Universidade de Lisboa
Instituto Superior de Economia e Gestão
Doctorate Degree (PhD) in Economics

Advanced Mathematical Economics – 1st Semester - 2025/2026

First Test - 14th of October 2025

Duration: $(75 + \varepsilon)$ minutes, $|\varepsilon| \leq 15$

Version B

Name: ....

Student ID #: ....

- Give your answers in exact form.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. Consider the following IVP (y is a function of $t \neq 0$):

$$\begin{cases} y' + \frac{y}{t} = e^t \\ y(1) = 1 \end{cases}$$

   (a) Write the integrating factor associated to the equation and sketch its graph.

   (b) Write the solution of the IVP, identifying its maximal domain.

2. (Logistic law by Verhulst) Let p be a C^∞ map such that (p depends on $t \in \mathbb{R}_0^+$):

$$\begin{cases} p' = 5p - p^2 \\ p(0) = a \end{cases}$$

   (a) Write all possible values of $a \in \mathbb{R}_0^+$ for which the solution of the IVP is monotonically increasing.

   (b) Solve explicitly the IVP for $a = 3$.

3. Consider the linear system of ODEs in $\mathbb{R}^2$ given by (x and y depend on $t \in \mathbb{R}$):

$$\begin{cases} \dot{x} = x + y \\ \dot{y} = 3x + 3y \end{cases}$$

   (a) Write the general form of the solution.

   (b) Sketch the phase portrait of the system. Locate, in the phase portrait, the **unique** solution such that $x(0) = 0$ and $y(0) = 3$. What is the limit of this solution when $t \to -\infty$?

4. Consider the differential equation of second order
(y is a function of $t \neq \pi/2 + k\pi, k \in \mathbb{Z}$):

$$y'' + y = \frac{1}{\cos t}$$

   (a) Find a fundamental system of solutions of the associated homogeneous equation, say $\{\varphi_1, \varphi_2\}$.

   (b) Compute $W[\varphi_1(t), \varphi_2(t)]$, where W denotes the Wronskian operator.

   (c) Using the method of variation of constants, find the general solution of the differential equation.

5. (Lotka-Volterra system) Consider the system of ODEs in $\mathbb{R}^2$ given by (x and y depend on $t \in \mathbb{R}$):

$$\begin{cases} \dot{x} = 2x - xy \\ \dot{y} = -4y + 8xy \end{cases}$$

   (a) Find the equilibria of the system.

   (b) Classify the origin $(0, 0)$ according to its Lyapunov-stability.

   (c) Sketch the phase portrait of the system in a small neighbourhood of $(0, 0)$. Please justify.

Credits:

1(a)	1(b)	2(a)	2(b)	3(a)	3(b)	4(a)	4(b)	4(c)	5(a)	5(b)	5(c)
1	1	0.5	1.5	1	1	0.5	0.5	1	0.5	0.5	1