Version C

Part I

a) $-2, \frac{1}{5}, 4$; $1, 1$; 1

b) $Q(x,y) = -8xy + 2y^2$

c) $\ln\left(y-(x-1)^2\right) + \frac{1}{(x-1)^2+(y-5)^2}$

d) $\{(-1,0),(1,0)\}$

e) $\{(x,y)\in\mathbb{R}^2 : x^2+(y+2)^2 = 4 \vee (x=0 \wedge y\in[-4,0])\}$

f) $1 < x^2+y^2 \leq 4$

Weierstrass ; compact/closed

g) sphere

h) $+\infty$

i) $u_n = \left(e^2+\frac{1}{n}, \frac{1}{n}\right)$

j) 1

$$\frac{12x^2}{\sqrt{12x^2+8y^2}} + \frac{8y^2}{\sqrt{12x^2+8y^2}} = 1 \cdot \sqrt{12x^2+8y^2}$$

k) $5t^4 + 2t$

l) $\pi - 1$

m) $\begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix}$

n) $(x-9)^3$

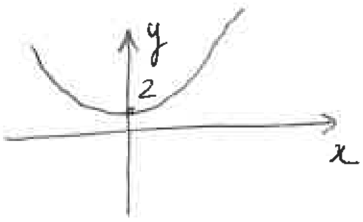
o) $0, 0, 9$

p) $\begin{cases} \dot{y} = y^2 \\ y\left(-\frac{1}{2}\right) = -2 \end{cases}$ or $\begin{cases} y' = -1/x^2 \\ y\left(-\frac{1}{2}\right) = 2 \end{cases}$

q) $e^2 - e$

r) $4, 0, \frac{1}{4}$

s) carrying capacity, decreasing, $p(t) = 1000\, e^{3t}$

t) 

Part $\underline{\text{II}}$ Sketch II.1

1

a)

$\Delta_1 = \alpha$ $\qquad\qquad$ $\alpha \neq \pm 3$

$\Delta_2 = \alpha^2 - 9$

$\Delta_3 = \alpha^2 - 9$

α	$-\infty$	-3		0		3	$+\infty$
Δ_1	$-$	//	$-$	0	$+$	//	$+$
Δ_2	$+$	//	$-$	$-$	$-$	//	$+$
Δ_3	$+$	//	$-$	$-$	$-$	//	$+$
Q	UND	//	UND			//	DP

$\alpha \in]-\infty ; 3[\setminus \{-3\} \Rightarrow Q$ is undefined

$\alpha > 3 \Rightarrow Q$ is positively defined

(using leading minors method)

b)

$$\begin{bmatrix} \alpha & 3 & 0 \\ 3 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \cancel{4}\,1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \alpha + 3 = \cancel{4}\,1 \\ 3 + \alpha = \cancel{4}\,1 \\ \text{—} \end{cases} \Leftrightarrow \begin{cases} \alpha = -2 \\ \text{—} \\ \text{—} \end{cases}$$

$$\boxed{\alpha = -2}$$

2a)

$(x,y) \in \mathbb{R}^2 \setminus \{(x,x), x \in \mathbb{R}\} \Rightarrow$

$\Rightarrow$ f is continuous because it is the product of maps whose denominator does not vanish.

$(x,y) \in \{(x,x), x \in \mathbb{R}\}$

if $(x,y) = (a,a) \neq (0,0)$

$$\lim_{(x,y)\to(a,a)} f(x,y) = 0 \text{ (along all directions)}$$

if $(x,y) = (0,0)$, the denominator vanishes. then

$$0 \leq \left| \frac{x^2(x-y)}{\sqrt{x^2+y^2}} \right| \leq \left| \frac{(x^2+y^2)(x-y)}{\sqrt{x^2+y^2}} \right| \leq \left| \left(\sqrt{x^2+y^2}\right)(x-y) \right|$$

Since $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\,(x-y) = 0$, then using the squeezing theorem, we get

$$\lim_{(x,y)\to(0,0)} \frac{x^2(x-y)}{\sqrt{x^2+y^2}} = 0.$$

2b)

if $y \leq x$, then $f(x,y) = 0$

if $y > x$, then $f(x,y) = \dfrac{x^2(x-y)}{\sqrt{x^2+y^2}} \leq 0$

because $x^2 \geq 0$, $\sqrt{x^2+y^2} > 0$ and $x - y < 0$.

Since at $f(a,a) = 0$ $(a \in \mathbb{R})$, then f admits

zero as global maximum.

2c) f is positively homogeneous of degree 2.

Proof.

$y \leq x \longrightarrow$ trivial

$y > x \longrightarrow f(\lambda x, \lambda y) = \dfrac{\lambda^2 x^2 (\lambda x - \lambda y)}{\sqrt{\lambda^2 x^2 + \lambda^2 y^2}} =$

$$= \frac{\lambda^3 x^2 (x-y)}{|\lambda| \sqrt{x^2+y^2}} \underset{\lambda > 0}{=} \lambda^2 f(x,y).$$

Using Euler Identity, the equality follows.

3 a) $f(x,0) = x^3$

3 b) Critical points: $(0,0)$ and $(-1,-1)$

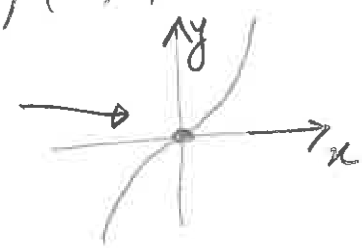
$$\nabla f(x,y) = \left(3x^2 + 3x^2y^3 ; 3y^2 + 3y^2x^3\right)$$

$$\nabla f(x,y) = (0,0)$$

$$\begin{cases} 3x^2 + 3x^2y^3 = 0 \\ 3y^2 + 3y^2x^3 = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2(1+y^3) = 0 \\ 3y^2(1+x^3) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} y = -1 \\ x = -1 \end{cases}$$

At $(0,0)$: Saddle because $f(x,0) = x^3$
and the graph of $f(x,0)$ is



In particular there are points near zero
where f has negative / positive values.

At $(-1,-1)$:

$$H_f(x,y) = \begin{pmatrix} 6x+6xy^3 & 9x^2y^2 \\ 9x^2y^2 & 6y+6yx^3 \end{pmatrix}$$

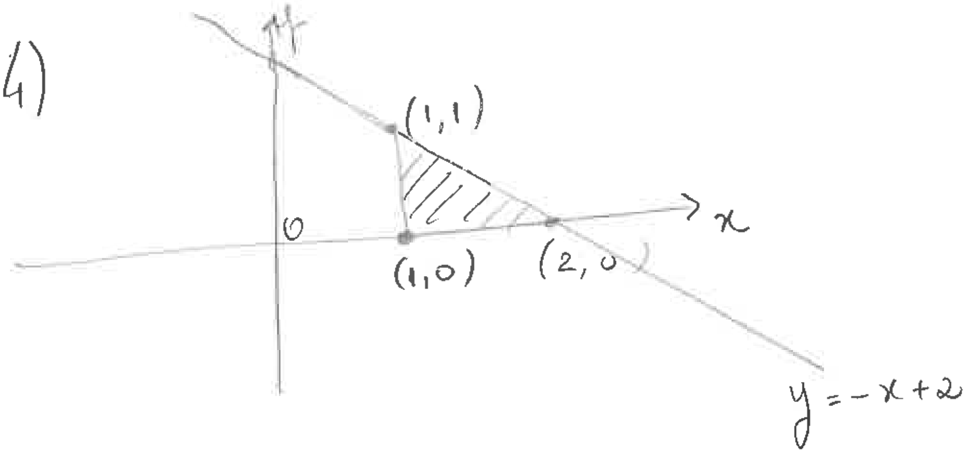
$$H_f(-1,-1) = \begin{pmatrix} -12 & 9 \\ 9 & -12 \end{pmatrix}$$

$$\left.\begin{array}{l} \Delta_1 = -12 \\ \Delta_2 = 12^2 - 81 > 0 \end{array}\right\} \text{ negatively defined}$$

$\Downarrow$

f has a maximum at $(-1,-1)$.

4)



$$\int_1^2 \int_0^{-x+2} xy \, dy \, dx = \int_1^2 \left[\frac{xy^2}{2} \right]_{x=0}^{x=-x+2} dx =$$

$$= \int_1^2 \frac{x(2-x)^2}{2}\,dx =$$

$$= \int_1^2 \frac{4x - 4x^2 + x^3}{2}\,dx =$$

$$= \left[x^2 - \frac{2}{3}x^3 + \frac{x^4}{8}\right]_1^2 =$$

$$= \left(4 - \frac{2}{3}\cdot 8 + \frac{16}{8}\right) - \left(1 - \frac{2}{3} + \frac{1}{8}\right)$$

$$= \left(\frac{18}{3} - \frac{16}{3}\right) - \left(\frac{24 - 16 + 3}{24}\right) = \frac{2}{3} - \frac{11}{24} = \frac{5}{24}$$

5)

$$x^2 y' + xy = x^3$$

$$\xrightarrow{x \neq 0}$$

$$y' + \frac{1}{x} y = x$$

$$\xrightarrow{\forall x\ \mu(x) > 0}$$

$$\mu(x) y' + \mu(x)\frac{1}{x} y = x\,\mu(x)$$

$$\mu'(x) = \mu(x)\,\frac{1}{x}$$

$$\Leftrightarrow \frac{\mu'(x)}{\mu(x)} = \frac{1}{x} \Leftrightarrow \frac{d}{dx}\ln|\mu(x)| = \frac{1}{x}$$

$\Leftrightarrow \ln|\mu(x)| = \ln x + C$

$\overset{\mu(x)>0}{\Leftrightarrow} \quad \mu(x) = Cx, \quad C \in \mathbb{R}\setminus\{0\}$

$\therefore \quad \frac{d}{dx}\left(\cancel{C}x \cdot y\right) = \cancel{C}\, x^2$

$$x\, y(x) = \frac{x^3}{3} + C, \quad C \in \mathbb{R}$$

$$\boxed{y(x) = \frac{x^2}{3} + \frac{C}{x}}, \quad C \in \mathbb{R}$$

Since $y(1) = 4/3 \Leftrightarrow \quad \frac{4}{3} = \frac{1}{3} + \frac{C}{1} \Leftrightarrow \quad C = 1$

$$\therefore \quad \boxed{y(x) = \frac{x^2}{3} + \frac{1}{x}}, \quad x > 0$$

<u>Check:</u>

$$x^2 \cdot y' + x y = x^2\left(\frac{2}{3}x - \frac{1}{x^2}\right) + x \cdot \left(\frac{x^2}{3} + \frac{1}{x}\right)$$

$$= \frac{2}{3}x^3 - 1 + \frac{x^3}{3} + 1 = x^3 \checkmark$$

$$3y(1) = 3\left(\frac{1}{3} + 1\right) = \cancel{3}\left(\frac{4}{\cancel{3}}\right) = 4. \quad //$$