1 Modeling and Solving Linear Programming Problems

1. For each one of the following problems write a linear programming model and solve it using the solver of the Excel.

   (a) Geppetto's Woodcarving, Inc., manufactures two types of wooden toys: soldiers and trains. A soldier sells for $27 and uses $10 worth of raw materials. Each soldier that is manufactured increases Geppetto's variable labor and overhead costs by $14. A train sells for $21 and uses $9 worth of raw materials. Each train build increases Geppetto's variable labor and overhead costs by $10. The manufacture of wooden soldiers and trains requires two types of skilled labor: carpentry and finishing. A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hour of finishing and 1 hour of carpentry labor. Each week, Geppetto can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours. Demand for trains is unlimited, but at most 40 soldiers are bought each week. Geppetto wants to maximize weekly profit (revenues – costs).

   (b) Farmer Giles must determine how many acres of corn and wheat to plant this year. An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week. An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week. All wheat can be sold at $4 a bushel, and all corn can be sold at $3 a bushel. Seven acres of land and 40 hours per week of labor are available. Government regulations require that at least 30 bushels of corn have to be produced during the current year.

      i. Let x_1 be number of acres of corn planted, and x_2 be number of acres of wheat planted. Using these decision variables, formulate an LP whose solution will tell Farmer Giles how to maximize the total revenue from wheat and corn.

      ii. Check the feasibility of the following points
      $(x_1 = 2, x_2 = 3)$
      $(x_1 = 4, x_2 = 3)$
      $(x_1 = 2, x_2 = 1)$
      $(x_1 = 3, x_2 = 2)$

      iii. Using the variables x_1 = number of bushels of corn produced and x_2 = number of bushels of wheat produced reformulate Farmer Giles's LP.

   (c) Truckco manufactures two types of trucks: 1 and 2. Each truck must go through the painting shop and assembly shop. If the painting shop were completely devoted to painting Type 1 trucks, then 800 per day could be painted; if the painting shop were completely devoted to painting Type 2 trucks, then 700 per day could be painted. If the assembly shop were completely devoted to assembling truck 1 engines, then 1,500 per day could be assembled; if the assembly shop were completely devoted to assembling truck 2 engines, then 1,200 per day could be assembled. Each Type 1 truck contributes $300 to profit; each Type 2 truck contributes $500. The profit of the company should be maximized.

(d) Al Ferris has \$60,000 that he wishes to invest now in order to use the accumulation for purchasing a retirement annuity in 5 years. After consulting with his financial adviser, he has been offered four types of fixed-income investments, which we will label as investments A, B, C, D. Investments A and B are available at the beginning of each of the next 5 years (call them years 1 to 5). Each dollar invested in A at the beginning of a year returns \$1.40 (a profit of \$0.40) 2 years later (in time for immediate reinvestment). Each dollar invested in B at the beginning of a year returns \$1.70 three years later. Investments C and D will each be available at one time in the future. Each dollar invested in C at the beginning of year 2 returns \$1.90 at the end of year 5. Each dollar invested in D at the beginning of year 5 returns \$1.30 at the end of year 5.

(e) A cargo plane has three compartments for storing cargo: front, center, and back. These compartments have capacity limits on both weight and space, as summarized below:

Compartment	Weight capacity (tons)	Space capacity (cubic feet)
Front	12	7000
Center	18	9000
Back	10	5000

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the airplane. The following quantities of four cargoes are available for shipment on an upcoming flight and have volume and profit as displayed:

Cargo	total Weight (tons) available	Volume (cubic feet/ton)	Profit (\$/ton)
1	20	500	320
2	16	700	400
3	25	600	360
4	13	400	290

Any portion of these cargoes can be accepted. The objective is to determine how much tons (if any) of each cargo should be accepted and how to distribute each among the compartments to maximize the total profit for the flight.

(f) A company has at its service 1000 skilled workers, semi-skilled 2300 workers and 800 unskilled workers. The semi-skilled workers can become skilled workers if they attend a one-year course, which costs 500 monetary units (m.u.) per worker. Unskilled workers can improve to be semi-skilled by training courses with a cost of 350 m.u. per worker. The company intends to plan the training of its staff over the next two years so that at the end of the planning period:

- unskilled workers cannot represent more than 10% of the total;
- at least 40% of the workers should attend a training course;
- at least 35% of the amount spent on training should be for unskilled workers

How much should the company spend on training courses?

(g) A farmer is raising pigs for market, and he wishes to determine the quantities of the available types of feed that should be given to each pig to meet certain nutritional requirements at a minimum total cost. The number of units of each type of basic nutritional ingredient contained within a kilogram of each feed type is given in the following table, along with the daily nutritional requirements and feed costs.

Nutritional Ingredient	Kg of Corn	Kg of Tankage	Kg of Alfalfa	Min Daily Requirement
Carbohydrates	90	20	40	200
Protein	30	80	60	180
Vitamins	10	20	60	150
Cost (m.u.)	42	36	30	

The objective is to minimize the cost of feeding one pig.

(h) A plant imports three types of thread - cotton, wool and fiber - to produce three different types of cloth - C1, C2 and C3. The cloths should follow the specifications below:

Cloth	Selling price (m.u./kg)	Specifications
C1	680	At least 60% of cotton and at most 20% of fiber
C2	570	At most 60% of fiber and at least 15% of wool
C3	450	At most 50% of fiber

The aim is to determine the production plan so that the profit is maximized, using the information about availabilities and cost given in the following table:

Thread	Availabilities (kg)	Cost (m.u./kg)
Cotton	2 000	700
Wool	2 500	500
Fiber	1 200	400

2. **Portfolio Optimization under Risk Constraints.** A company is given three financial assets for investment. The goal is to allocate a total investment budget across these assets in order to maximize the portfolio's expected return, subject to risk and diversification constraints.

The expected returns and linearized risk contributions for each asset are:

Asset i	Expected Return μ_i	Risk Contribution ρ_i	Bounds $[l_i, u_i]$
1	8.0%	0.05	[0, 0.7]
2	6.0%	0.02	[0, 0.7]
3	10.0%	0.10	[0, 0.7]

The total portfolio must satisfy:

- all capital must be invested (i.e., the weights must sum to 1),
- the total linear risk exposure must not exceed $R^{\max} = 0.06$.

(a) Formulate the problem as a linear program.

(b) Solve the LP using a software tool of your choice (e.g., Python + `scipy.optimize.linprog`, Python + PuLP, Excel Solver, XPRESS, Gurobi, etc.).

(c) Report the optimal solution, the expected return and risk of the optimal portfolio.

(d) Interpret the results: which asset is most preferred? How tight is the risk constraint?

(e) Try slightly increasing $R^{\max}$ (e.g., to 0.065). How does the optimal solution change? What does this tell you about the sensitivity of the portfolio to risk appetite?

3. Solve the following problems using the graphical method, the Solver of Excel and the Simplex method.

(a) $\max z = x_1 + 2x_2$
s. a
$$\begin{aligned} & x_1 - 2x_2 \leq 3 \\ & x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(b) $\max z = 3x_1 + 4x_2$
s. a
$$\begin{aligned} & x_1 - 2x_2 \geq 4 \\ & x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(c) $\max z = x_1 + x_2$
s. a
$$\begin{aligned} & x_1 - x_2 \leq 2 \\ & x_1 - x_2 \geq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(d) $\max z = x_1 - x_2$
s. a
$$\begin{aligned} & x_1 - x_2 \leq 2 \\ & x_1 - x_2 \geq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(e) $\max z = 3x_1 + 6x_2$
s. a
$$\begin{aligned} & x_1 + 2x_2 \leq 4 \\ & x_1 - x_2 \geq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(f) $\min z = x_1 + x_2$
s. a
$$\begin{aligned} & x_1 - x_2 \leq 2 \\ & x_1 - x_2 \geq -2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(g) $\max z = -10x_1 - 5x_2$
s. a
$$\begin{aligned} & x_1 - x_2 \leq 5 \\ & x_1 + \tfrac{8}{5}x_2 \geq -3 \\ & x_1 \in \mathbb{R},\ x_2 \geq 0 \end{aligned}$$

(h) $\min z = x_1 + x_2$
s. a
$$\begin{aligned} & 2x_1 + x_2 \geq 4 \\ & x_1 - x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Some solutions

1.1a $x^* = (20, 60),\ z^* = 180,$

x_1 number of soldiers to produce per week

x_2 number of trains to produce per week

$$\begin{aligned} \max \quad & z = 3x_1 + 2x_2, \\ \text{s. to:} \quad & 2x_1 + x_2 \le 100, \\ & x_1 + x_2 \le 80, \\ & x_1 \qquad \le 40, \\ & x_1, x_2 \ge 0. \end{aligned}$$

1.1b (i) $x^* = (3, 2.8),\ z^* = 370$. (iii) $x^* = (30, 70),\ z^* = 370$.

1.1c $x^* = (0, 700),\ z^* = 350000,$

x_j number of trucks of type j to produce per day, $j = 1, 2,$

$$\begin{aligned} \max \quad & z = 300x_1 + 500x_2, \\ \text{s. to:} \quad & \frac{1}{800}x_1 + \frac{1}{700}x_2 \le 1, \\ & \frac{1}{1500}x_1 + \frac{1}{1200}x_2 \le 1, \\ & x_1, x_2 \ge 0. \end{aligned}$$

1.1d On year 1 invest 60000 on A, on year 3 invest 84000 on A, and on year 5 invest 117600 on D; $x^* = (60000, 0, 0, 0, 0, 84000, 0, 0, 117600),\ z^* = 152880$.

1.1e

1.1f 840 semi-skilled workers will become skilled workers, and 800 unskilled workers will become semi-skilled workers, $x^* = (840, 800, 0),\ z^* = 700000$.

1.1g $x^* = (1.143; 0; 2.429),\ z^* = 120.857$.

1.1h $x^* = (2000; 0; 0; 666.667; 1833.333; 0; 666.667; 533.333; 0),\ z^* = 485666.667$.

1.2 Data:

μ_i	expected return of asset i, $i = 1, \ldots, n$
l_i, u_i	minimum and maximum portfolio weights for asset i
ρ_i	linearized measure of risk exposure for asset i
$R^{\max}$	maximum total risk exposure allowed

Decision variables:

$$w_i \ge 0 \quad \text{portfolio weight assigned to asset } i,\ i = 1, \ldots, n.$$

Formulation:

$$\begin{aligned}
\max_{w} \quad & \sum_{i=1}^{n} \mu_i w_i && \text{(maximize expected return)} \\
\text{s.t.} \quad & \sum_{i=1}^{n} w_i = 1 && \text{(capital fully invested)} \\
& l_i \leq w_i \leq u_i && \forall i \quad \text{(weight bounds)} \\
& \sum_{i=1}^{n} \rho_i w_i \leq R^{\max} && \text{(risk constraint)} \\
& w_i \geq 0 && \forall i
\end{aligned}$$

Notes:

- This is a linearized version of the Markowitz model.
- Variance or VaR constraints are approximated by linear inequalities.

Optimal solution: $x^* = (0.7000,\ 0.0625,\ 0.2375)$.

Expected return: 8.35%.

Total risk exposure: 0.0600.

1.3.(a) unique optimal solution $x^* = (0, 3),\ z^* = 6$.

1.3.(b) empty feasible region, problem impossible.

1.3.(c) unbounded feasible region, problem unbounded.

1.3.(d) alternative optimal solutions, semi-line $x^* = (2.0) + \beta(1, 1), \beta \geq 0,\ z^* = 2$. The extreme point $(2, 0)$ is an optimal solution and all the points in the semi-line $(2, 0) + \beta(1, 1)$ are also optimal.

1.3.(e) alternative optimal solutions, the segment line between the points $(4, 0)$ and $(\frac{4}{3}, \frac{4}{3})$, $z^* = 12$. We can write the segment line as $x^* = \alpha(\frac{4}{3}, \frac{4}{3}) + (1 - \alpha)(4, 0), \alpha \in [0, 1]$ and, alternatively, we can also write $x^* = \alpha_1(4, 0) + \alpha_2(\frac{4}{3}, \frac{4}{3}),\ \alpha_1 + \alpha_2 = 1, \alpha_1, \alpha_2 \geq 0$.

1.3.(f) unique optimal solution, $x^* = (0, 0),\ z^* = 0$.

1.3.(g) unique optimal solution, $x^* = (-3, 0),\ z^* = 30$.

1.3.(h) unique optimal solution, $x^* = (\frac{5}{3}, \frac{2}{3}),\ z^* = \frac{7}{3}$.