Test 1 – Advanced Mathematical Economics

14th October 2025

Question 1:

$$\begin{cases} y' + \frac{1}{t} y = e^t \\ y(1) = 1 \end{cases}$$

a) $\mu(t) \rightarrow$ integrant factor $(\mu(t) > 0, \forall t \in \mathbb{R}^+)$

$$\mu(t)\, y'(t) + \frac{1}{t} y(t)\mu(t) = \mu(t) e^t$$

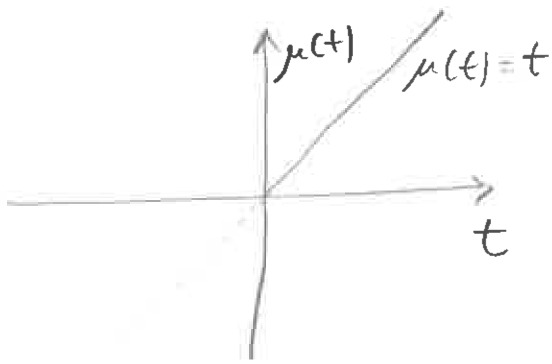
We are going to force:

$$\mu'(t) = \frac{1}{t}\mu(t)$$

$$\frac{\mu'(t)}{\mu(t)} = \frac{1}{t} \Leftrightarrow \frac{d}{dt} \ln|\mu(t)| = \frac{1}{t}$$

$$\Leftrightarrow \ln|\mu(t)| = \ln t$$

$$\Leftrightarrow \boxed{\mu(t) = t}$$



1b) $\frac{d}{dt}\left[\mu(t)\, y(t)\right] = e^t \cdot \mu(t)$

$\overset{\mu(t)=t}{\Leftrightarrow} \quad \frac{d}{dt}\left[t\, y(t)\right] = e^t \cdot t$

$\Leftrightarrow \quad t\, y(t) = e^t \cdot t - e^t + k, \quad k \in \mathbb{R}$

$\Leftrightarrow \quad \boxed{y(t) = e^t - \frac{e^t}{t} + \frac{k}{t}} \quad k \in \mathbb{R}.$

$\int \underbrace{e^t}_{g'} \underbrace{t}_{f}\, dt = e^t t - \int e^t dt = e^t t - e^t$

Since $y(1) = 1 \Leftrightarrow 1 = \cancel{e} - \frac{\cancel{e}}{1} + \frac{k}{1} \Rightarrow k = 1$

$\boxed{y(t) = e^t - \frac{e^t}{t} + \frac{1}{t}} \quad t \in \mathbb{R}^+$

Maximal domain.

2 a) $\begin{cases} p' = 5p - p^2 \\ p(0) = a \end{cases}$

$p' = \underbrace{p(5-p)}_{f(p)}$ $\quad f'(p) = 5 - 2p \quad f'(0) = 5 \quad f'(5) = -5$

Equilibria : $\{0, 5\}$

If $a \in]0,5[$, then $p(t)$ is monotonically increasing.

b) $p' = p(5-p) \Leftrightarrow \dfrac{p'}{p(5-p)} = 1$

$$\frac{1}{p(5-p)} = \frac{A}{p} + \frac{B}{5-p}$$

$$\Rightarrow 1 = A.(5-p) + Bp \Leftrightarrow \begin{cases} A = 1/5 \\ B = 1/5 \end{cases}$$

$$\int \frac{p'}{p(5-p)} = \int 1\, dt$$

$$\Leftrightarrow -\frac{1}{5}\ln p - \frac{1}{5}\ln(5-p) = t + C, \quad C \in \mathbb{R}$$

$$\Leftrightarrow \ln\left(\frac{p}{5-p}\right) = 5t + C, \quad C \in \mathbb{R}$$

$$\frac{p}{5-p} = e^{5t} \cdot k, \quad k \in \mathbb{R}$$

$$\frac{5-p}{p} = \frac{1}{e^{5t} \cdot k} \Leftrightarrow \frac{5}{p} = 1 + \frac{1}{e^{5t} \cdot k}$$

$$\Rightarrow p = \frac{e^{5t} k \cdot 5}{e^{5t} k + 1} \Leftrightarrow p = \frac{5}{1 + \tilde{k} e^{-5t}}, \tilde{k} \in \mathbb{R}$$

$$p(0) = 3 \Leftrightarrow 3 = \frac{5}{1+\tilde{k}} \Leftrightarrow 3 + 3\tilde{k} = 5 \Leftrightarrow \tilde{k} = 2/3$$

$$\therefore p(t) = \frac{5}{1 + \frac{2}{3} e^{-5t}}, \quad t \in \mathbb{R}^+$$

3 a)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$P(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 \\ 3 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 3 = \lambda^2 - 4\lambda$$
$$= \lambda(\lambda - 4)$$

$$E_0 \quad \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad E_0 = \langle (1,-1) \rangle$$

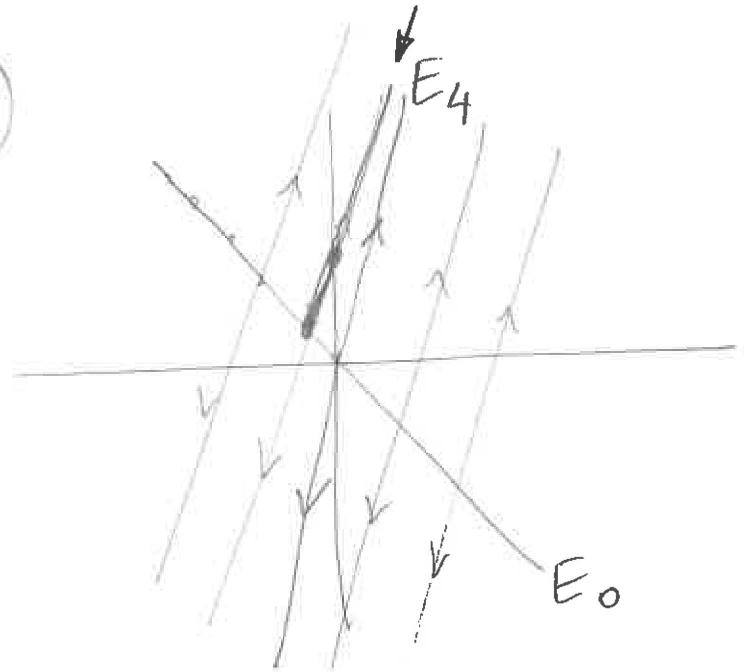
E_4 $\quad \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad E_4 = \langle (1,3) \rangle$

General form:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{4t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} x(t) = k_1 e^{4t} + k_2 \\ y(t) = 3k_1 e^{4t} - k_2 \end{cases}$$

b)



$$\begin{cases} y = 3x+3 \\ y = -x \end{cases} \Leftrightarrow \begin{cases} -x = 3x+3 \\ \text{---} \end{cases} \Leftrightarrow \begin{cases} -3 = 4x \\ \frac{3}{4} = y \end{cases}$$

the limit is the point $\left(-\frac{3}{4}, \frac{3}{4}\right)$.

4a) $y'' + y = 0$

$P(\lambda) = \lambda^2 + 1 \qquad P(\lambda) = 0 \Leftrightarrow \lambda = \pm i$

$\boxed{\varphi_1(t) = \cos t} \qquad \boxed{\varphi_2 = \sin t}$

b) $W[\varphi_1(t), \varphi_2(t)] = \varphi_1(t)\varphi_2'(t) - \varphi_1'(t)\varphi_2(t)$

$= \cos t \cdot \cos t + \sin t \cdot \sin t = 1$

c) $\psi(t) = \mu_1(t)\cos t + \mu_2(t)\, g(t)$

$g(t) = \frac{1}{\cos t}$

$$\begin{cases} \mu_1'(t) = \dfrac{-g(t)\,\varphi_2(t)}{W[\varphi_1(t), \varphi_2(t)]} \\ \mu_2'(t) = \dfrac{g(t)\,\varphi_1(t)}{W[\varphi_1(t), \varphi_2(t)]} \end{cases}$$

$$\begin{cases} \mu_1'(t) = \dfrac{-\frac{1}{\cos t} \cdot \sin t}{1} \\ \mu_2'(t) = \dfrac{\frac{1}{\cos t} \cos t}{1} \end{cases} \Rightarrow \begin{cases} \mu_1'(t) = \dfrac{-\sin t}{\cos t} \\ \mu_2'(t) = 1 \end{cases}$$

$$\begin{cases} u_1(t) = \ln|\cos(t)| \\ u_2(t) = t \end{cases}$$

$\therefore$ general form of the solution

$$y(t) = C_1 \cos t + C_2 \sin t + [\ln|\cos(t)| . \cos t + t \sin t]$$

$C_1, C_2 \in \mathbb{R}$.

Question 5

$$\begin{cases} \dot{x} = 2x - xy \\ \dot{y} = -4y + 8xy. \end{cases}$$

a) $\begin{cases} 2x - xy = 0 \\ -4y + 8xy = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} y = 2 \\ x = 1/2 \end{cases}$

Equilibria: $(0,0), (\frac{1}{2}, 2)$.

$\Rightarrow$

b

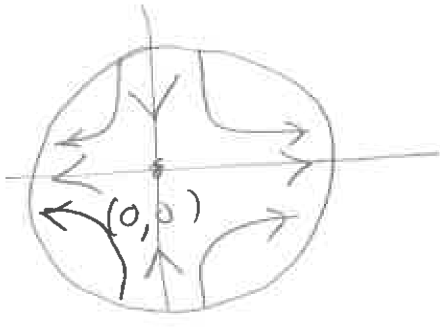
b) $\text{Jac} f(x,y) = \begin{pmatrix} 2-y & -x \\ 8y & -4+8x \end{pmatrix}$

$\text{Jac} f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$

eigenvalues of $df|_{(0,0)}$: $2, -4$

Since $2>0$, by the Lyapunov criterion, we get that $(0,0)$ is unstable.

c)



$(0,0)$ is hyperbolic

$\Downarrow$

by Hartman-Grobman theorem, the phase portrait is conjugate to its linear part.

$$\begin{cases} \dot{x} = 2x \\ \dot{y} = -4y. \end{cases}$$

The end!