2.2. Short Rate Models

2.2.1. Interest Rate Trees

2.2.2. Continuous-time Single-factor models

2.2.3. Continuous-time Multi-Factor models

2.2.1. Interest Rate Trees

- Focus: How to model the term structure by specifying the behavior of the short-term interest rate?

- Bond and interest rate derivative prices depend on the behavior of the risk-free short-term interest rate (or instantaneous short rate).

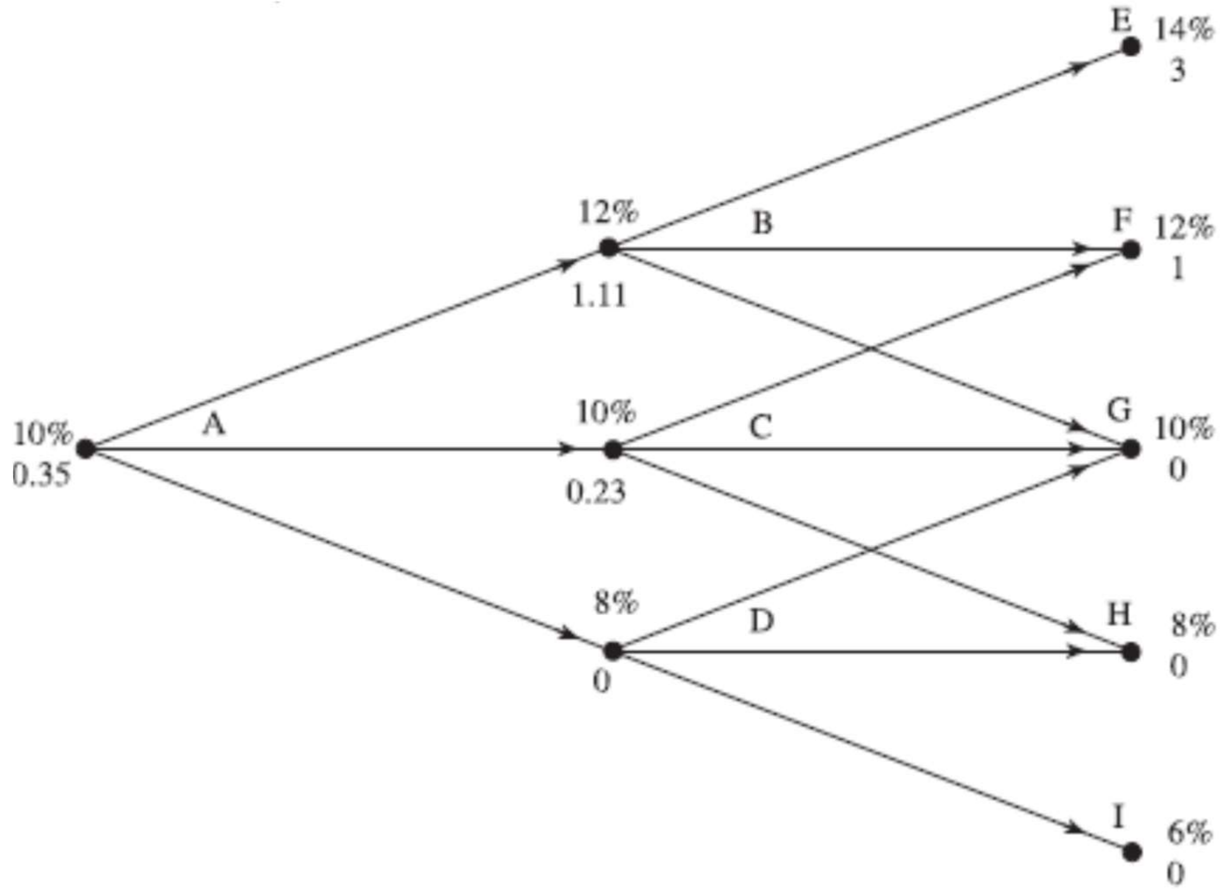
- The variable to be modeled by trees will be the instantaneous short rate.

- Why do we use trees?

A tree is a discrete-time representation of the stochastic process.

- Binomial trees are often used, even though trinomial trees are recommended to value interest rate derivatives.
- At the final nodes, the value of the derivative equals its pay-off.
- At previous nodes, the value of the derivative is calculated through a rollback procedure, calculating the expected value of the derivative according to the probabilities attached to the different scenarios and discounting this expected value using the interest rate at that node.

Figure 32.4 Example of the use of trinomial interest rate trees. Upper number at each node is rate; lower number is value of instrument.



The tree is used to value a derivative that provides a payoff at the end of the second time step of

$$\max[100(R - 0.11), 0]$$

Assumption:

Probabilities of up, middle and down are 0.25, 0.5 and 0.25, respectively.

Derivative value at Node B:

$$[0.25 \times 3 + 0.5 \times 1 + 0.25 \times 0]e^{-0.12 \times 1} = 1.11$$

Derivative value at Node C:

$$(0.25 \times 1 + 0.5 \times 0 + 0.25 \times 0)e^{-0.1 \times 1} = 0.23$$

Derivative value at Node D = 0

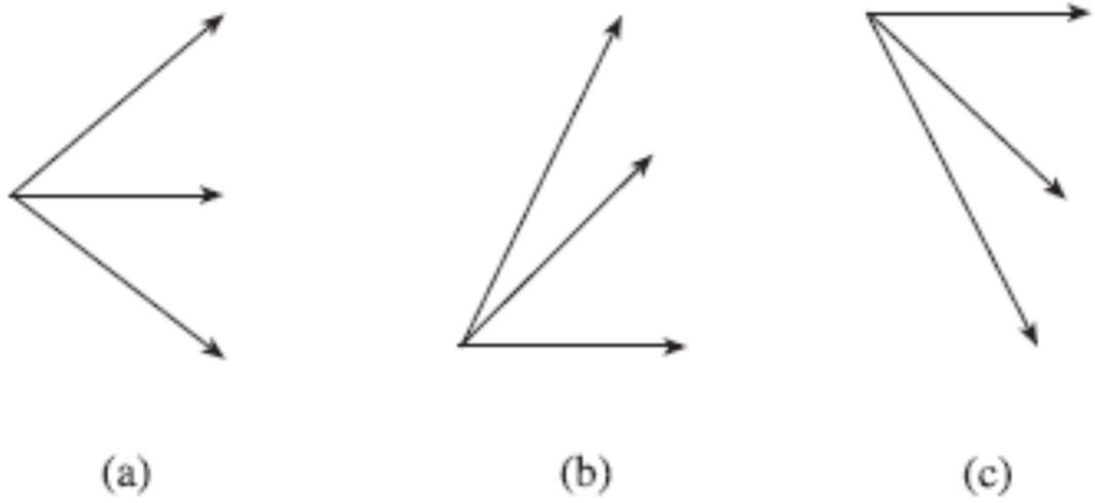
Derivative value at Node A:

$$(0.25 \times 1.11 + 0.5 \times 0.23 + 0.25 \times 0)e^{-0.1 \times 1} = 0.35$$

Source: Hull, John (2018), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 10th Edition

➢ Non-standard branching

Figure 32.5 Alternative branching methods in a trinomial tree.



Source: Hull, John (2018), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 10th Edition

- (b) and (c) are useful to represent mean-reverting interest rates when interest rates are either very low (and are not supposed to move even lower) or very high (and are not supposed to move even higher), respectively.

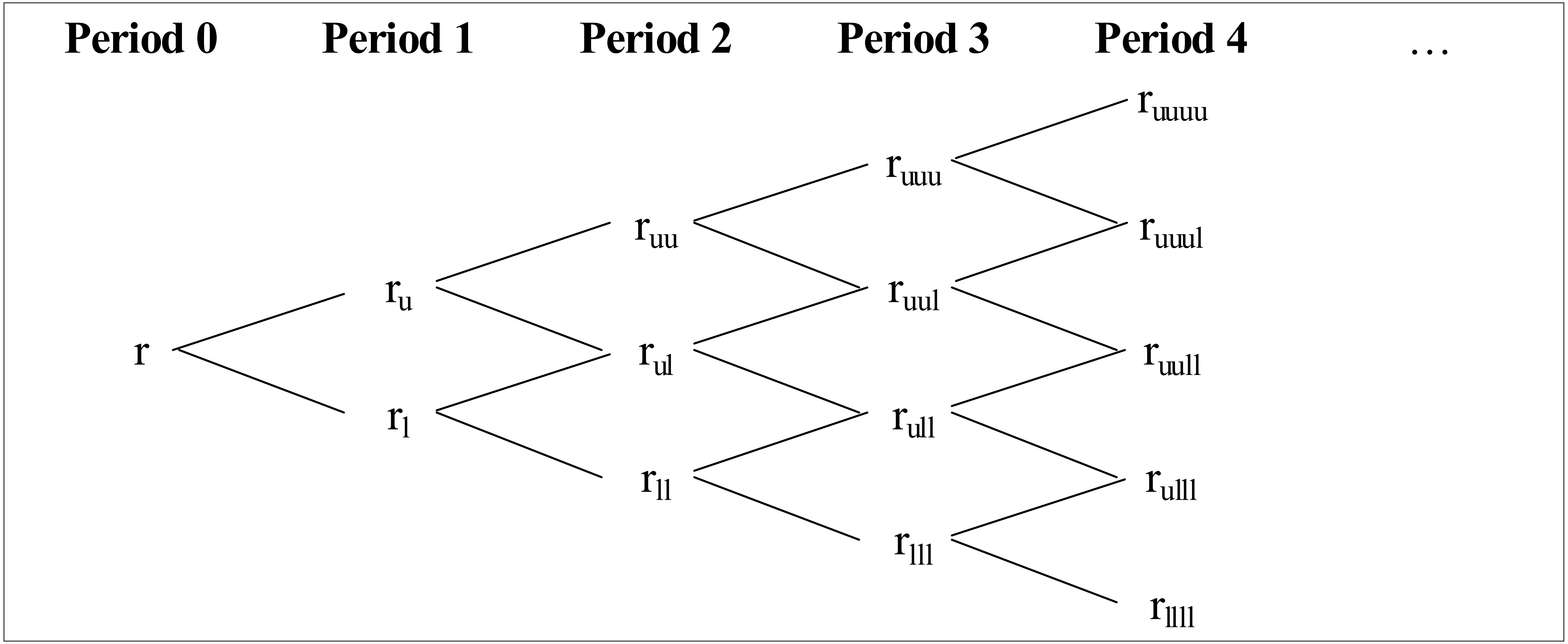
General binomial model

- Given the current level of short-term rate r, the next-period short rate can take only two possible values: an upper value r_u and a lower value r_l, with equal probability 0.5
- In the following period, the short-term interest rate can take on four possible values: r_{uu}, r_{ul}, r_{lu}, r_{ll}
- More generally, in period *n*, the short-term interest rate can take on 2^n values => very time-consuming and computationally inefficient.

Recombining trees

- Means that an upward-downward sequence leads to the same result as a downward-upward sequence (regardless being binomial or trinomial trees)
- For example, $r_{ul} = r_{lu}$
- Only (n+1) different values at period n

INTEREST RATE TREE - Recombining



INTEREST RATE TREE – analytical

- We may write down the binomial process as:

$$\Delta r_t \equiv r_{t+1} - r_t = \sigma \varepsilon_t$$

Itô process

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

$$\Delta r_t \equiv r_{t+\Delta t} - r_t = \mu\left(t, \Delta t, r_t\right) + \sigma\left(t, \Delta t, r_t\right)\varepsilon_t$$

- Specific case – assuming that the drift and the variance are proportional to the time increment:

$$\Delta r_t \equiv r_{t+\Delta t} - r_t = \mu \Delta t + \sigma \sqrt{\Delta t}\,\varepsilon_t$$

- Continuous-time limit (Merton (1973)):

$$dr_t \equiv r_{t+dt} - r_t = \mu dt + \sigma dW_t$$

2.2.2. CT Single Factor Models

- In the first term structure models, the *short rate* is the only driver of the yield curve, being assumed as a continuous and stochastic or random variable.
- Therefore, a single-factor continuous-time model specifies the dynamics of the sort-term rate:

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t$$

- The term *W* denotes a Brownian motion - process with independent normally distributed increments:

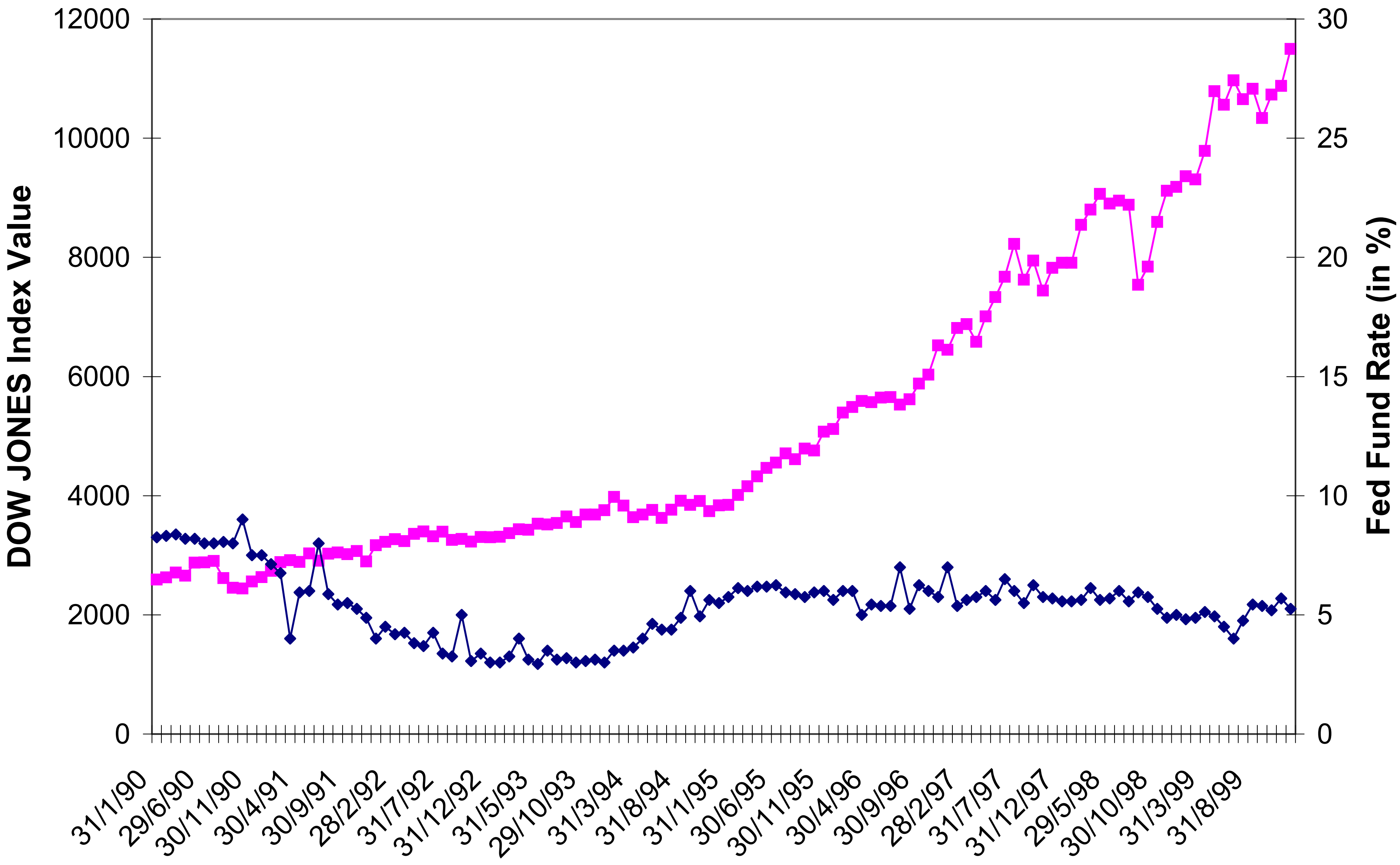
$$dW_t = \varepsilon_t \sqrt{dt}$$

  - *dW* represents the instantaneous change.
  - It is stochastic (uncertain)
  - It is a stochastic variable with a normal distribution with zero mean and variance *dt*
- A latter class of models – HJM (Heath-Jarrow-Morton) models – describe the dynamics of the forward rate.

What is a good Model?

- A good model is a model that is consistent with reality
- Stylized facts about the dynamics of the term structure:
  - Fact 1: (nominal) interest rates are (usually) positive
  - Fact 2: interest rates are mean-reverting
  - Fact 3: interest rates with different maturities are imperfectly correlated
  - Fact 4: the volatility of interest rates evolves (randomly) in time
- A good model should also be:
  - Tractable
  - Parsimonious

Empirical Facts 1, 2 and 4



Note: Interest Rate in blue

Empirical Fact 3

	1M	3M	6M	1Y	2Y	3Y	4Y	5Y	7Y	10Y
1M	1									
3M	0.999	1								
6M	0.908	0.914	1							
1Y	0.546	0.539	0.672	1						
2Y	0.235	0.224	0.31	0.88	1					
3Y	0.246	0.239	0.384	0.808	0.929	1				
4Y	0.209	0.202	0.337	0.742	0.881	0.981	1			
5Y	0.163	0.154	0.255	0.7	0.859	0.936	0.981	1		
7Y	0.107	0.097	0.182	0.617	0.792	0.867	0.927	0.97	1	
10Y	0.073	0.063	0.134	0.549	0.735	0.811	0.871	0.917	0.966	1

EQUILIBRIUM VS NO-ARBITRAGE MODELS OF THE SHORT RATE

Equilibrium models:

(i) the initial yield curve is given by an analytical formula, as a function of the short-term rate, according to the model parameters and assuming that the economy is in equilibrium.

(ii) don’t automatically fit the current TSIR, even though they can provide an approximate fit to many observed TSIRs, if the parameters are properly chosen ⇔ the current TSIR is an output;

(iii) the drift of the short rate is not usually a function of time;

(iv) if the instantaneous short rate follows a Markov process, all rates can be calculated at all times as a function of the short rate.

Equilibrium vs No-Arbitrage Models of the Short Rate

No-arbitrage models:

(i) the current TSIR is an input

(ii) the drift is, in general, dependent on time, as the shape of the initial spot curve governs the average path taken by the short rate in the future – positively sloped zero curve => positive drift for the short rate.

⬇

Equilibrium models can be transformed into no-arbitrage models by including a function of time in the drift of the short rate.

EQUILIBRIUM ONE-FACTOR MODELS OF THE SHORT RATE

- The short-term rate is the single factor => endogenous models:

$$dr = m(r)\,dt + s(r)\,dz$$ → The drift is not a function of time

In a one-factor equilibrium model, the process for *r* involves only one source of uncertainty – the short-term rate.

$m(r) = \mu r;\; s(r) = \sigma r$ (Rendleman and Bartter model)

$m(r) = a(b - r);\; s(r) = \sigma$ (Vasicek model)

$m(r) = a(b - r);\; s(r) = \sigma\sqrt{r}$ (Cox, Ingersoll, and Ross model)

EQUILIBRIUM ONE-FACTOR MODELS OF THE SHORT RATE

1. Rendleman and Bartter

- The short-term interest rate follows a GMB: $dr = \mu r\, dt + \sigma r\, dz$

Rendleman, R. and B. Bartter (1980). "The Pricing of Options on Debt Securities". *Journal of Financial and Quantitative Analysis*. **15**: 11–24).

Pros:

- More tractable model, as it follows a GMB.

Cons:

- Assumes that interest rates follow a stochastic process similar to stocks, while they usually exhibit a mean-reversion behavior.

EQUILIBRIUM ONE-FACTOR CT MODELS OF THE SHORT RATE

2. Vasicek (1977) $dr = a(b - r)\,dt + \sigma\,dz$

Vasicek, O., 1977, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, 177–188.

Also known as Hull and White (1990) model or an Ornstein–Uhlenbeck process.

Hull, J., and White, A., "Pricing Interest Rate Derivative Securities", *Review of Financial Studies*, 1990, pp. 573–592.

Pros:

- More tractable model, due to constant volatility.
- Interest rates are mean-reverting (to *b*), at a reversion rate (pace) *a*.

Cons:

- Gaussian distributions for interest rates are not compatible with market implied distributions.
- Interest rate volatility is often variable, namely during periods of higher uncertainty, when the estimation of interest rates becomes more complex but also more useful.

EQUILIBRIUM ONE-FACTOR CT MODELS OF THE SHORT RATE

3. Cox, Ingersoll and Ross (CIR)

$$dr = a(b - r)\,dt + \sigma\sqrt{r}\,dz$$

Stochastic volatility model => higher volatility with higher interest rates.

Cox, Ingersoll, and Ross. 1985, "A Theory of the Term Structure of Interest Rates", *Econometrica*, Vol 53, March.

Pros:

- Model closer to reality, as interest rates have stochastic volatilities (higher volatilities with higher interest rates).

Cons:

- Model becomes less tractable, as it requires the single factor to be positive.

NO-ARBITRAGE SHORT RATE CT MODELS

1. **Ho-Lee (1986)**

Ho, T.S.Y., and S.-B. Lee, "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, 41 (December 1986): 1011–29.

$$dr = \theta(t)\,dt + \sigma\,dz$$

$\theta(t)$ defines the average direction that r moves at time t:

2. **Hull-White One-Factor Model (1990)**

Hull, J. C., and A. White, "Pricing Interest Rate Derivative Securities," *The Review of Financial Studies*, 3, 4 (1990): 573–92.

Extended version of Vasicek, to provide an exact fit to the initial TSIR: $dr = [\theta(t) - ar]\,dt + \sigma\,dz$ or $dr = a\left[\frac{\theta(t)}{a} - r\right]dt + \sigma\,dz$

Corresponds to Ho-Lee model, with mean reversion at rate a.

NO-ARBITRAGE SHORT RATE CT MODELS

3. Black-Derman-Toy (1990)

Black, F., E. Derman, and W. Toy, "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Prices," *Financial Analysts Journal*, January/February 1990: 33–39.

$$d\ln r = [\theta(t) - a(t)\ln r]\,dt + \sigma(t)\,dz$$

with $a(t) = -\dfrac{\sigma'(t)}{\sigma(t)}$ and $\sigma'(t)$ is the derivative of σ with respect to *t*.

- It is similar to Hull-White One-Factor Model, but in logs and with mean reversion rate *a* being time-dependent.
- It doesn't allow negative interest rates.

Constant volatility => $\sigma'(t)$ = 0 => a(t)=0 => BDT model: $d\ln r = \theta(t)\,dt + \sigma\,dz$

Log-normal version of Ho-Lee model

NO-ARBITRAGE SHORT RATE CT MODELS

4. Black-Karasinski (1991)

Black, F., and P. Karasinski, "Bond and Option Pricing When Short Rates Are Lognormal," *Financial Analysts Journal*, July/August (1991): 52–59.

Extended version of BDT (1990) model, where the reversion rate and volatility are determined independently of each other:

$$d\ln r = [\theta(t) - a(t)\ln r]\,dt + \sigma(t)\,dz$$

The model is the same as BDT (1990), but with no relation between *a(t)* and $\sigma(t)$.

2.2.3. CT MULTI FACTOR MODELS

1. **Fong and Vasicek (1991) model -** short rate and its volatility (*v*) as two state variables

H. G. Fong and O. A. Vasicek: Fixed-income volatility management. Journal of Portfolio Management, 41-56, 1991.

$$dr = \alpha(\bar{r} - r)dt + \sqrt{v}dz_1$$

$$dv = \gamma(\bar{v} - v)dt + \xi\sqrt{v}dz_2$$

2. Longstaff and Schwartz (1992) model

Longstaff, F. A. and E. S. Schwartz, "Interest Rate Volatility and the Term Structure: A Two Factor General Equilibrium Model," *Journal of Finance*, 47, 4 (September 1992): 1259–82.

- Longstaff and Schwartz (1992) uses the same two state variables (the short rate and its volatility), but with a different specification.
- The starting point is a two-factor model, where the drift is governed by two factors or state variables, while the variance is a function of only one of them:

$$\frac{dQ}{Q} = (\mu X + \theta Y)\, dt + \sigma\sqrt{Y}\, dZ_1$$

- With this specification, it is ensured that the drift and the variance are not perfectly correlated.
- The dynamics of the state variables are as follows:

$$dX = (a - bX)\, dt + c\sqrt{X}\, dZ_2$$

$$dY = (d - eY)\, dt + f\sqrt{Y}\, dZ_3 :$$

- With the rescaling of the state variables to $x = X/c^2$ and $y = Y/f^2$,the dynamics of state variables are as follows:

$$dr = \left(\alpha\gamma + \beta\eta - \frac{\beta\delta - \alpha\xi}{\beta - \alpha} r - \frac{\xi - \delta}{\beta - \alpha} V \right) dt$$

$$+ \alpha \sqrt{\frac{\beta r - V}{\alpha(\beta - \alpha)}}\, dZ_2 + \beta \sqrt{\frac{V - \alpha r}{\beta(\beta - \alpha)}}\, dZ_3,$$

$$dV = \left(\alpha^2\gamma + \beta^2\eta - \frac{\alpha\beta(\delta - \xi)}{\beta - \alpha} r - \frac{\beta\xi - \alpha\delta}{\beta - \alpha} V \right) dt$$

$$+ \alpha^2 \sqrt{\frac{\beta r - V}{\alpha(\beta - \alpha)}}\, dZ_2 + \beta^2 \sqrt{\frac{V - \alpha r}{\beta(\beta - \alpha)}}\, dZ_3$$

where $\gamma = a/c^2$, $\delta = b$, $\eta = d/f^2$, $\xi = e$, r is the instantaneous riskless rate,

where $\alpha = \mu c^2$ and $\beta = (\theta - \sigma^2) f^2$

- Relevant features:

(i) All parameters are positive;

(ii) *r* is non-negative, since both state variables follow square root processes;

(iii) *r* has a long-run stationary distribution with mean and variance:

$$E[r] = \frac{\alpha\gamma}{\delta} + \frac{\beta\eta}{\xi} \qquad Var[r] = \frac{\alpha^2\gamma}{2\delta^2} + \frac{\beta^2\eta}{2\xi^2}$$

(iv) Volatilitv also has a stationarv distribution with mean:

$$E[V] = \frac{\alpha^2\gamma}{\delta} + \frac{\beta^2\eta}{\xi} \qquad Var[V] = \frac{\alpha^4\gamma}{2\delta^2} + \frac{\beta^4\eta}{2\xi^2}$$

(v) *r* depends on volatility, but volatility also depends on *r*

- Closed-form expressions for riskless discount bond prices with τ maturity (τ = 0 => F = 1)

$$F(r, V, \tau) = A^{2\gamma}(\tau) B^{2\eta}(\tau) \exp(\kappa\tau + C(\tau) r + D(\tau) V),$$

where

$$A(\tau) = \frac{2\phi}{(\delta + \phi)(\exp(\phi\tau) - 1) + 2\phi},$$

$$B(\tau) = \frac{2\psi}{(\nu + \psi)(\exp(\psi\tau) - 1) + 2\psi},$$

$$C(\tau) = \frac{\alpha\phi(\exp(\psi\tau) - 1) B(\tau) - \beta\psi(\exp(\phi\tau) - 1) A(\tau)}{\phi\psi(\beta - \alpha)}$$

$$D(\tau) = \frac{\psi(\exp(\phi\tau) - 1) A(\tau) - \phi(\exp(\psi\tau) - 1) B(\tau)}{\phi\psi(\beta - \alpha)},$$

and

$$\nu = \xi + \lambda,$$

$$\phi = \sqrt{2\alpha + \delta^2},$$

$$\psi = \sqrt{2\beta + \nu^2},$$

$$\kappa = \gamma(\delta + \phi) + \eta(\nu + \psi).$$

- YTM of riskless discount bonds with τ maturity:

$$Y_\tau = -(\kappa\tau + 2\gamma \ln A(\tau) + 2\eta \ln B(\tau) + C(\tau)r + D(\tau)V)/\tau$$

- For a given τ maturity, the yield is a linear function of *r* and *V*.

- It can be shown that:

$\tau \rightarrow 0 \Rightarrow Y_t \rightarrow r$

$\tau \rightarrow \infty \Rightarrow Y_t$ tends to a constant $\gamma(\phi - \delta) + \eta(\psi - \nu)$

- The current values of *r* and *V* become less relevant for very distant cash-flows.

- The current term structure is irrelevant for the determination of very long interest rates.

- This model offers a much larger variety of shapes than single factor models, with one inflexion point for the slope and the convexity.

- Instantaneous expected return for a discount bond:

$$r + \lambda \frac{(\exp(\psi\tau) - 1)B(\tau)}{\psi(\beta - \alpha)}(\alpha r - V)$$

- Subtracting r from the previous result, one obtains the risk premium.

- For a given τ maturity, the term premium is a linear function of *r* and *V*, depending on λ (market price of risk):
  - λ <0 => term premium > 0.
  - λ =0 => term premium = 0 => Expectations theory holds

- For small τ, the term premium is an increasing function of *r*.

3. Balduzzi et al. (1996) models

Balduzzi, P., S. R. Das, S. Foresi, and R. Sundaran, 1996, "A Simple Approach to Three-Factor Affine Term Structure Models," *The Journal of Fixed Income,* 6, 14–31.

- Balduzzi et al. (1996) suggest the use of a 3-factor model by adding the mean of the short-term rate (θ) to a 2-factor model.

$$dr = \mu_r(r, \theta, t)dt + \sigma_r(r, V, t)dz$$

$$d\theta = \mu_\theta(\theta, t)dt + \sigma_\theta(\theta, t)dw$$

$$dV = \mu_V(V, t)dt + \sigma_V(V, t)dy$$

$$dr = \kappa(\theta - r)dt + \sqrt{V}\,dz$$

$$d\theta = \alpha(\beta - \theta)dt + \eta dw$$

$$dV = a(b - V)dt + \phi\sqrt{V}\,dy$$