is equal to 0).
- X is a mixed random variable?

We can define two random variables:

X_1 ="Amount received by the company in millions in S1"

X_2 = "Amount received by the company in millions in S2"

Mixed random variable

Example

Since $P(X_1 = 1) = 1$, then

$$F_{X_1}(x) = \begin{cases} 0, & x < 1 \\ 1, & x \geq 1 \end{cases}$$

On the other hand, in scenario 2, the firm gets back a random amount uniformly distributed between 0 and 3 million €. Therefore,

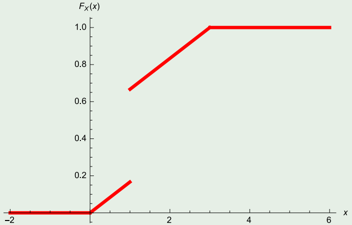
$$f_{X_2}(x) = \begin{cases} \frac{1}{3}, & x \in [0,3] \\ 0, & \text{otherwise} \end{cases}, \quad \text{and} \quad F_{X_2}(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{3}, & 0 \leq x < 3 \\ 1, & x \geq 3, \end{cases}$$

Since S1 holds with probability $\frac{1}{2}$ and S2 holds with $\frac{1}{2}$, we have that

Mixed random variable

Example

$$F_X(x) = \frac{1}{2}F_{X_1}(x) + \frac{1}{2}F_{X_2}(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{6}, & 0 \leq x < 1 \\ \frac{1}{2} + \frac{x}{6}, & 1 \leq x < 3 \\ 1, & x \geq 3, \end{cases}$$



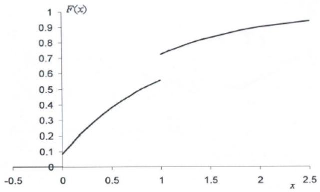
$D_X = \{1\}$, because

$$F_X(1) - F_X(1^-) = \frac{2}{3} - \frac{1}{2}$$
$$= \frac{1}{2} = P(X = 1) < 1$$

Mixed random variable

Exercise: Let

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{12} + \frac{3}{4}\left(1 - e^{-x}\right) & 0 \leq x < 1 \\ \frac{1}{4} + \frac{3}{4}\left(1 - e^{-x}\right) & x \geq 1 \end{cases}$$



Compute $P(X = 0)$, $P(X = 1)$, $P(0.5 < X < 1)$ and $P(0.5 < X < 2)$.

Answer:

$$P(X = 0) = \frac{1}{12}, \quad P(X = 1) = \frac{2}{12}$$

$$P(0.5 < X < 1) = F_X(1) - F_X(0.5) - P(X = 1) = \frac{3}{4}\left(e^{-0.5} - e^{-1}\right)$$

$$P(0.5 < X < 2) = F_X(2) - F_X(0.5) = \frac{2}{12} + \frac{3}{4}\left(e^{-0.5} - e^{-2}\right)$$

Distribution of functions of random variables

Motivation: Assume that the random variable D represents the demand of a given product in a store. The profit of this store is represented by the random variable $L = 4D - 5$. If the probability function of D is given by

$$P(D = d) = \begin{cases} 0.3, & d = 0 \\ 0.2, & d = 1 \\ 0.3, & d = 2 \\ 0.2, & d = 3 \end{cases},$$

what is the probability of having $L > 2$?

$$P(L > 2) = P\left(D > \frac{7}{4}\right) = P(D = 2) + P(D = 3) = 0.5$$

Since L is a random variable, it should be possible to find its distribution. How to do it?

- Let X be a known random variable with known cumulative distribution function $F_X(x)$.
- Consider a new random variable $Y = g(X)$, where $g : \mathbb{R} \to \mathbb{R}$ is a known function. Let $F_Y(y)$ be the cumulative distribution function of Y. How can we derive $F_Y(y)$ from $F_X(x)$?.
- The derivation of $F_Y(y)$ is based on the equality

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \in A_y^*)$$

where $A_y^* = \{x : g(x) \leq y\}$

Examples: Derive the cumulative distribution functions of $Y = aX + b$, where $a > 0$ and $Z = X^2$.

- $Y = aX + b$

$$F_Y(y) = P(Y \leq y) = P(aX + b \leq y)$$
$$= P\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

- $Z = X^2$

For $z \geq 0$,

$$F_Z(z) = P(Z \leq z) = P(X^2 \leq z)$$
$$= P\left(-\sqrt{z} \leq X \leq \sqrt{z}\right)$$
$$= F_X\left(\sqrt{z}\right) - F_X\left(-\sqrt{z}\right) + P(X = -\sqrt{z})$$

Examples: Assume that in the previous example X is a continuous random variable such that

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases},$$

then the following holds:

- $Y = aX + b$

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right) = \begin{cases} 0, & \frac{y-b}{a} < 0 \\ \frac{y-b}{a}, & 0 \leq \frac{y-b}{a} < 1 \\ 1, & \frac{y-b}{a} \geq 1 \end{cases}$$

$$= \begin{cases} 0, & y < b \\ \frac{y-b}{a}, & b \leq y < a+b \\ 1, & y \geq a+b \end{cases}$$

Examples: Assume that in the previous example X is a continuous random variable such that

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases},$$

then the following holds:

- $Z = X^2$

If $z < 0$ then $F_Z(z) = P(Z \leq z) = 0$. When $z \geq 0$

$$F_Z(z) = F_X\left(\sqrt{z}\right) - F_X\left(-\sqrt{z}\right) + \underbrace{P(X = -\sqrt{z})}_{=0,\text{ because } X \text{ is continuous}}$$

$$= F_X\left(\sqrt{z}\right) - \underbrace{F_X\left(-\sqrt{z}\right)}_{=0 \text{ because } -\sqrt{z} \text{ is negative}}$$

$$= \begin{cases} 0, & z < 0 \\ \sqrt{z}, & 0 \leq z < 1 \\ 1, & z \geq 1 \end{cases}$$

Discrete random variables

- When X is a discrete random variable, it is easier to find the distribution of $Y = g(X)$. In this case, we will derive the probability function.
- Let $D_X = \{x_1, x_2, x_3 \ldots\}$ be the set of discontinuities of $F_X(x)$, then $D_Y = \{g(x_1), g(x_2), g(x_3) \ldots\}$ is the set of discontinuities of $F_Y(y)$.
- The probability function of Y is given by

$$\begin{aligned} f_Y(y) &= P(Y = y) = P(g(X) = y) \\ &= P(X \in \{x \in D_X : g(x) = y\}) \\ &= \sum_{x_i \in \{x \in D_X : g(x) = y\}} f(x_i) \end{aligned}$$

Discrete random variables

Consider the discrete random variable X with probability function

x	-2	-1	0	1	2
$f_X(x)$	$12/60$	$15/60$	$10/60$	$6/60$	$17/60$

Let $Y = X^2$, what is $f_Y(y)$?

Firstly: The set of discontinuities D_Y is $D_Y = \{0, 1, 4\}$

x	-2	-1	0	1	2
$y = x^2$	4	1	0	1	4

Consequently

- $f_Y(0) = P(Y = 0) = P(X^2 = 0) = P(X^2 = 0) = \frac{10}{60}$.
- $f_Y(1) = P(Y = 1) = P(X^2 = 1) = P(X = 1) + P(X = -1) = 6/60 + 15/60 = 21/60$.
- $f_Y(4) = P(Y = 4) = P(X^2 = 4) = P(X = 2) + P(X = -2) = 17/60 + 12/60 = 29/60$.