Foundations of Financial Economics
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Midterm
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Total time: 2 hours and 30 minutes. Total points: 20

Instructions:

Please sit in alternate seats. This is a closed-book, closed-note exam. Please get rid of everything but pen/pencil. In your answer explain all the steps in your reasoning. Keep answers short; I don't give more credit for long answers, and I can take points off if you add things that are wrong or irrelevant.

Formulas:

If x and y are random variables then

$$E\left(xy\right) = ExEy + cov\left(x, y\right); \, \sigma^2\left(x\right) = Ex^2 - \left(Ex\right)^2,$$

$$\sigma^2(ax+by) = a^2\sigma^2(x) + b^2\sigma^2(y) + 2abcov(y, x),$$

$$cov\left(ax, by\right) = abcov\left(x, y\right); \, \sigma^2\left(kx\right) = k^2\sigma^2\left(x\right)$$

$$-1 \leq corr(x, y) \leq 1; \; corr\left(x, y\right) = \frac{cov\left(x, y\right)}{\sigma\left(x\right)\sigma(y)}$$

If x is normal distributed then $\exp(x)$ is lognormal, and

$$E\exp(x) = \exp(Ex + 0.5\sigma^2\left(x\right))$$

I (12 pts.)

1. (12 pts) This group of questions concerns basic concepts.

(1 pt) a. What is a stochastic discount factor?

Answer: Is a random variable that discounts future payoffs to their present value. The price of an asset is: $P_t = E_t\left[m_{t+1}x_{t+1}\right]$ where m_{t+1} is the stochastic discount factor, and x_{t+1} the payoff of the asset.

(1 pt) b. What is the equity premium puzzle?

Answer: The equity premium, i.e. the extra return that investors expect to earn from investing in stocks (equities) over a risk-free asset (typically government bonds), is historically large, about 6%. The puzzle refers to the fact that standard economic models cannot explain why that happens.

(1 pt) c. What is the risk-free puzzle?

Answer: Standard economic models need a very high risk aversion coefficient to explain the equity premium. However, in that case the risk-free return (such as returns on government bonds) implied by the model is much higher than in the data.

(1 pt) d. What is the CAPM?

Answer: The Capital Asset Pricing Model (CAPM) is a financial model that describes the relationship between systematic risk and expected return for assets. It is widely used to estimate an investment's required return based on its risk relative to the overall market: $E[R_i] = R_f + \beta_i [E[R_m] - R_f]$, where $E[R_i]$ is the expected return on the asset, R_f the risk-free rate, R_m the expected return of the market, and β_i is the beta of the asset (a measure of its sensitivity to market movements)

(1 pt) e. What is the expected return, according to CAPM, on an asset that has a beta of 1.5, when the market return is 6%, and the risk-free rate is 2%?

Answer: $2\% + 1.5(6\% - 2\%) = 2\% + 6\% = 8\%$

(1 pt) f. How is the systematic risk measured in the CAPM?

Answer: The beta measures the systematic risk or the volatility of a security relative to the market. It tells how much the security's returns are expected to change in response to changes in the returns of the market. Formal definition: $\beta_i = Cov(R_i, R_m)/Var(R_m)$

(1 pt) g. What is the mean variance frontier?

Answer: It represents the set of portfolios that offer the highest expected return for a given level of risk (variance or standard deviation) or the lowest risk for a given level of expected return.

(1 pt) h. What is an Arrow-Debreu security?

Answer: It is a state-contingent security that pays one unit of the "good" if a specific state of the world occurs and zero otherwise.

(1 pt) i. What is the relationship between the riskless security and the Arrow-Debreu securities?

Answer: A riskless bond is equivalent to holding one unit of every Arrow security in all possible future states. This ensures a guaranteed constant payment.

(1 pt) j. Consider the case when all assets have independent returns. Is it true that a portfolio with more assets has always lower volatility (i.e. standard deviation of returns)? Explain.

Answer: No. Example: Consider assets A and B with variance of returns $\sigma^2(R_A)$ and $\sigma^2(R_B)$. A portfolio with weights ϖ_A and ϖ_B on assets A and B has a variance: $\sigma^2(\varpi_A R_A + \varpi_B R_B) = \varpi_A^2 \sigma^2(R_A) + \varpi_B^2 \sigma^2(R_B)$. Consider a portfolio C, with $\varpi_A = \varpi_B = 0.5$. Assume $\sigma^2(R_A) = 1$ and $\sigma^2(R_B) = 8$. The volatility of portfolio C's returns is: $\left(\frac{1+8}{4}\right)^{0.5} = \frac{3}{2}$ and the volatility of the returns of portfolio A is 1. Asset B has a higher volatility than A but it B has a higher return than A it might be held in equilibrium.

(1 pt) k. Show that when the returns of the assets are independent and have equal volatilities then an investor that minimizes portfolio returns volatility should hold in his/her portfolio all the assets available.

Answer: Let N be the total number of assets an agent holds in his/her portfolio. If an investor holds an equal amount of each asset then the variance of the portfolio returns is $\sigma_{R_p}^2 = \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 \sigma_R^2 = \frac{1}{N}\sigma_R^2$. The $\sigma_{R_p}^2$ decreases with N.

(1 pt) l. Can there be arbitrage opportunities when markets are complete? Explain.

Answer: In a complete market, the law of one price holds, identical securities (or their duplications) must have the same price, preventing risk-free profit opportunities.

II (4 pts.)

2. Consider a representative agent economy. The consumer maximizes $E_0 \sum_{t=0}^{\infty} \beta^t u(C_t)$, where C_t is consumption of period t. There is a risk free asset that pays return R_{t+1}^f, known at time t. There is also a risky asset that pays return R_{t+1}, unknown at time t and realized at time $t+1$. The representative household chooses to invest quantity B_t in the riskless asset, and quantity S_t in the risky asset, to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}, \;\; 0 < \beta < 1$$

subject to the budget constraint

$$B_{t+1} + S_{t+1} + C_t = R_t^f B_t + R_t S_t, \text{ for all } t \geq 0.$$

(1 pt) a. Write the Lagrangian of the household's problem.
Answer: The Lagrangian is:

$$\begin{aligned} L \;\; = \;\; & E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t}{1-\gamma} \right]^{1-\gamma} + \\ & \sum_{t=0}^{\infty} \lambda_t \left[\begin{array}{c} R_t^f B_t + R_t S_t \\ -B_{t+1} - S_{t+1} - C_t \end{array} \right] \end{aligned}$$

(1 pt) b. What are the first order conditions of the consumer's problem?
Answer:

$$B_{t+1} : C_t^{-\gamma} = R_{t+1}^f \beta E_t \left[C_{t+1}^{-\gamma} \right]$$

$$S_{t+1} : C_t^{-\gamma} = \beta E_t \left[C_{t+1}^{-\gamma} R_{t+1} \right]$$

Rearranging we have:

$$1 = R_{t+1}^f E_t \left[\frac{\beta C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \right]$$

$$1 = E_t \left[\frac{\beta C_{t+1}^{-\gamma}}{C_t^{-\gamma}} R_{t+1} \right]$$

(2 pt) c. Define $\Delta c_{t+1} \equiv \ln C_{t+1} - \ln C_t$. Assume that Δc_{t+1} is an i.i.d. normal distribution with mean μ and variance σ^2. Obtain the analytical solution for the riskless return.

Answer:

$$C_t^{-\gamma} = R_{t+1}^f E_t \beta C_{t+1}^{-\gamma}$$

Can rewrite as

$$1 = \beta E_t \exp\left(-\gamma \Delta c_{t+1} + r_{t+1}^f\right)$$

where $r_{t+1} = \ln(R_{t+1})$. Since $\exp(\Delta c_{t+1})$ is lognormal then

$$1 = \beta \exp\left(-\gamma\mu + r_{t+1}^f + \frac{1}{2} var(-\gamma \Delta c_{t+1})\right),$$

get the risk free rate is:

$$r_{t+1}^f = -\log\beta + \gamma\mu - \frac{\gamma^2}{2}\sigma^2$$

III (2 pts.)

The problem of an investor that wants to create a portfolio with minimum variance return, using asset A and asset B, is the following:

$$\min_{\{w_A, w_B\}} \sigma^2(w_A R_A + w_B R_B)$$

$$\text{s.t.}$$

$$w_A + w_B = 1$$

(1 pt) a. Determine the w_A that solves the problem.

Answer: Differentiate the portfolio variance

$$w_A^2 \sigma_{R_A}^2 + (1 - w_A)^2 \sigma_{R_B}^2 + 2 w_A (1 - w_A) cov(R_A, R_B)$$

w.r.t. w_A and equate to zero

$$w_A = \frac{\sigma_{R_B}^2 - cov(R_A, R_B)}{\sigma_{R_A}^2 + \sigma_{R_B}^2 - 2cov(R_A, R_B)}$$

(1 pt) b. Assume that $cov(R_A, R_B) > 0$ and $\sigma_{R_B} < \sigma_{R_A}$. Show that w_A is higher, the higher is the volatility of asset B.

Answer:

$$\frac{\partial w_A}{\partial \sigma_{R_B}^2} = \frac{\sigma_{R_A}^2 - cov(R_A, R_B)}{\left[\sigma_{R_A}^2 + \sigma_{R_B}^2 - 2cov(R_A, R_B)\right]^2}$$

Since $-1 \leq corr(x, y) = \frac{cov(x,y)}{\sigma(x)\sigma(y)} \leq 1$, then $\sigma_{R_A}^2 - cov(R_A, R_B) > \sigma_{R_A}^2 - \sigma_{R_A}\sigma_{R_B} = \sigma_{R_A}(\sigma_{R_A} - \sigma_{R_B}) > 1$. It follows that $\frac{\partial w_A}{\partial \sigma_{R_B}^2} > 1$.

IV (2 pts.)

Assume there are S states of nature in the economy and that markets are complete.

(1 pt) a. Characterize formally the stochastic discount factor.

Answer: Let q_s be the price of an Arrow-Debreu security:

$$q_s = E_t \{m_{t+1} e^s\}$$

where e^s is the Arrow-Debreu security of state s. It follows that in state s the stochastic discount factor is

$$m_s = \frac{q_s}{\pi_s}.$$

(1 pt) b. Let the payoffs of a security be $X(s) = s$, where $s = 1, 2, ..., S$. What is the price of this security?

Answer:

$$P(X) = \sum_{s=1}^{S} s q_s$$