PART II

INTEREST RATE MODELS

1

STATIC INTEREST RATE MODELS

1.1. INTRODUCTION

FUNDAMENTAL ASSET PRICING FORMULA

$$P = \sum_{t=1}^{N} \frac{C_t}{(1+s_t)^t} + \frac{M}{(1+s_N)^N}$$

Main question: Where do we get s_t from?

- Any relevant information concerning how to price a financial asset must be primarily obtained from market sources

- Spot rate - annualized rate of a pure risk-free discount (or zero-coupon) bond

- As there aren't enough zero-coupon bonds for most countries and currencies, this information will have to be extracted from coupon-paying bonds.

1.2. Fitting the TS of Interest Rates

1.2.1. Direct Methods

- Bootstrapping method
- Carleton and Cooper (1976)
- Interpolation methods:
  - Linear
  - polynomial

BOOTSTRAPPING

- Consider 2 securities (nominal value = 100€):
  - 1-year pure discount bond, with P = 95€.
  - 2-year coupon-paying bond, with coupon rate = 8% and P = 99€.

- 1-year spot rate:

$$95 = \frac{100}{(1 + R_{0,1})}; \; R_{0,1} = 5.26\%$$

- 2-year spot rate:

$$99 = \frac{8}{(1 + R_{0,1})} + \frac{108}{(1 + R_{0,2})^2} = \frac{8}{1.0526} + \frac{108}{(1 + R_{0,2})^2};$$

$$R_{0,2} = 8.7\%$$

- The same type of reasoning can be developed for any number of bonds, e.g. 4 bonds (*d*(*k*) is the discount factor for cash-flows to be paid *k* years from now).

- Solve the following system recursively to obtain d(k):

  101 = 105d(1)

  101.5 = 5.5d(1) + 105.5d(2)

  99 = 5d(1) + 5d(2) + 105d(3)

  100 = 6d(1) + 6d(2) + 6d(3) + 106d(4)

- R(k) is obtained from d(k): d(k)=1/{[1+R(k)]^k} => R(k)=[1/d(k)]^(1/k)-1:

Maturity (k)	Price	Coupon rate	d(k)	R(k)
1	101	5,0%	0,9619	3,960%
2	101,5	5,5%	0,9119	4,717%
3	99	5,0%	0,8536	5,417%
4	100	6,0%	0,7890	6,103%

- Limitation: usually, one cannot find enough bonds with coincident coupon payment dates and longer maturities.

- Moreover, bond maturities are often different from whole number (when measured in years).

- A usual practical way to estimate the yield curve by bootstrapping involves the employment of interbank money market rates for different maturities:

Maturity	Price	Coupon rate	R(k)
O/N			4,40%
1m			4,50%
2m			4,60%
3m			4,70%
6m			4,90%
9m			5,00%
1y			5,10%
1y2m	103,7	5,00%	5,41%
1y9m	102,0	6,00%	5,69%
2y	99,5	5,50%	5,79%

1y2m rate:

$$103.7 = \frac{5}{(1+4.6\%)^{1/6}} + \frac{105}{(1+x)^{1+1/6}}$$

Conclusions

- If one can find different bonds with coincident cash-flow dates and one of them only has one remaining cash-flow date, then one can get the spot rates directly.

- These are spot rates instead of yields (for the shortest bond, the yield is equal to the spot rate) and consequently they do not face the consistency problems of yields.

- Therefore, **we have a single spot rate for each maturity**.

- One can also calculate spot rates by using money market rates.

Carlton and Cooper (1976)

- Estimation of the discount factors by OLS method if the number of bonds is larger than the number of discount factors to be estimated.

$$\underset{(ix1)}{P} = \underset{(ixt)}{CF} \cdot \underset{(tx1)}{d}$$

Where

i = 1, …, *k* - riskless government bonds considered

t = 1,…, *n* - the cash-flows for which the discount factors are calculated.

P = vector of the prices of the *i* bonds (a column vector with *i* rows);

CF = matrix of the cash-flows of the *i* bonds for the *t* cash-flows (*i* rows and *t* columns);

d = vector of the discount factors for the *t* cash-flows (a column vector with *t* cash-flows).

in Carleton. Willard R. and Ian Cooper (1976), "Estimation and Uses of the Term Structure of Interest Rates", Journal of Finance, September, pp. 1067-1083.

- This method has several drawbacks:

(i) it only allows the estimation of some points of the discount function (for the maturities of the cash-flows considered);

(ii) it does not impose any smoothness on the discount function, allowing meaningless shapes; and

(iii) It faces multicolinearity problems resulting from the linear dependence between the cash-flows of, at least, some of the securities considered.

Interpolation - Linear

- Interpolations may be useful if we don't have all market information required to get spot rates for the same maturities.

- Simplest approach - linear interpolations:

  – Assuming that we know discount rates for maturities t_1 and t_2, the rate for maturity t, being $t_1 < t < t_2$, corresponds to the weighted average of the adjacent rates, being the weights higher for the maturity closer to t (e.g. if $t = t_2$, t_1 will not have any relevance to calculate t):

$$R(0,t) = \frac{(t_2 - t)R(0,t_1) + (t - t_1)R(0,t_2)}{(t_2 - t_1)}$$

  – Linear interpolations provide good proxies for near maturities.

  – However, for distant maturities, the shape of the resulting yield curve tends to be kinked.

  – By definition, linear interpolation doesn't allow to get estimates for maturities longer than those observed.

INTERPOLATION – POLYNOMIAL

- Polynomial interpolations of the interest rates allow to obtain smoother yield curves, with interest rates as polynomial functions of maturities.
- A very common polynomial interpolation is the cubic => one can estimate the full term structure just by knowing the spot rates for 4 maturities.
- Therefore, if $R(0, t_1)$, $R(0, t_2)$, $R(0, t_3)$ and $R(0, t_4)$ are known, one can solve the following system in order to the 4 coefficients of the 3rd order polynomial.

$$\begin{cases} R(0,t_1) = at_1^3 + bt_1^2 + ct_1 + d \\ R(0,t_2) = at_2^3 + bt_2^2 + ct_2 + d \\ R(0,t_3) = at_3^3 + bt_3^2 + ct_3 + d \\ R(0,t_4) = at_4^3 + bt_4^2 + ct_4 + d \end{cases} \Rightarrow R = T \cdot A\,,$$

being $R = \begin{bmatrix} R(0,1) \\ R(0,2) \\ R(0,3) \\ R(0,4) \end{bmatrix}, T = \begin{bmatrix} t_1^3 & t_1^2 & t_1 & 1 \\ t_2^3 & t_2^2 & t_2 & 1 \\ t_3^3 & t_3^2 & t_3 & 1 \\ t_4^3 & t_4^2 & t_4 & 1 \end{bmatrix}, A = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

- If one uses more than 4 spot rates, these coefficients are estimated by econometric techniques (as we will have degrees of freedom), e.g. OLS (as the functions are linear in the coefficients).
- Otherwise, $R = T \cdot A \Leftrightarrow A = T^{-1} \cdot R$

EXAMPLE

- The calculation of *a*, *b*, *c* and *d* allows to obtain the spot rate for any maturity *t*: $R(0,t) = at^3 + bt^2 + ct + d$
- Assuming the following rates are known:
  - R(0,1) = 3%
  - R(0,2) = 5%
  - R(0,3) = 5.5%
  - R(0,4) = 6%

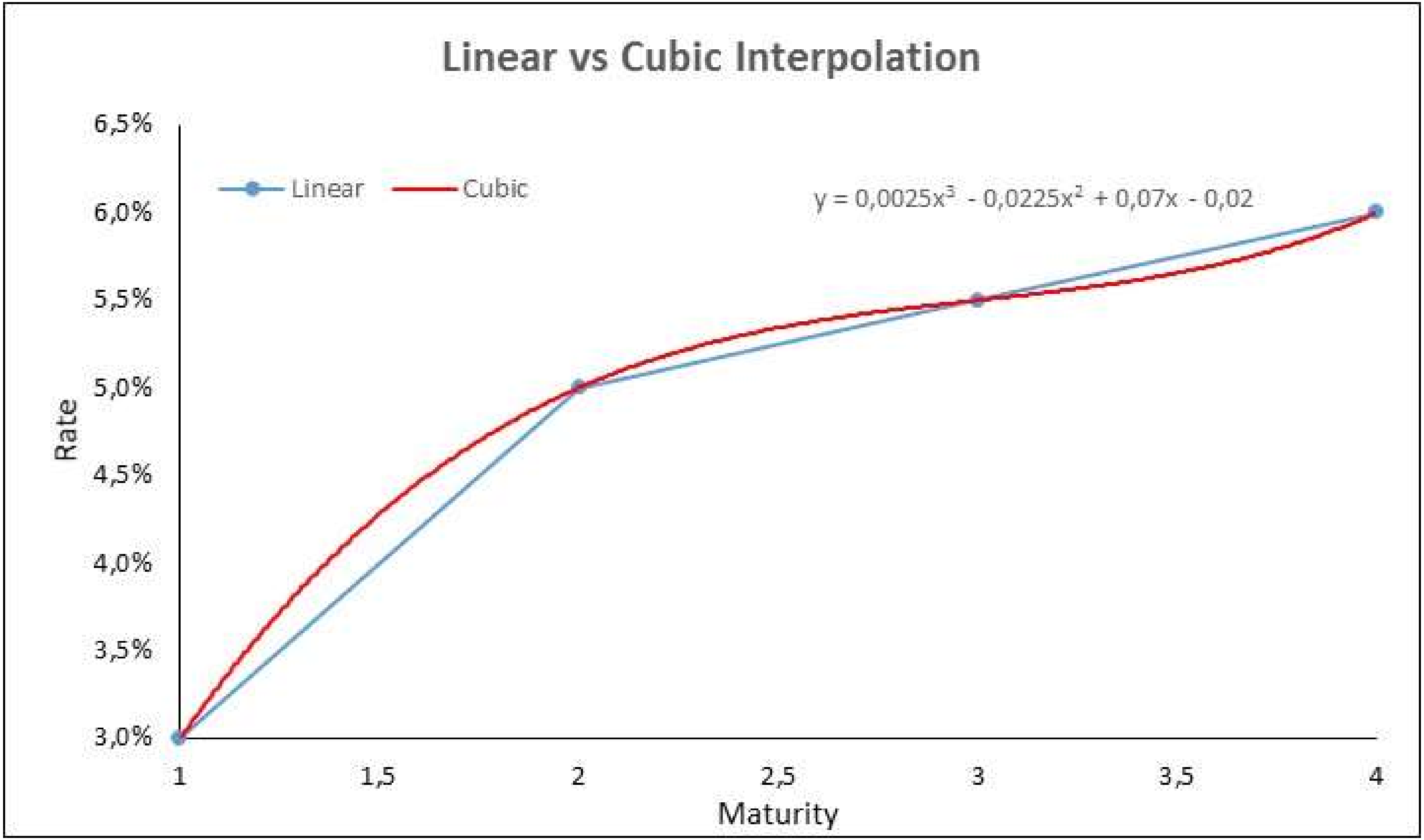
$$\begin{cases} R(0,1) = a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d \\ R(0,2) = a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d \\ R(0,3) = a \cdot 3^3 + b \cdot 3^2 + c \cdot 3 + d \\ R(0,4) = a \cdot 4^3 + b \cdot 4^2 + c \cdot 4 + d \end{cases}$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3\% \\ 5\% \\ 5.5\% \\ 6\% \end{pmatrix} = \begin{pmatrix} 0.0025 \\ -0.0225 \\ 0.07 \\ -0.02 \end{pmatrix}$$

- **Goal - Compute the 2.5 year rate:**

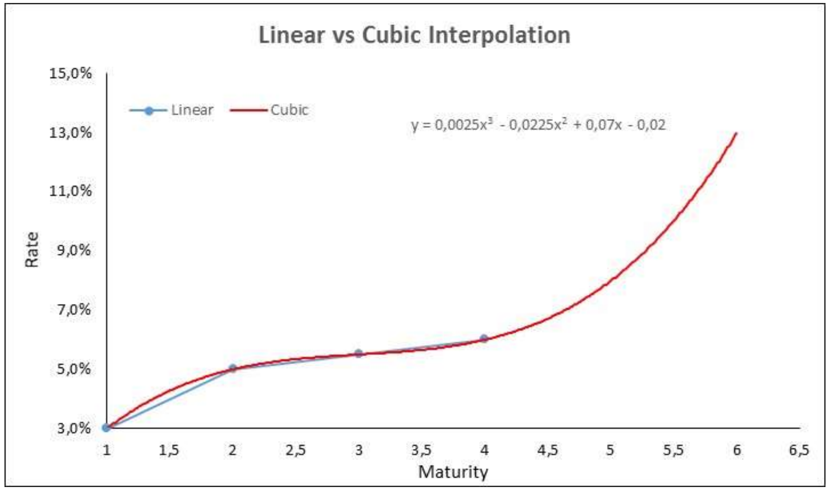
R(0,2.5) = a x 2.5^3 + b x 2.5^2 + c x 2.5^1 + d = 5.34375%

ILLUSTRATION: LINEAR VERSUS CUBIC



CONCLUSIONS

- **The resulting spot curve using 3rd order polynomial methods tends to be too irregular**, namely when:
  - it is used to estimate a rate for a maturity much higher than the maximum maturity used to calculate the polynomial coefficients (e.g. in the previous example the 10-year would be 93%!)
  - the difference between two consecutive maturities is too large.



- **Polynomial splines** improve the adjustment, by allowing different specifications for the polynomials in different maturity buckets.
- Nonetheless, the explosive behavior of the resulting curves is kept.

1.2.2. Spline Methods

Polynomial Functions

- Discount factors (p) as polynomial functions of the maturity (s), with **all coefficients differing in the different maturity buckets**:

$$p(s) = \begin{cases} p_0(s) = d_0 + c_0 s + b_0 s^2 + a_0 s^3, s \in [0,5] \\ p_5(s) = d_1 + c_1 s + b_1 s^2 + a_1 s^3, s \in [5,10] \\ p_{10}(s) = d_2 + c_2 s + b_2 s^2 + a_2 s^3, s \in [10,20] \end{cases}$$

- Imposing continuity constraints and given the fact that the discount factor for zero maturity is 1, the number of parameters is reduced:

$$p_0(5) = p_5(5)$$
$$p_5(10) = p_{10}(10)$$
$$p_0(0) = 1$$

- The number of parameters may be even further reduced if it is assumed that only one of the parameters is different in the several maturity buckets => McCulloch (1971, 1975) splines.*

* McCulloch, J., (1971) "Measuring the term structure of interest rates",The Journal of Business,19-31.
McCulloch, J., 1975. The tax-adjusted yield curve. The Journal of Finance 30, 811-830.

McCulloch Splines

- Dividing the maturity spectrum in *k-2* intervals, with *k-3* vertices or knots, the discount function can be defined as a cubic function, adding a factor (spline) to the 3rd order component, being k = No. of parameters:

$$d(t) = 1 + a_{2,1}t + a_{3,1}t^2 + a_{4,1}t^3 + \sum_{h=1}^{k-3} a_{4,h+1}(t - t_h)^3 \cdot D_h(t)$$

where $D_h(t)$ for h=1,2,…, k-3 are functions defined on the basis of the vertices of the intervals, as follows:

$D_h(t)$ =0, if $t<t_h$, $D_h(t)$ =1, if $t \geq t_h$, for h=1,….,k-3.

- The discount function is continuous ⇔ for all vertices, the values for the discount function are given as: $d(t) = a_0 + \sum_{h=1}^{k} a_h g_h(t)$

- **How to choose the number of parameters/intervals and the vertices:**

– If the number of intervals is very low, the spline adjustment becomes close to the simple polynomial.

– *K*-3 = 1 =>

  - No. of intervals (k-2) =2
  - No. of vertices (k-3) =1
  - No. of parameters =4

- McCulloch proposes the No. intervals (k-2) = square root of the number of observations (bonds), rounded to the nearest integer, with the vertices chosen to ensure all intervals have the same No. observations (or the difference between the No. observations in each interval is not higher than 1).

↓

- With 10 interest rates observed, we should have 3 intervals and 2 vertices.

- Alternative methodology (used more often) - fixing the vertices of the intervals in maturity dates corresponding to the maturities in which the market is traditionally "divided": 1, 3, 5 and 10 years.

- If the vertices of the intervals correspond to the maturities in which the market is traditionally "divided" - 1, 3, 5 and 10 years – we have:
  - No. Intervals: k-2 = 5 (0-1, 1-3, 3-5, 5-10 and > 10y)
  - No. Vertices: k-3 = 4 (1, 3, 5 and 10)
  - No. Parameters: k = 7

$$d(t) = 1 + a_{2,1}t + a_{3,1}t^2 + a_{4,1}t^3 + \sum_{h=1}^{k-3} a_{4,h+1}(t - t_h)^3 \cdot D_h(t)$$

$$d(t) = 1 + a_{2,1}t + a_{3,1}t^2 + a_{4,1}t^3 + a_{4,2}(t-1)^3 \cdot D_1(t) + a_{4,3}(t-3)^3 \cdot D_2(t)$$
$$+ a_{4,4}(t-5)^3 \cdot D_3(t) + a_{4,5}(t-10)^3 \cdot D_4(t)$$

$D_1(t) = 0$, if *t<1*, $D_1(t) = 1$, if $t \geq 1$ $D_2(t) = 0$, if *t<3*, $D_2(t) = 1$,if $t \geq 3$ $D_3(t) = 0$, if *t<5*, $D_3(t) = 1$, if $t \geq 5$ $D_4(t) = 0$, if *t<10*, $D_4(t) = 1$, if $t \geq 10$

- The method of polynomial splines provides us better estimates in sample, i.e. up to the longest observed maturity, comparing to polynomial functions.
- However, the estimation problems outside the sample remain, as the discount function tends to assume irregular shapes from the longest maturity onwards, and it may even become negative.
- Whenever the yield curve assumes complex shapes, the use of a high number of parameters leads the estimated curve to adjust excessively to outliers => yield curve becomes even more irregular.
- This is particularly inconvenient if the objective is, as it usually happens, the estimation of the term structure of interest rates for a fixed or standardised range of maturities, or to calculate forward rates.
- Therefore, more complex specifications will be required.
- Langetieg and Smoot [1981] discuss extensions of McCulloch's spline methodology, namely fitting cubic splines to the spot rates rather than the discount function, and varying the location of the spline **knots.**

1.2.3. DETERMINISTIC METHODS

- Also known as parsimonious or parametric methods, usually involving 3 steps:

  - **Step 1**: select a set of K bonds with prices P^j paying cash-flows $F^j(t_i)$ at dates $t_i>t$
  - **Step 2**: select a deterministic interest rate model for the functional form of the discount factors $p(t,t_i;ß)$, or the discount rates $R(t,t_i;ß)$ (or alternatively spot or forward rates), where $ß$ is a vector of unknown parameters, and generate prices.

$$\hat{P}^j(t) = \sum_{i=1}^{N} CF^j(t_i)p(t,t_i;\beta) = \sum_{i=1}^{N} CF^j(t_i)e^{-(t_j-t)R(t,t_i;\beta)}$$

  - **Step 3**: estimate the parameters $ß$ as the ones making the theoretical prices as close as possible to market prices:

$$\beta = \arg\min \sum_{j=1}^{K} \left(\hat{P}^j(t) - P^j(t)\right)^2$$

- **Key advantages:**

- Parsimonious models, i.e. do not involve many parameters
  - As Friedman (1977) stated, "Students of statistical demand functions might find it more productive to examine how the whole term structure of yields can be described more compactly by a few parameters.“ (Friedman, Milton. 1977. Time perspective in demand for money. Unpublished paper.Chicago: University of Chicago, pp.22).

  - Ensure stable functions

  - Adjust to many possible shapes of the TS

  - Some parameters have economic interpretation

NELSON AND SIEGEL (1987)

- Nelson and Siegel (1987)* proposed to fit the yield curve of US Treasury bills with a flexible and smooth parametric function for the instantaneous forward rate, corresponding to a 3 unobserved factor model (as pointed out in Diebold and Li (2005)):

* Nelson, C R and A F Siegel (1987): "Parsimonious modeling of yield curves", *Journal of Business*, 60, pp 473-89.

$$ {}_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau)} + \beta_2 \cdot \left[(m/\tau) \cdot e^{(-m/\tau)} \right] $$

$$ s_m = \beta_0 + (\beta_1 + \beta_2) \cdot \left[1 - e^{(-m/\tau)} \right] / (-m/\tau) - \beta_2 \cdot e^{(-m/\tau)} $$ as $$ s_m = \frac{1}{m} \cdot \int_0^m {}_\mu f_0 \, d\mu $$

$$ d_m = e^{\left[-\beta_o m - (\beta_1 + \beta_2)\tau \left(1 - e^{-\frac{m}{\tau}} \right) + \beta_2 m \cdot e^{\left(-\frac{m}{\tau} \right)} \right]} $$

β_0 : level parameter - the long-term spot or instantaneous forward rate ($lim_{m \to \infty} s$ or $lim_{m \to \infty} f$)

$\beta_0 + \beta_1$: short-term rate ($lim_{m \to 0} s$ or $lim_{m \to 0} f$)

β_1 : (-) slope parameter

β_2: curvature parameter

τ : influences the speed of convergence of the curve towards the asymptotic value (decay factor).

$\left(1 - \frac{\beta_1}{\beta_2}\right)\tau$: point of inflection of the slope of the forward curve

$\left(2 - \frac{\beta_1}{\beta_2}\right)\tau$: point of inflection of the concavity of the forward curve

- This model is a simplification of a more flexible version, with two time decay factors, τ_1 and τ_2, instead of a single factor:

$$ {}_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau_1)} + \beta_2 \cdot \left[(m/\tau) \cdot e^{(-m/\tau_2)} \right] $$

$$ s_m = \beta_0 + \beta_1 \cdot \left[1 - e^{(-m/\tau_1)} \right] / (m/\tau_1) + \beta_2 \cdot \left[\left[1 - e^{(-m/\tau_2)} \right] / (m/\tau_2) - \left[e^{(-m/\tau_2)} \right] \right] $$

- According to Bliss (1996),* this version is more adequate when a larger set of maturities is fitted.

* Bliss, R.R. (1996), “Testing Term Structure Estimation Methods”, Working Paper 96-12a, Federal Reserve Bank of Atlanta.

- This curve can be written as a 3-factor model, as:

$$s_m = \beta_0 + (\beta_1 + \beta_2) \cdot \left[1 - e^{(-m/\tau)}\right]/(m/\tau) - \beta_2 \cdot e^{(-m/\tau)} \Leftrightarrow$$

$$s_m = \beta_0 + \beta_1 \cdot \left[1 - e^{(-m/\tau)}\right]/(m/\tau) + \beta_2 \cdot \left\{\left[1 - e^{(-m/\tau)}\right]/(m/\tau) - e^{(-m/\tau)}\right\}$$

- Replacing $1/\tau$ by λ and adding a time argument => Dynamic NS (DNS) or Diebold and Li (2006) model: (Diebold, Francis X., and Canlin Li (2006) 'Forecasting the term structure of government bond yields.' *Journal of Econometrics* 130, pp.337–364)

$$s_{t,m} = \beta_{0,t} + \beta_{1,t} \cdot \left[1 - e^{(-\lambda_t m)}\right]/(\lambda_t m) + \beta_2 \cdot \left\{\left[1 - e^{(-\lambda_t m)}\right]/(\lambda_t m) - e^{(-\lambda_t m)}\right\}$$

- β_0, β_1 and β_2 are the 3 latent factors, with the following factor loadings:

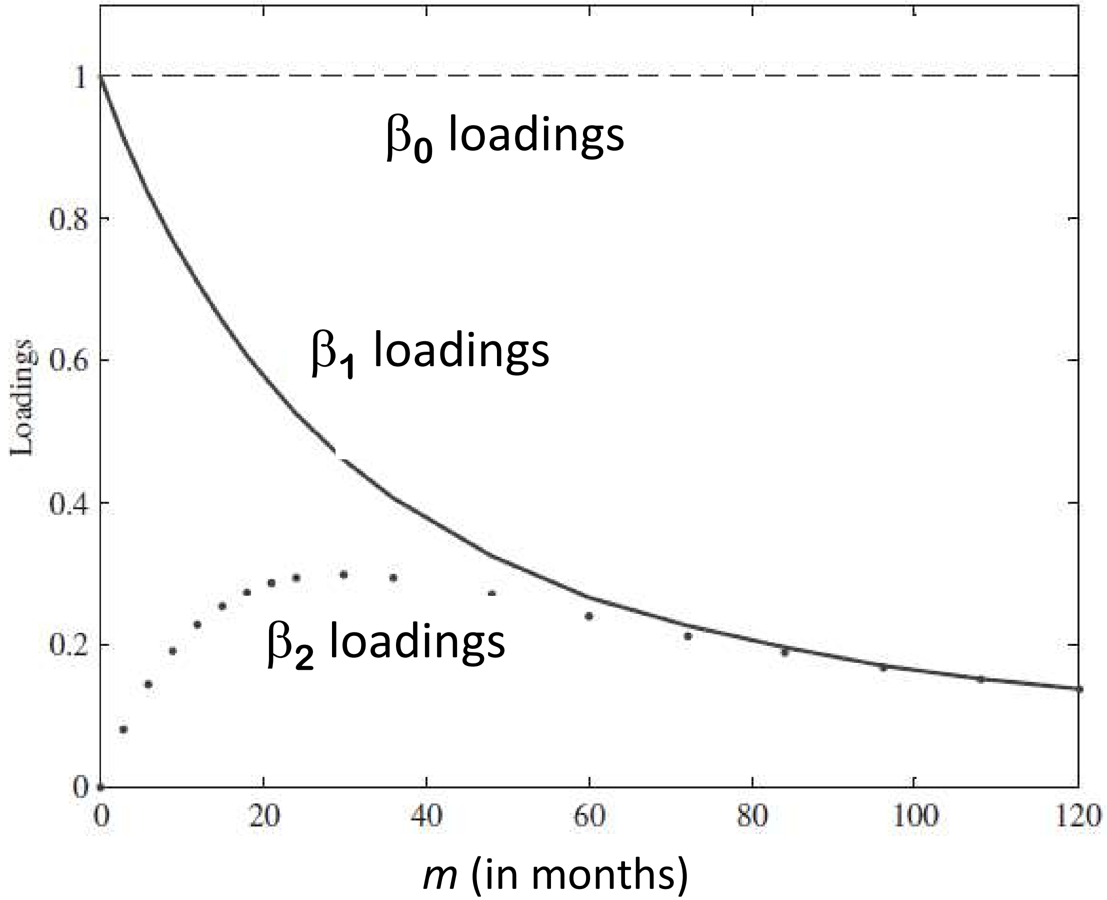
β_0: constant factor loading = 1, i.e. not decaying with maturity => long-term factor

β_1: $\left[1 - e^{(-\lambda_t m)}\right]/(\lambda_t m)$ - factor loading starting at 1 ($lim_{m\to 0}$) and decaying monotonically to 0 ($lim_{m\to\infty}$), governing the slope.

β_2: $\left[1 - e^{(-\lambda_t m)}\right]/(\lambda_t m) - e^{(-\lambda_t m)}$ - factor loading starting at 0 ($lim_{m\to 0}$), increases initially and then decays to 0 ($lim_{m\to\infty}$), driving the curvature.

- Therefore, while from a cross-section perspective, β_0, β_1 and β_2 are the parameters, they are the variables (the factors) from a time-series perspective.

- Factor loadings (with λ = 0,0609):



Source: Diebold, F.X., and C. Li (2006), "Forecasting the Term Structure of Government Bond Yields," Journal of Econometrics, 130, 337–364.

- Diebold and Li (2006) proposed fixing the value of τ in the mean observed value throughout the original sample => model becomes linear and can be estimated by OLS

$$s_{t,m} = \beta_{0,\mathrm{t}} + \beta_{1,\mathrm{t}} \cdot \left[1 - e^{(-\lambda_t m)}\right]/(\lambda_t m) + \beta_2 \cdot \left\{\left[1 - e^{(-\lambda_t m)}\right]/(\lambda_t m) - e^{(-\lambda_t m)}\right\}$$

$$s_{t,m} = \begin{bmatrix}1 & \left[1 - e^{(-\lambda_t m)}\right]/(\lambda_t m) & \left[1 - e^{(-\lambda_t m)}\right]/(\lambda_t m) - e^{(-\lambda_t m)}\end{bmatrix} \begin{bmatrix}\beta_{0,\mathrm{t}} \\ \beta_{1,\mathrm{t}} \\ \beta_2\end{bmatrix}$$

- The main in-sample problem with the regular Nelson-Siegel yield curve for a static fitting is that, for reasonable choices of λ (which are empirically in the range from 0.5 to 1 for U.S. Treasury), the factor loading for the slope and the curvature factor decay rapidly to zero as a function of maturity.

⬇

- Only the level factor is available to fit yields with longer maturities.

⬇

- To address this problem in fitting the cross section of yields, Svensson (1994) introduced an extended version of the Nelson-Siegel yield curve with an additional curvature factor.

- Nelson-Siegel model faces estimation difficulties whenever the yield curve has more than one point of inflection of the slope or concavity.
- This is usually observed after disturbances in money markets.

- Several more flexible NS specifications have been proposed in the literature to improve the fit to more complex shapes, namely with multiple inflection points, introducing additional factors and parameters.
- A popular term-structure estimation method among central banks (see BIS, 2005) to address is the 4-factor Svensson (1994) model, that accommodates 2 changes in the slope or in the concavity.

- Svensson (1994) proposes to increase the flexibility and fit of the NS model by adding a second hump-shape factor with a separate decay parameter – extended NS model.

Svensson (1994)

- The resulting 4-factor forward curve is given by:

$$ {}_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau_1)} + \beta_2 \cdot \left[(m / \tau_1) \cdot e^{(-m/\tau_1)} \right] + \beta_3 (m / \tau_2) e^{(-m/\tau_2)} $$

- Thus, the spot rate will be given by the following expression:

$$ \begin{aligned} s_m &= \beta_0 + \beta_1 \cdot \left[1 - e^{(-m/\tau_1)}\right] / (m/\tau_1) \\ &+ \beta_2 \cdot \left\{ \left[1 - e^{(-m/\tau_1)}\right] / (m/\tau_1) - e^{(-m/\tau_1)} \right\} \\ &+ \beta_3 \cdot \left\{ \left[1 - e^{(-m/\tau_2)}\right] / (m/\tau_2) - e^{(-m/\tau_2)} \right\} \end{aligned} $$

β_0 : level parameter - the long-term rate
β_0 **+** β_1: short-term rate
β_1 : **(-)** slope parameter
β_2, β_3: curvature parameters
τ_1, τ_2 : influences the speed of convergence of the curve towards the asymptotic value.

- Just like with the NS model, there is a dynamic version of the Svensson model (replacing $1/\tau_1$ by λ_1 and $1/\tau_2$ by λ_2) – the DNSS model:

$$ \begin{aligned} s_{t,m} &= \beta_{0,t} + \beta_{1,t} \cdot \left[1 - e^{(-\lambda_{1t} m)}\right] / (\lambda_{1t} m) \\ &+ \beta_2 \cdot \left\{ \left[1 - e^{(-\lambda_{1t} m)}\right] / (\lambda_{1t} m) - e^{(-\lambda_{1t} m)} \right\} \\ &+ \beta_3 \cdot \left\{ \left[1 - e^{(-\lambda_{2t} m)}\right] / (\lambda_{2t} m) - e^{(-\lambda_{2t} m)} \right\} \end{aligned} $$

ESTIMATION

- These methods can be implemented by using non-linear least squares, minimizing a decision variable, that can be the sum of the squared differences between the observed and the estimated yields (or bond prices), or the log-likelihood function.
- NS can even be estimated by OLS, if the τ parameter value is fixed beforehand.
- Usually, yields are used in the estimation, instead of bond prices, as:

(i) the goal is to obtain a fitted yield curve;

(ii) when bond prices are used, the estimation procedure will tend to generate poorer yield estimates, due to the different duration of the bonds.

- Conversely, if yields are used, poorer estimates of bond prices are obtained.

ESTIMATION

- Even though the Svensson method is more adequate to estimate the term structure of interest rates for monetary policy purposes, given its higher adjustment capacity in the segment of the shorter maturities, **when the yield curve assumes simple shapes in the short segment, the estimation by the NS method seems preferable since it is more parsimonious.**

- In fact, the NS model is a restricted version of the Svensson model with the restriction $\beta_3 = 0$ and/or $\tau_2 \rightarrow 0$. Thus, we can test the null hypothesis corresponding to those restrictions:

$$H_0: \quad \beta_0 = \beta_1 = \ldots = \beta_q = 0$$

where: v = likelihood function of the adjustment with restrictions; $v*$ = likelihood function of the adjustment without restrictions; q = number of restrictions.

- The test is based on the following log-likelihood ratio test: $\lambda = -2 \cdot (\ln v - \ln v^*) \approx \chi^2(q)$

- Maximum Likelihood Estimation Problem: $\hat{\beta} : \underset{\beta}{Max} \left[-\frac{n}{2} \ln(2\pi\sigma_\varepsilon) - \frac{1}{2} \sum_{j=1}^{n} \left(\frac{y_j - Y_j(\beta)}{\sigma_\varepsilon} \right)^2 \right]$

ESTIMATION

- In this case, *v* corresponds to the likelihood function of the NS model (the restricted model), while *v** is the likelihood function of the Svensson model.

- Thus, if the logarithm of the likelihood function of the Svensson model is large enough (i.e., is statistically above that of the NS model), the Svensson model will be selected.

⬇

- **H_0 is rejected if $\lambda > \chi^2$ ⇔ Svensson model must be chosen.**

- A potential problem with the Svensson model is that it is highly non-linear, which can make the estimation of the model difficult (see Bolder and Stréliski (1999) for a discussion).

- Nonetheless, one can implement it even in a spreadsheet!

BJÖRK AND CHRISTENSEN

- One alternative model to the Svensson model was developed by Björk and Christensen (1999),* adding a 4th factor to the instantaneous forward curve, but with a different specification for this 4th factor, that depends on a parameter (τ) that is the same in the 3rd factor:

  * Bjork, T. and Christensen B.J. (1999): "Interest rate dynamics and consistent forward rate curves", Mathematical Finance.

$$f_t(\tau) = \beta_{1,t} + \beta_{2,t}\exp\left(-\frac{\tau}{\lambda_t}\right) + \beta_{3,t}\left(\frac{\tau}{\lambda_t}\right)\exp\left(-\frac{\tau}{\lambda_t}\right) + \beta_{4,t}\exp\left(-\frac{2\tau}{\lambda_t}\right)$$

$${}_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau_1)} + \beta_2 \cdot \left[(m/\tau_1)\cdot e^{(-m/\tau_1)}\right] + \beta_3 (m/\tau_2) e^{(-m/\tau_2)}$$

← Svensson (1994)

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t}\left[\frac{1-\exp\left(-\frac{\tau}{\lambda_t}\right)}{\left(\frac{\tau}{\lambda_t}\right)}\right] + \beta_{3,t}\left[\frac{1-\exp\left(-\frac{\tau}{\lambda_t}\right)}{\left(\frac{\tau}{\lambda_t}\right)} - \exp\left(-\frac{\tau}{\lambda_t}\right)\right]$$

$$+ \beta_{4,t}\left[\frac{1-\exp\left(-\frac{2\tau}{\lambda_t}\right)}{\left(\frac{2\tau}{\lambda_t}\right)}\right]$$

The 4th component, resembles the 2nd component, as it also mainly affects short-term maturities.
The difference is that it decays to zero at a faster rate.

BJÖRK AND CHRISTENSEN

Properties:

- The factor in $\beta_{4,t}$ can be interpreted as a second slope factor.
- As a result, Björk and Christensen model captures the slope of the term structure by the (weighted) sum of $\beta_{2,t}$ and $\beta_{4,t}$.
- The instantaneous short rate in this case is given by :

$$y_t(0) = \beta_{1,t} + \beta_{2,t} + \beta_{4,t}$$

DIEBOLD, PIAZZESI AND RUDEBUSCH (2005)

- **Conversely, some authors argue that even the NS model has to many parameters to be estimated, as** the variation in interest rates can be explained mostly by 2 common factors:

- Diebold, Piazzesi, and Rudebusch (2005)* examine a 2-factor NS model, even though they recognize that more than 2 factors may "be needed in order to obtain a close fit to the entire yield curve at any point in time".

- Compared to the 3-factor NS model, the 2-factor model only contains the level and slope factor => only 3 parameters have to be estimated:

$$s_m = \beta_0 + \beta_1 \cdot \left[1 - e^{(-m/\tau)}\right]/(m/\tau)$$

* Diebold, Francis X., Monika Piazzesi and Glenn D. Rudebusch (2005), "Modeling Bond Yields in Finance and Macroeconomics", American Economic Review, 95, pp. 415-420.

CONCLUSIONS

- Despite the drawback that deterministic interest rate models **lack theoretical background**, the BIS concluded that 9 out of 13 central banks which report their curve estimation methods to the BIS use these models (BIS (2005), “Zero-coupon yield curves: technical documentation”, BIS Papers, No 25, Monetary and Economic Department, October 2005).
- According to this study, **most central banks have adopted either the NS (1987) model or the extended version by Svensson (1994)**, with the exception of Canada, Japan, Sweden, UK and the US, which all apply variants of the “smoothing splines” method.
- Deterministic interest rate models are also widely used among market practitioners.
- Given that these models are usually non-linear in the parameters, attention has to be paid to their starting values.