Problem 1.

(a) The absolute risk aversion coefficient is $ara = -\frac{u''(c)}{u'(c)}$. *Sometimes is defined as* $\frac{u''(c)}{u'(c)}$. We scale by $u'(c)$ because expected utility is only defined up to linear transformations – $a+bu(c)$ gives the same predictions as $u(c)$ – and this measure of the second derivative is invariant to linear transformations. *It is a measure of the intensity of an individual's aversion to risk. The higher it is, the higher the risk premium require to induce full investment in a risky investment.* Show that the utility function with constant absolute risk aversion is $u(c) = -e^{-\alpha c}$.

Answer:

$$-\frac{u''(c)}{u'(c)} = \frac{\alpha^2 e^{-\alpha c}}{\alpha e^{-\alpha c}} = \alpha$$

(b) The coefficient of relative risk aversion in a one-period model (i.e. when consumption equals wealth) is defined as $rra = -\frac{cu''(c)}{u'(c)}$. $rra = c \cdot ara$. *For instance under increasing relative risk aversion, that is when* $\frac{\partial rra}{\partial c} > 0$ *the proportion of an individual's wealth invested in the risky asset decreases as his wealth increases. Under constant relative risk aversion* $\frac{\partial rra}{\partial c} = 0$, *that proportion does not depend on the wealth of the individual.*

For power utility $u(c) = c^{1-\gamma}$, show that the risk aversion coefficient equals γ. What if $u(c) = c^{-\gamma}$?

Answer:

For $u(c) = c^{1-\gamma}$:

$$rra = -c\frac{u''(c)}{u'(c)} = -c\frac{(1-\gamma)(-\gamma)\,c^{-\gamma-1}}{(1-\gamma)\,c^{-\gamma}} = \gamma$$

For $u(c) = c^{-\gamma}$

$$rra = -c\frac{u''(c)}{u'(c)} = -c\frac{-(1+\gamma)(-\gamma)\,c^{-\gamma-2}}{(-\gamma)\,c^{-\gamma-1}} = 1+\gamma$$

(c) The elasticity of intertemporal substitution is defined as $\xi^I \equiv -\frac{\frac{d(c_1/c_2)}{c_1/c_2}}{dR/R}$.

Show that with power utility $u(c) = c^{1-\gamma}$, the intertemporal substitution elasticity is equal to $1/\gamma$.

Solution:

Answer: The elasticity of intertemporal substitution is defined as $\xi^I \equiv -\frac{\frac{d(c_1/c_2)}{c_1/c_2}}{\frac{dR}{R}}$

$$d\left(\frac{c_1}{c_2}\right) = \frac{\partial\left(\frac{c_1}{c_2}\right)}{\partial c_1}dc_1 + \frac{\partial\left(\frac{c_1}{c_2}\right)}{\partial c_2}dc_2$$
$$= \frac{dc_1}{c_2} - \frac{c_1 dc_2}{(c_2)^2}$$

$$\frac{d\left(\frac{c_1}{c_2}\right)}{\frac{c_1}{c_2}} = \frac{dc_1}{c_1} - \frac{dc_2}{c_2}$$

Thus:

$$\xi^I \equiv -\frac{\frac{d(c_1/c_2)}{c_1/c_2}}{\frac{dR}{R}} = -\frac{\frac{dc_1}{c_1} - \frac{dc_2}{c_2}}{\frac{dR}{R}}$$

The budget constraints in a 2 period nonstochastic model are:

$$c_1 + s = e_1$$
$$c_2 = e_2 + Rs$$

where e_1 and e_2 are the endowments in period 1 and 2. The intertemporal budget constraint is

$$c_1 + \frac{c_2}{R} = e_1 + \frac{e_2}{R}$$

We can state the investor's maximization problem in a 2 period nonstochastic model as

$$\max_{\{c_1,c_2\}} u(c_1) + \beta u(c_2)$$

subject to the intertemporal budget constraint

The Lagrangian is

$$L = u(c_1) + \beta u(c_2) + \lambda \left(e_1 + \frac{e_2}{R} - c_1 - \frac{c_2}{R} \right)$$

The first order conditions are

$$\begin{aligned} u'(c_1) &= \lambda \\ \beta u'(c_2) &= \frac{\lambda}{R}. \end{aligned}$$

Total differentiation of the first order conditions,

$$d\left(u'(c_1)\right) = \frac{\partial u'(c_1)}{\partial c_1} dc_1 = u''(c_1) dc_1 = d\lambda$$

Divide the equation by $u'(c_1)$ the l.h.s. and by λ the r.h.s.

$$c_1 \frac{u''(c_1)}{u'(c_1)} \frac{dc_1}{c_1} = -\gamma \frac{dc_1}{c_1} = \frac{d\lambda}{\lambda}$$

Total differentiation of the first order conditions,

$$\begin{aligned} d\left(\beta u'(c_2)\right) &= \frac{\partial \beta u'(c_2)}{\partial c_2} dc_2 + \frac{\partial \beta u'(c_1)}{\partial \beta} d\beta = \frac{\partial \frac{\lambda}{R}}{\partial \lambda} d\lambda + \frac{\partial \frac{\lambda}{R}}{\partial R} dR \\ \Leftrightarrow & \ \beta u''(c_2) dc_2 = \frac{d\lambda}{R} - \frac{\lambda dR}{R^2} \end{aligned}$$

Divide the equation by $\beta u'(c_2)$ the l.h.s. and by $\frac{\lambda}{R}$ the r.h.s.

$$\beta c_2 \frac{u''(c_2)}{\beta u'(c_2)} \frac{dc_2}{c_2} = \left(\frac{\lambda}{R}\right)^{-1} \left(\frac{d\lambda}{R} - \frac{\lambda dR}{R^2}\right)$$

$$-\gamma \frac{dc_2}{c_2} = \frac{d\lambda}{\lambda} - \frac{dR}{R}$$

Substituting in ξ^I,

$$\begin{aligned} \xi^I &= -\frac{\frac{dc_1}{c_1} - \frac{dc_2}{c_2}}{\frac{dR}{R}} \\ &= -\frac{-\frac{1}{\gamma}\frac{d\lambda}{\lambda} + \frac{1}{\gamma}\left(\frac{d\lambda}{\lambda} - \frac{dR}{R}\right)}{\frac{dR}{R}} \\ &= 1/\gamma, \end{aligned}$$

we obtain the result, i.e. $\xi^I = 1/\gamma$ with a power utility function.

Problem 2.

The first order conditions for an infinitely lived consumer who can buy an asset with dividend stream $\{D_t\}$ are

$$p_t = E_t \left\{ \sum_{s=1}^{\infty} \beta^s \frac{u'(c_{t+s})}{u'(c_t)} D_{t+s} \right\} \tag{39}$$

The first order conditions for buying a security with price p_t and payoff $x_{t+1} = D_{t+1} + p_{t+1}$ are

$$p_t = E_t \left\{ \beta \frac{u'(y_{t+1})}{u'(y_t)} (D_{t+1} + p_{t+1}) \right\} \tag{40}$$

(a) Derive (40) from (39)

Answer:

(a) Rather obviously, use the equation at t and $t+1$, i.e. start with

$$p_t = E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} D_{t+1} + \beta^2 \frac{u'(c_{t+2})}{u'(c_t)} D_{t+2} + ... \right\}$$

and

$$p_{t+1} = E_{t+1} \left\{ \beta \frac{u'(c_{t+2})}{u'(c_{t+1})} D_{t+2} + \beta^2 \frac{u'(c_{t+3})}{u'(c_{t+1})} D_{t+3} + ... \right\}$$

These 2 equations together imply

$$p_t = E_t \left(\beta \frac{u'(c_{t+1})}{u'(c_t)} D_{t+1} \right) + E_t \left(\beta \frac{u'(c_{t+1})}{u'(c_t)} p_{t+1} \right)$$

(b) Derive (39) from (40). You need an extra condition. Show that this extra condition is a first order condition for maximization. To do this, think about what strategy the consumer could follow to improve utility if the condition did not hold.

Answer:

(b) Substitute recursively,

$$\begin{aligned}
p_t &= E_t \left(\beta \frac{u'(c_{t+1})}{u'(c_t)} p_{t+1} \right) + E_t \left(\beta \frac{u'(c_{t+1})}{u'(c_t)} D_{t+1} \right) \\
&= E_t \left(\beta^2 \frac{u'(c_{t+2})}{u'(c_t)} p_{t+2} \right) + E_t \left(\beta^2 \frac{u'(c_{t+2})}{u'(c_t)} D_{t+2} \right) + E_t \left(\beta \frac{u'(c_{t+1})}{u'(c_t)} D_{t+1} \right) \\
&\dots \\
&= E_t \left\{ \sum_{s=1}^{\infty} \beta^s \frac{u'(c_{t+s})}{u'(c_t)} D_{t+s} \right\} + \lim_{T \longrightarrow \infty} E_t \left(\beta^T \frac{u'(c_{t+T})}{u'(c_t)} p_{t+T} \right)
\end{aligned}$$

The last term is not automatically zero. For example, if $u'(c)$ is a constant, then $p_t = \beta^t$ or greater growth will lead to such a term. It also has an interesting economic interpretation. Even if there are no dividends, if the last term is present, it means the price today is driven entirely by the expectation that someone else will pay a higher price tomorrow. People think they see this behavior in "speculative bubbles" and some models of money work this way. The absence of the last term is a first order condition for optimization of an infinitely-lived consumer. If $p_t < (>) \, E_t \left\{ \sum_{s=1}^{\infty} \beta^s \frac{u'(c_{t+s})}{u'(c_t)} D_{t+s} \right\}$, he can buy (sell) more of the asset, eat the dividends as they come, and increase utility. This lowers c_t, increases c_{t+s} until the condition is filled.

If markets are complete — if he can also buy and sell claims to the individual dividends — then he can do even more. For example, if $p_t >$, then he can sell the asset, buy claims to each dividend, pay the dividend stream of the asset with the claims, and make a sure, instant profit. He does not have to wait forever. (Advocates of bubbles point out that you have to wait a long time to eat the dividend stream, but they often forget the opportunities for immediate arbitrage that a bubble can induce. The plausibility of bubbles relies on incomplete markets.) Bubble type solutions show up often in models with overlapping generations, no bequest motive, and incomplete markets. The OG gets rid of the individual first order condition that removes bubbles, and the incomplete markets gets rid of the arbitrage opportunity. The possibility of bubbles figures in the evaluation of volatility tests.

Problem 3: If $\log x = \mu + \sigma z$ and $z \sim N(0,1)$ so that $y \equiv \log x \sim N(\mu, \sigma^2)$ then $Ex = E\exp(y) = \exp(\mu + \frac{\sigma^2}{2})$.

Answer: $Ex = E\exp(\mu + \sigma z) = \exp(\mu) \int_{-\infty}^{+\infty} \exp(\sigma z) f(z) dz$, where $f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$; thus

$$Ex = \exp(\mu) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\sigma z - \frac{\sigma^2}{2} + \frac{\sigma^2}{2} - \frac{z^2}{2}\right) dz$$

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\sigma z - \frac{\sigma^2}{2} - \frac{z^2}{2}\right) dz$$

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-\sigma)^2}{2}\right) dz$$

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right),$$

since the integral is the normal cumulative distribution function with mean σ and variance 1, i.e. $z \sim N(\sigma, 1)$. Remember if

$$z \sim N\left(\mu, \sigma^2\right)$$

then

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right).$$

Comment:
Let $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, $u'(c_t) = c_t^{-\gamma}$ and assume $\log \dfrac{c_{t+1}}{c_t} \sim N(\mu, \sigma^2)$.

Observation: the combination of power utility function and log-normal distribution is the usual trick to get an analytical solution to the pricing equation.

$$p_t = E_t\left(\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right)$$

Let $c_{t+1} = \left(1 + \mu_{c,t}\right) c_t$, then $\log\left(\dfrac{c_{t+1}}{c_t}\right) = \log\left(1 + \mu_{c,t}\right) \approx \mu_{c,t}$ (net growth rate)

If $\left(\dfrac{c_{t+1}}{c_t}\right)$ has a lognormal distribution

$$\log\left(\frac{c_{t+1}}{c_t}\right) \sim N(\mu, \sigma^2)$$

then

$$-\gamma \log\left(\frac{c_{t+1}}{c_t}\right) \sim N(-\gamma\mu_t, \gamma^2\sigma_t^2)$$

$$\Longrightarrow E \exp\left(-\gamma \log\left(\frac{c_{t+1}}{c_t}\right)\right) = \exp\left(-\gamma\mu_t + \frac{\gamma^2\sigma_t^2}{2}\right)$$

since

$$1 = E_t\left(\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R_{t+1}^f\right)$$

$$\Longrightarrow R_{t+1}^f = \frac{1}{\beta E_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]} = \frac{1}{\beta E_t\left[\exp\left(\log\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right)\right]}$$

$$= \frac{1}{\beta E_t\left[\exp\left(-\gamma \log\left(\frac{c_{t+1}}{c_t}\right)\right)\right]} = \left[\beta \exp\left(-\gamma\mu_t + \frac{\gamma^2\sigma_t^2}{2}\right)\right]^{-1}$$

Take logarithms

$$r_{t+1}^f \equiv \log R_{t+1}^f = \log\left[\exp(-\delta)\exp\left(-\gamma\mu_t + \frac{\gamma^2\sigma_t^2}{2}\right)\right]^{-1}$$

where $\beta = \exp(-\delta)$

$$= \delta + \gamma\mu_t - \frac{\gamma^2\sigma_t^2}{2}.$$

The r_{t+1}^f is high when:

impatience δ is high. People want to save less because they prefer to consume earlier.

when *consumption growth,* μ_t, is high, and higher ***risk aversion*** γ makes interest rates more sensitive to consumption growth, μ_t. People want to save less to smooth consumption and smoothing is more important the more risk averse they are.

The term σ_t^2 captures precautionary savings. When consumption is more volatile, people are more worried about the low consumption states, than they are pleased by the high consumption states. Therefore, people want to save more now, driving down interest rates.

Problem 4: Is it true that the pricing of an asset does not depend on the volatility of the asset's return? why?

Answer: Yes.

$$p_t = E_t\left(m_{t+1}x_{t+1}\right)$$
$$\Longrightarrow p_t = E_t(m_{t+1})E_t(x_{t+1}) + cov_t(m_{t+1}, x_{t+1})$$
$$\Longrightarrow p_t = E_t(x_{t+1})/R^f_{t+1} + cov_t(m_{t+1}, x_{t+1}),$$

using $R^f_{t+1} = 1/E_t(m_{t+1})$ or

$$p_t = \frac{E_t(x_{t+1})}{R^f_{t+1}} + cov_t\left(\beta\frac{u'(c_{t+1})}{u'(c_t)}, x_{t+1}\right)$$

The **first term** is the standard discounted present value formula. This is the asset's price in a risk-neutral world – if consumption is constant or if utility is linear.

The **second term** is a risk adjustment. The price is high if the payoff covaries negatively with consumption and is low if it covaries positively.

Problem 5: What is the relation between the holding return $R^B_{2,t+1} = \frac{B_{1,t+1}}{B_{2,t}}$ and the riskless return $R_{1,t} = \frac{1}{B_{1,t}}$? Which is larger in expected value?

Answer:

$$B_{2,t} = E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} B_{1,t+1} \right\}$$

$$\Longrightarrow 1 = E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{B_{1,t+1}}{B_{2,t}} \right\}$$

$$1 = E_t \left(\beta \frac{u'(c_{t+1})}{u'(c_t)} \right) E_t \frac{B_{1,t+1}}{B_{2,t}} + cov_t \left(\beta \frac{u'(c_{t+1})}{u'(c_t)}, \frac{B_{1,t+1}}{B_{2,t}} \right) \qquad (*)$$

also

$$B_{1,t} = E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \cdot 1 \right\}$$

equation (*) can be writtten as

$$\Longrightarrow \frac{E_t \dfrac{B_{1,t+1}}{B_{2,t}}}{\dfrac{1}{B_{1,t}}} = 1 - \frac{cov_t \left(\beta \frac{u'(c_{t+1})}{u'(c_t)}, B_{1,t+1} \right)}{B_{2,t}}$$

or

$$\frac{E_t R^B_{2,t+1}}{R_{1,t}} = 1 - \frac{cov_t \left(\beta \frac{u'(c_{t+1})}{u'(c_t)}, E_{t+1} \beta \frac{u'(c_{t+2})}{u'(c_{t+1})} \right)}{B_{2,t}}$$

Interpretation?

The *expected excess holding return is positive* i.e. $E_t R^B_{2,t+1} > (<) R_{1,t}$ if $cov_t (m_{t+1}, E_{t+1} m_{t+2}) > (<) 0$. Typically, growth rate of consumption as positive autocorrelation, i.e. $cov \left(\frac{c_{t+1}}{c_t}, \frac{c_{t+2}}{c_{t+1}} \right) > 0$.

Problem 6: A stochastic process $\{p_t\}$ is a *martingale* if $E_t \{p_{t+1}\} = p_t$. In a short period horizon is the price of a security (approximately) a martingale?

Answer: In a short period horizon a security pays no dividends between t and $t+1$, β is close to one and $c_t \sim c_{t+1}$. The pricing equation is $p_t = E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} p_{t+1} \right\}$, under the conditions of the exercise $p_t = E_t \{p_{t+1}\}$.