

# Advanced Microeconomics:

## Game Theory (II) - Dynamic games

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These class notes draw extensively from Jehle, G. A., & Reny, P. J. (2011). *Advanced microeconomic theory* (3rd ed.). Pearson. Unless otherwise noted, all figures and diagrams presented herein are adapted from or directly sourced from this text.

# Roadmap of the lecture

Dynamic games of complete information

Mathematical and graphical representation

Extensive form game strategies

Games of perfect information and backward induction strategies

Repeated Games

# Introduction to Sequential Games

- I Players no longer make simultaneous decisions.
- I Games unfold sequentially, where players:
  - I Have perfect or incomplete information about past actions.
- I Sequential games introduce **time** and **information** into decision-making.
- I Extensive form games are used to model these types of games.
- I Extensive form games are defined by: players, actions, nodes, chance, decision rules, and payoffs.
- I Each player's information and decisions depend on the sequence of previous moves.
- I **Credibility** becomes the central element to finding the equilibria of the game.

# Extensive Form Games

An extensive form game consists of:

1. **Players:** A finite set of participants,  $N$ .
2. **Actions:** The set of all possible actions,  $A$ , including those that may not be chosen.
3. **Nodes (Histories):** Nodes represent sequences of actions taken so far.
  - I Each history starts at an initial node  $x_0$ .
  - I The initial node is unique
  - I Every action leads to a new node.
  - I Node cannot be reached by more than one branch.

# Extensive Form Games

4 **History**  $X$ : complete description of the actions that have been taken so far in the game

- I  $X$  contains a distinguished element,  $x_0$ , called the initial node, or empty history,
- I each  $x \in X \setminus \{x_0\}$  takes the form  $x = (a_1; a_2; \dots; a_k)$  for some finitely many actions  $a_i \in A$ , and
- I if  $(a_1; a_2; \dots; a_k) \in X \setminus \{x_0\}$  for some  $k > 1$ , then  $(a_1; a_2; \dots; a_{k-1}) \in X \setminus \{x_0\}$  (removing one action is also in the original sequence).

I

$$A(x) = \{a \in A \mid (x; a) \in X\}$$

denote the set of actions available to the player whose turn it is to move after the history  $x \in X \setminus \{x_0\}$ .

# Extensive Form Games

- 5 **Available Actions**  $A(x_0)$  **A and probability:** A set of actions,  $A(x_0)$   $A$ , and a probability distribution,  $\mu$ , on  $A(x_0)$  to describe the role of chance in the game.

I Chance always moves first, and just once, by randomly selecting an action from  $A(x_0)$  using the probability distribution  $\mu$ .

I  $(a_1; a_2; \dots; a_k) \in A(x_0)$  implies that  $a_i \in A(x_0)$  for  $i = 1$  and only  $i = 1$  (i.e., first action is determined by chance).

- 6 **End Nodes**  $E \subseteq X$  **for all**  $a \in A_g$ : Nodes where the game ends, with no further actions available.

- 7 **Decision Function** :

I Specifies which player makes the move at each node.

I

$$X_i = \{x \in X \mid (E \cap \{x\}) \neq \emptyset \text{ and } i(x) = i\}$$

decision nodes belonging to player  $i$

# Extensive Form Games

- 8 A partition  $I$  of decision nodes  $X \cap (E \setminus \{x_0\})$  into **information sets**. For player  $i$ , we denote her set of information sets  $I_i$

- I Two nodes  $x$  and  $x'$  are in the same information set if:
  - I (i) The same player  $\phi(x) = \phi(x')$  moves at both nodes.
  - I (ii) The available actions  $A(x) = A(x')$  are the same.
  - I (iii) Once the set containing  $x$  and  $x'$  is reached, the player does not know if she is on  $x$  or  $x'$ . She only knows the node has been reached.
- I The information set  $I(x)$  describes what the player knows when it's their turn to act.
- I  $I(x)$  represents the histories the player cannot distinguish between at node  $x$ .
- I **Singleton**: information set containing only one node
- I **Imperfect information**: not all information sets are singleton.
- I **Perfect recall**: no player forgets what they learn.
  - I For nodes  $x$  and  $y$  in the same information set, they must share the same history of the player's own actions.
  - I Players never forget their own past actions.

# Extensive Form Games

9 For each  $i \in N$  a **Payo Function**  $u_i$ :

- I von Neumann-Morgenstern payoff function whose domain is the set of end nodes
- I Describes the payoff to each player for every possible complete play of the game.

An extensive form game is written as

$\Gamma = \langle N; A; X; E; ; ; I; (u_i)_{i \in N} \rangle$ . If the sets of actions  $A$  and nodes  $X$  are finite, the game is called a **finite extensive form game**.



## Example 1: Coin Removal Game

There are two players, denoted by  $N = \{1, 2\}$  and 21 coins. Each player can remove up to three coins on their turn, represented by the actions  $r_1$ ,  $r_2$ , and  $r_3$ , corresponding to the removal of one, two, or three coins, respectively. There is no chance in this game, we define  $A(x_0) = \{r_1, r_2, r_3\}$ , indicating that chance has only one move.

I Full set of actions:

I **History Example:**

I **Information:**

## Example 1: Coin Removal Game

Suppose we have these two stories in the end of the game (all 21 coins are removed).

$$I \quad e_1 = (\bar{a}; r_1; r_2; r_1; r_3; r_3; r_3; r_3; r_3; r_2)$$

$$I \quad e_2 = (\bar{a}; r_3; r_3; r_3; r_3; r_3; r_3; r_2; r_1)$$

**I Game Outcome:**

$$I \quad e_1:$$

$$I \quad e_2:$$

**I Payo s:**

$$I$$

$$I$$

# Game Tree

Following this mathematical description, we can represent extensive form games graphically through game trees.

Game trees are directed graphs in which:

1. Game starts with a unique initial node ( $x_o$ ). We indicate if it's chance of a player's turn.
2. Each node is associated with the player that plays it.
3. The possible actions at a node for a player are indicated as branches.
4. At the end nodes the payoffs associated with the previous sequence of moves are indicated.
5. Each node can only be reached by one branch.
6. Information sets are denoted by ellipses.

# Game tree

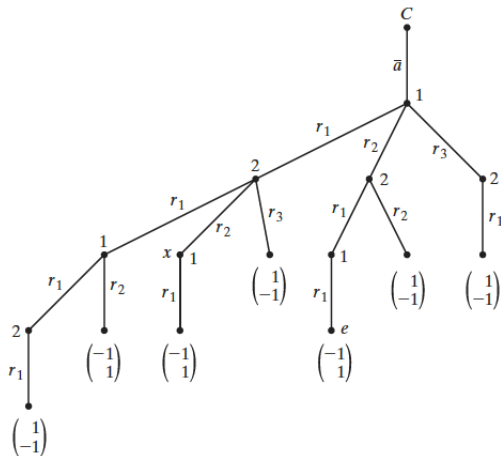


Figure: Game tree corresponding to the remove-coins example

# Extensive form game strategies

A (pure) strategy for a player in an extensive form game is a complete description of the choices the player would make in every possible contingency.

It specifies what the player would do at each decision point, as if these choices were to be executed by someone else on the player's behalf.

Formally, a pure strategy for player  $i$  in an extensive form game  $\Gamma$  is defined as a function:

$$s_i : I_i \rightarrow A_i$$

satisfying:

$$s_i(I(x)) \in A(x) \quad \forall x \text{ with } (x) = i:$$

# Characteristics of Pure Strategies

- I Specifies an action for each information set. Complete plan of action, detailing responses to every possible move in the game.

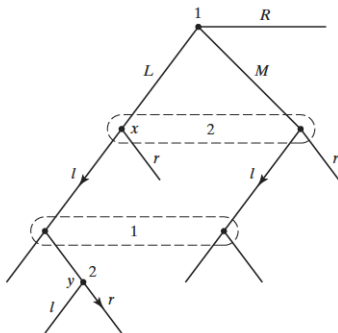


Figure: Extensive form game strategy

# Backward Induction Strategies

For finite games of perfect information, we use the concept of backward induction to rule out unreasonable paths.

**Backward induction** ensures that each action at every decision node is optimal, considering both past and future.

- I In finite extensive-form games with perfect information, strategies from backward induction form a Nash equilibrium.
- I The algorithm always **terminates** because the game is finite.
- I Every finite extensive-form game with perfect information has a pure strategy Nash equilibrium.
- I However, not all Nash equilibria arise from backward induction strategies.

# Backward Induction in Practice

A **backward induction strategy** is constructed by sequentially choosing optimal actions, starting from the last decision points and moving backward through the game.

1. Identify the **penultimate nodes**: the player chooses their best action considering the payoffs at the immediate end nodes that follow them.
2. Assign **optimal actions** at each penultimate node.

For each penultimate node  $x$ , let  $s_{i(x)}(x)$  denote the action that maximizes player  $i(x)$ 's payoff. This results in a payoff vector  $u^x$  corresponding to  $x$ .

3. **Collapse the game**, simplifying the decision tree.

Replace all nodes strictly following each penultimate node  $x$  with  $x$  itself and assign  $u^x$  as the payoff. Now  $x$  becomes an end node with the payoff  $u^x$ .

4. **Repeat** this process until every node has an action.

Continue this process | moving backward and assigning optimal actions at each stage | until every decision node in the game has an action assigned to it.



## Example: Grenade Game

1. P1 chooses an action  $a_1$  from the feasible set  $A_1$
2. P2 observes  $a_1$  and then chooses an action  $a_2$  from the feasible set  $A_2$
3. Payoffs:  $u_1(a_1; a_2)$  and  $u_2(a_1; a_2)$ .

## Example: Grenade Game

We start with backward induction. When player 2 moves, she observes what player 1 has done and decides  $a_2$  to maximize her utility:

$$\max_{a_2 \in A_2} u_2(a_1; a_2)$$

Assume this has a unique solution and let us denote it as  $R_2(a_1)$ . This is 2's reaction or best-response to 1's.

## Example: Grenade Game

Now, one can reason in the same way and will realize that 2 is going to respond to whatever  $a_1$  she does with  $R_2(a_1)$ . Thus, one will choose  $a_1$  to maximize her utility, given 2's response

$$\max_{a_1 \in A_1} u_1(a_1; R_2(a_1)) \quad (1)$$

Suppose this has a unique solution  $a_1$ .

Then, the **backward induction outcome** of the game is  $(a_1; R_2(a_1))$ .

Notice that the outcome does not involve non-credible threats. 1 anticipates 2's reaction to any of her choices. So 1 does not believe 2 will not do any action that is not  $R_2(a_1)$  (i.e., will not do something that is not in her best-interest).

# Dynamic games of imperfect information

Backward induction does not apply to games with **imperfect information**, as players lack knowledge about certain actions or events within an information set.

To solve such games, players treat subgames prior to the information set as regular Nash equilibria and apply backward induction in a modified form.

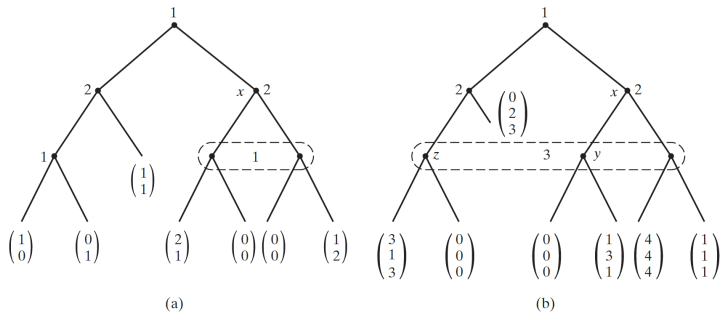


Figure: Subgames in trees

# Subgames and Subgame Perfect Equilibrium

A node  $x$  defines a subgame if:

$$I(x) = fxg$$

- I It starts at a singleton  $x$
- I Contains all decision nodes after  $x$ , and only those.
- I Players know if the node  $x$  has been reached.
- I Does not break any information set (i.e., if an information  $x^0$  set follows  $x$ , all nodes in  $x^0$  belong to the subgame defined by  $x$ ). This implies that all information prior to the subgame's initial node is known.
- I In Figure 3(a), node  $x$  defines a subgame because all decision points follow  $x$ .
- I In Figure 3(b), node  $x$  does not define a subgame, as not all nodes following  $x$  can be reached from  $x$ .

# Subgame Perfect Equilibrium

A strategy profile is a subgame perfect equilibrium if it results in a Nash equilibrium in every subgame.

- I Subgame perfection refines Nash equilibrium, ensuring optimal play in all subgames.
- I Any subgame perfect equilibrium is also a Nash equilibrium of the entire game.
- I **Theorem of existence:** Every finite dynamic game has at least a subgame perfect Nash equilibrium.

# Subgame Perfection and Backward Induction

- I Subgame perfection generalizes **backward induction**.
- I In finite extensive form games with perfect information, backward induction strategies are equivalent to subgame perfect equilibria.
- I Randomization may be necessary to ensure the existence of subgame perfect equilibria,

# Subgame perfection and backward induction

Notice that in our grenade game the backward induction outcome was  $(a_1; R_2(a_1))$  but the subgame-perfect Nash equilibrium is  $(a_1; R_2(a_1))$

- I  $a_1$  is a strategy for 1. She only plays at the beginning.
- I  $R_2(a_1)$  is an action not a strategy because it responds to  $a_1$
- I The strategy for 2 is the best-response  $R_2(a_1)$
- I In the game we have a subgame for each  $a_1$ . So we need to show that  $(a_1; R_2(a_1))$  is the Nash equilibrium and that the strategies are an equilibrium of all subgames (this is satisfied by  $R_2(a_1)$ ).
- I  $(a_1; R_2(a_1))$  is NE because the strategies are best responses to each other.



# Repeated games

## Finitely repeated game:

- I Given a stage game,  $G$ ,  $G(T)$  represents the finitely repeated game in which  $G$  is played  $T$  times, with the outcomes of all preceding plays observed before the next play begins. The payoffs are the sum of the payoffs from the  $T$  stage games (can incorporate discount factor).
- I That is, players at  $t$  know the result of the previous game played.
- I Player's take into account that their actions at  $t$  can have an effect in the future. Need to look ahead the current stage and think about repetitions.
- I Proposition: if  $G$  has a unique Nash Eq'm then, for any finite  $T$ ,  $G(T)$  has a unique subgame-perfect outcomes: the Nash Equ'm of  $G$  in every stage. Analogous if  $G$  is a dynamic game of complete information.

## Repeated games

If  $G$  has multiple Nash Eq'm., then  $G(T)$  has multiple Subgame Perfect Nash Eq'. Some of them may include not Nash Eq.

Let's consider an example:

|       | $L_2$ | $M_1$ | $R_2$ |
|-------|-------|-------|-------|
| $L_1$ | 1; 1  | 5; 0  | 0; 0  |
| $M_1$ | 0; 5  | 4; 4  | 0; 0  |
| $R_1$ | 0; 0  | 0; 0  | 3; 3  |

Now we have 2 Nash Eq:  $(L_1; L_2)$  and  $(R_1; R_2)$ . Suppose the game is played twice, observing the first stage outcome before playing the second.

Key: anticipate that different first-stage outcomes will be followed by different equilibria in the second stage.

## Repeated games

Suppose the following example: second stage outcome  $(R_1; R_2)$  if in the first-stage they do  $(M_1; M_2)$ , but  $(L_1; L_2)$  in the second-stage if they fail to do  $(M_1; M_2)$  in the first.

Adding the payoff of  $(R_1; R_2)$  to the stage game for  $(M_1; M_2)$  and the payoff of  $(L_1; L_2)$  to the rest :

|       | $L_2$ | $M_2$ | $R_2$ |
|-------|-------|-------|-------|
| $L_1$ | 2; 2  | 6; 1  | 1; 1  |
| $M_1$ | 1; 6  | 7; 7  | 1; 1  |
| $R_1$ | 1; 1  | 1; 1  | 4; 4  |

Nash Eq. of the stage game:  $f(L_1; L_2); (M_1; M_2); (R_1; R_2)g$

Subgame-perfect outcome of repeated game:

$f(L_1; L_2; L_1; L_2); (R_1; R_2; L_1; L_2); (M_1; M_2; R_1; R_2)g$ :

# Repeated games

Notice that the sub-game perfect outcome  $(M_1; M_2; R_1; R_2)$  is different from the rest because there is cooperation.

With multiple stage equilibria, one can build credible promises and threats about the future which help reshaping behaviour.

In general, if  $G$  is a static game of complete information with multiple Nash Eq., there may be subgame-perfect outcomes in which for some stage  $t < T$  the outcome is not a Nash Eq.

# Infinitely Repeated Games

An infinitely repeated game,  $G(1; \delta)$  consists in playing  $G$  infinite times in a row. Players, at any  $t$ , can know the result of the  $t - 1$  previous games played. Payoffs, labeled  $u_{it}$  are discounted with a discount rate  $\delta \in (0; 1)$ .

A player's payoff is then the present value of the payoffs from the infinite sequence of stage games:

$$U_i = \sum_{t=1}^{\infty} \delta^{t-1} u_{it}$$

Strategies are defined as a complete plan of action. They specify a feasible action for the player in every contingency. That is, strategies specify, for each possible history of play, the actions to be taken.

# Infinitely repeated games

- I Because of the infinite stages, there are infinite strategies.
- I Subgames are defined as what's left to be played at any point at which we know everything that happened before.
- I Each subgame is identical to the original game. There are infinite subgames that will be different depending on the history of play until reaching the subgame.
- I Theorem: a strategy will constitute a Subgame-perfect Nash equilibrium if there are no profitable one-information node deviations. That is, if I wouldn't deviate in one period and keep the rest unchanged, that is a subgame-perfect Nash equilibrium.

## Repeated Prisoner's Dilemma with infinite horizon

Suppose our Prisoner's Dilemma will be played an infinite number of times. At each stage the payoffs are:

|       | $L_2$ | $R_2$ |
|-------|-------|-------|
| $L_1$ | 1; 1  | 5; 0  |
| $R_1$ | 0; 5  | 4; 4  |

Assume players have a discount factor  $\delta$  and care about the present value of the payoffs.

Just as in the previous example with multiple Nash Eq. in the stage game, cooperation is possible.

Let's think of the following strategy (**trigger strategy** or **Grim strategy**):

$s : (R_1; R_2)$  in  $t=1$  and at any  $t$  provided that at  $t-1$  the outcome was  $(R_1; R_2)$ ; otherwise play  $L$  forever.

# Infinitely Repeated Games

To be able to generalize the result to other infinitely repeated games, we need two additional definitions.

**Feasible Payoffs:** for a stage game  $G$  are the linear combination of the pure strategy payoffs of  $G$  (i.e., convex hull of the 4 payoffs of the Prisoner's dilemma)

**Average Payoff ( $\bar{v}$ ):** payoff that would have to be received in every stage to yield the same present value.

$$\bar{v} = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} v_t$$

- I It re-scales present value, so maximizing  $\bar{v}$  is the same as maximizing the present value.
- I Directly comparable to stage-game payoffs.



# Infinitely Repeated Games

## \Folk" Theorem (Friedman,1971):

Let  $G$  be a finite, static game with complete information, and let  $(e_1; \dots; e_n)$  represent the payoffs for the players in a Nash equilibrium of  $G$ . Consider another feasible set of payoffs  $(x_1; \dots; x_n)$  from the game  $G$ , where each player's payoff  $x_i$  is strictly greater than their equilibrium payoff  $e_i$ , i.e.,  $x_i > e_i$  for every player  $i$ .

If the discount factor  $\delta$  is sufficiently close to 1 (i.e., the players are patient and value future payoffs highly), then there exists a **subgame-perfect Nash equilibrium** (SPNE) in the infinitely repeated version of the game that achieves the payoffs  $(x_1; \dots; x_n)$  as the **average payoff** over time.