Foundations of Financial Economics
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Second Exam
June, 2, 2025
Total time: 2 hours. Total points: 20

Instructions:

Please sit in alternate seats. This is a closed-book, closed-note exam. Please get rid of everything but pen/pencil. In your answer explain all the steps in your reasoning. Keep answers short; I don't give more credit for long answers, and I can take points off if you add things that are wrong or irrelevant.

Formulas:

If x and y are random variables then

$$E(xy) = ExEy + cov(x,y); \; \sigma^2(x) = Ex^2 - (Ex)^2,$$

$$\sigma^2(ax+by) = a^2\sigma^2(x) + b^2\sigma^2(y) + 2abcov(y,x),$$

$$cov(ax,by) = abcov(x,y); \; \sigma^2(kx) = k^2\sigma^2(x)$$

$$-1 \leq corr(x,y) \leq 1; \; corr(x,y) = \frac{cov(x,y)}{\sigma(x)\sigma(y)}$$

If x is normal distributed then $\exp(x)$ is lognormal, and $E\exp(x) = \exp(Ex + 0.5\sigma^2(x))$.

I (12 pts.)

1. (12 pts) This group of questions concerns basic concepts.

(1 pt) a. What is an arbitrage opportunity?

(1 pt) b. What are complete markets?

(1 pt) c. Why doesn't idiosyncratic risk matter in complete markets?

(1 pt) d. What is the relationship between complete markets and Pareto optimality (in a pure endowment economy)?

(1 pt) e. What is the expected return, according to CAPM, on an asset that has a beta of 2, when the market return is 5%, and the risk-free rate is 1%?

(1 pt) f. State four assumptions of the CAPM.

(1 pt) g. What are factor pricing models?

(1 pt) h. What is the Fama-French three-factor model?

(1 pt) i. Explain why is never optimal to exercise an American call option on a security that pays no dividends before the expiration date.

(1 pt) j. What is the expression for the price of an European put option.

(1 pt) k. What is the Black-Scholes formula?

(1 pt) l. What is the volatility smile

II (6 pts.)

Consider a representative agent economy. Let $m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$ be the stochastic discount factor. There is a risk free asset that pays return R^f_{t+1}, known at time t. There are N risky assets with returns, $R_{i,t+1}$ for $i = 1, ..., N$.

(2 pts) a. Show that

$$\begin{aligned} E_t(R_{i,t+1}) &> R^f_{t+1} \text{ if and only if } cov_t\{R_{i,t+1}, m_{t+1}\} < 0; \\ E_t(R_{i,t+1}) &< R^f_{t+1} \text{ if and only if } cov_t\{R_{i,t+1}, m_{t+1}\} > 0. \end{aligned}$$

(1 pt) b. What is the intuition for the result above?

(2 pts) c. The overall return generated by a portfolio that includes all available risky assets in the market, weighted by their market values, is equal to $R_{m,t+1} = c_{t+1}/p_{m,t}$, where c_{t+1} is the representative agent consumption and $p_{m,t}$ is the cost of the market portfolio's. Assume that the temporary utility function is quadratic, $u(c_t) = c_t - \frac{1}{2}c_t^2$. Show that

$$E_t(R_{i,t+1}) - R^f_{t+1} = \frac{cov_t(R_{i,t+1}, R_{m,t+1})}{var_t(R_{m,t+1})}\left[E_t(R_{m,t+1}) - R^f_{t+1}\right]$$

(1 pt) d. Provide the intuition for the equation above.

Solution:

(a) $1 = E_t(m_{t+1}) E_t(R_{i,t+1}) + cov_t(m_{t+1}, R_{i,t+1})$

since $E_t(m_{t+1}) = \dfrac{1}{R^f_{t+1}}$

$\Longrightarrow 1 = \dfrac{E_t(R_{i,t+1})}{R^f_{t+1}} + cov_t(m_{t+1}, R_{i,t+1})$

$\Longrightarrow E_t(R_{i,t+1}) - R^f_{t+1} = -R^f_{t+1} cov_t(m_{t+1}, R_{i,t+1})$

$\Longrightarrow E_t(R_{i,t+1}) > R^f_{t+1}$ if and only if $cov_t(m_{t+1}, R_{i,t+1}) < 0$

and $E_t(R_{i,t+1}) < R^f_{t+1}$ if and only if $cov_t(m_{t+1}, R_{i,t+1}) > 0$

(b)

Assets whose returns covary positively with consumption make consumption more volatile, and so must promise higher expected returns to induce investors to hold them. Thus, assets that covary negatively with consumption have expected rates of return that are lower than the risk-free rate.

$cov_t(m_{t+1}, R_{i,t+1}) = \dfrac{\beta}{u'(c_t)} cov_t(u'(c_{t+1}), R_{i,t+1}) < 0$ if and only if returns, $R_{i,t+1}$, covary negatively with the marginal utility of consumption, $u'(c_{t+1})$, or equivalently vary positively with consumption, c_{t+1}.

(c) From

$E_t(R_{i,t+1}) - R^f_{t+1} = -R^f_{t+1} cov_t(m_{t+1}, R_{i,t+1})$

$\Longrightarrow E_t(R_{m,t+1}) - R^f_{t+1} = -R^f_{t+1} cov_t(m_{t+1}, R_{m,t+1})$

Substituting for R^f_{t+1} in the first equation

$\Longrightarrow E_t(R_{i,t+1}) - R^f_{t+1} = \dfrac{cov_t(m_{t+1}, R_{i,t+1})}{cov_t(m_{t+1}, R_{m,t+1})}\left[E_t(R_{m,t+1}) - R^f_{t+1}\right]$

Assuming $u(c_t) = c_t - \frac{1}{2}c_t^2$

$\Longrightarrow m_{t+1} = \beta \frac{1-c_{t+1}}{1-c_t}$

Replacing in the equation above

$E_t\left(R_{i,t+1}\right) - R^f_{t+1} = \dfrac{cov_t\left(c_{t+1}, R_{i,t+1}\right)}{cov_t\left(c_{t+1}, R_{m,t+1}\right)} \left[E_t\left(R_{m,t+1}\right) - R^f_{t+1}\right]$

Dividing the numerator and denominator by $p_{m,t}$ we get

$E_t\left(R_{i,t+1}\right) - R^f_{t+1} = \dfrac{cov_t\left(R_{m,t+1}, R_{i,t+1}\right)}{var_t\left(R_{m,t+1}\right)} \left[E_t\left(R_{m,t+1}\right) - R^f_{t+1}\right]$

(d) This equation is the familiar CAPM relation stated in the expected return/beta language. It says that the risk premium in security i is proportional to that of the market portfolio. The proportionality is equal to the ratio of the covariance between the market portfolio's return and the security i's return and the variance of the market portfolio's return.

III (2 pts.)

The stochastic discount factor with the Epstein and Zin utility is

$$m_{t+1} = \beta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma\theta} R^{\theta-1}_{m,t+1},$$

where γ and θ are parameters.

(1 pt) a. Assume that $\log\left(\frac{C_{t+1}}{C_t}\right) \equiv \Delta c_{t+1}$ (growth rate of consumption) is normal distributed, $\log\left(R_{m,t+1}\right) = r_{m,t+1}$ (market return) is normal distributed and the two processes are independently distributed. What is the expression for the riskless interest rate?

(1 pt) b. Can these preferences explain the risk free rate puzzle?

Solution:

Then $\exp(\Delta c_{t+1})$ and $\exp(r_{m,t+1})$ are lognormal distributed.

$$\left(R^f_{t+1}\right)^{-1} = E_t\left(m_{t+1}\right)$$

$$\left(R^f_{t+1}\right)^{-1} = E_t \exp\left(\log\left(\beta^\theta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma\theta} R^{\theta-1}_{m,t+1}\right)\right)$$

$$\left(R^f_{t+1}\right)^{-1} = e^{\log \beta^\theta} E_t e^{-\gamma\theta(\Delta c_{t+1})} E_t e^{-(1-\theta) r_{m,t+1}}$$

using the fact that for independent variables $Exy = ExEy$

$$\left(R^f_{t+1}\right)^{-1} = e^{\log \beta^\theta} e^{-\gamma\theta E_t(\Delta c_{t+1}) + \frac{1}{2}(\gamma\theta)^2 \sigma^2(\Delta c_{t+1})} e^{-(1-\theta)E_t r_{m,t+1} + \frac{1}{2}(1-\theta)^2\sigma^2(r_{m,t+1})}$$

taking logarithms

$$\begin{aligned} r^f_{t+1} &= -\theta \log \beta + \gamma\theta E_t\left(\Delta c_{t+1}\right) - \frac{1}{2}\left(\gamma\theta\right)^2 \sigma^2\left(\Delta c_{t+1}\right) \\ &\quad + \left(1-\theta\right) E_t r_{m,t+1} - \frac{1}{2}\left(1-\theta\right)^2 \sigma^2\left(r_{m,t+1}\right) \end{aligned}$$

if $r_{m,t+1}$ and Δc_{t+1} are jointly lognormal distributed (not independent) then there is an additional term

$$r^f_{t+1} = ... - \gamma\theta\,(1-\theta)\,cov_t(\Delta c_{t+1}, r_{m,t+1})$$