

Problem Set - Game Theory

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[1, JR 7.17-18] Consider the strategic form game depicted below. Each of two countries must simultaneously decide on a course of action. Country 1 must decide whether to keep its weapons or to destroy them. Country 2 must decide whether to spy on country 1 or not. It would be an international scandal for country 1 if country 2 could prove that country 1 was keeping its weapons. The payoff matrix is as follows:

	Spy	Don't Spy
Keep	$(-1, 1)$	$(1, -1)$
Destroy	$(0, 2)$	$(0, 2)$

- (a) Does either player have a strictly dominant strategy?
- (b) Does either player have a weakly dominant strategy?
- (c) Find a Nash equilibrium in which neither player employs a weakly dominant strategy.

Now suppose that country 1 can be one of two types, 'aggressive' or 'non-aggressive'. Country 1 knows its own type. Country 2 does not know country 1's type but believes that country 1 is aggressive with probability $\varepsilon > 0$. The aggressive type places great importance on keeping its weapons. If it does so and country 2 spies on the aggressive type, this leads to war, which the aggressive type wins and justifies because of the spying, but which is very costly for country 2. When country 1 is non-aggressive, the payoffs are as before (i.e., as in the previous exercise). The payoff matrices associated with each of the two possible types of country 1 are given below.

Country 1 is 'aggressive' (Probability ε)

	Spy	Don't Spy
Keep	$(10, -9)$	$(5, -1)$
Destroy	$(0, 2)$	$(0, 2)$

Country 1 is 'non-aggressive' (Probability $1 - \varepsilon$)

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	Spy	Don't Spy
Keep	$(-1, 1)$	$(1, -1)$
Destroy	$(0, 2)$	$(0, 2)$

- (c) What action must the aggressive type of country 1 take in any Bayesian-Nash equilibrium?
- (d) Assuming that $\varepsilon < \frac{1}{5}$, find the unique Bayes-Nash equilibrium.

Solution

- (a) The best-response functions for each country are as follows:

- $BR_1(S) = D$; $BR_1(DS) = K$
- $BR_2(K) = S$; $BR_2(D) = \{S, DS\}$

Thus, no country has a strictly dominant strategy because no strategy is unambiguously better for either player across all of the other player's actions.

- (b) Country 2 (B) has a weakly dominant strategy, which is to play *Spy* (S). This is because spying provides the same payoff as not spying when Country 1 destroys its weapons, but yields a strictly higher payoff when Country 1 keeps its weapons.
- (c) The only pure-strategy Nash equilibrium of the game is (Destroy, Spy), but in this equilibrium, Country 2 employs a weakly dominant strategy. To find an equilibrium where no player employs a weakly dominant strategy, we need to consider mixed strategies.

Let p represent the probability that Country 1 chooses K (Keep), and q represent the probability that Country 2 chooses S (Spy). The expected payoffs for each country when the other is randomizing are:

$$\begin{aligned}
 EU_1(K, q) &= q(-1) + (1 - q)(1), \\
 EU_1(D, q) &= q(0) + (1 - q)(0) = 0, \\
 EU_2(p, S) &= p(1) + (1 - p)(2), \\
 EU_2(p, DS) &= p(-1) + (1 - p)(2).
 \end{aligned}$$

For a mixed-strategy Nash equilibrium, the players must be indifferent between their pure strategies. This happens for Country 2 when $EU_2(p, S) = EU_2(p, DS)$. Since this holds when $p = 0$ (i.e., Country 1 destroys its weapons), Country 2 will be indifferent and can because of the condition $EU_1(K, q) = EU_1(D, q)$, if $q = 1/2$ and 1 randomizes between keeping and destroying, but in this scenario.

We can also see this computing the best responses and their intersection.

$$EU_1(p, q) = -pq + p(1 - q) = p(1 - 2q);$$

$$EU_1(p, q) = pq + p(1 - q) + q[(q(1 - p) + (1 - p)(1 - q))] = 2pq + 2 - 3p.$$

To compute the best response functions notice that:

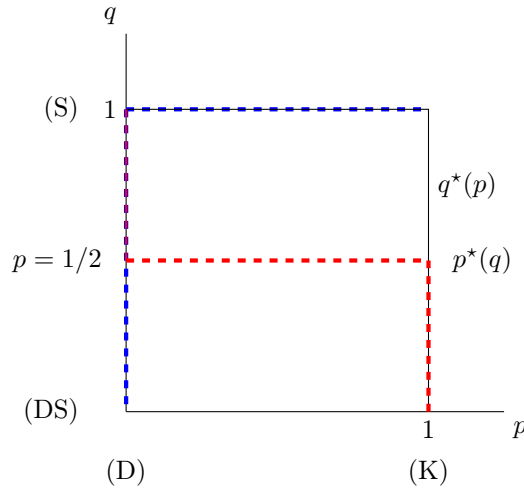
1. For 1, if $1 - 2q$ is negative, then it will try to minimize that by setting $p = 0$
2. If instead $1 - 2q > 0$ it will maximize it setting $p = 1$
3. If $1 - 2q = 0$, then he's indifferent and sets any $p \in [0, 1]$
4. For Country 2, for any $p > 0$, $q = 1$ is the best response.
5. Instead, if $p = 0$, Country 2 is indifferent and chooses any $q \in [0, 1]$

This gives the following 2 best-response functions:

$$p((q, (1 - q))) = p^*(q) = \begin{cases} p = 1 & q < 1/2, \\ p \in [0, 1] & q = 1/2 \\ p = 0 & q > 1/2. \end{cases}$$

$$q((p, (1 - p))) = q^*(p) = \begin{cases} q = 1 & p > 0, \\ q \in [0, 1] & p = 0 \end{cases}$$

Depicting this best-response functions:



Notice that both intersect in the pure-strategy Nash Equ'm $(q = 1, p = 0)$ $\{(\text{Destroy, Spy})\}$ and in the mixed strategy we had also found before $p = 0, q = 1/2$.

Thus, a Nash equilibrium where no player plays a weakly dominant strategy is $p = 0$ (Country 1 always destroys) and $q = 1/2$ (Country 2 mixes with equal probability). This would be a mixed-strategy Nash equilibrium.

- (d) Keep because it is a strictly dominant strategy for the country 1 if it is of the aggressive type.
- (e) Let us redefine the game. Notice that the set of strategies for country 2 remains unchanged: $S_2 = S, DS$. Instead, for country 1 now we have four possible strategies indicating what they would do depending on their type (a, w) : $S_1 = \{K^a K^w, K^a D^w, D^a K^w, D^a D^w\}$. For instance, the strategy $(K^a K^w)$ means that country 1 will keep if it is aggressive and weak. Notice that because keeping is a strictly dominant strategy if 1 is aggressive, D will never be played, so we can rule out equilibria in which the strategies for 1 of the type aggressive involve D . That is, we would be left with $\{K^a K^w, K^a D^w\}$

Let's look at the expected utility of country 2 taking this into account:

$$\begin{aligned} EU_2(K^a K^w, S) &= \varepsilon(-9) + (1 - \varepsilon)1 = 1 - 10\varepsilon \\ EU_2(K^a K^w, DS) &= \varepsilon(-1) + (1 - \varepsilon)(-1) = -1 \\ EU_2(K^a D^w, S) &= \varepsilon(-9) + (1 - \varepsilon)2 = 2 - 11\varepsilon \\ EU_2(K^a D^w, DS) &= \varepsilon(-1) + (1 - \varepsilon)2 = 2 - 3\varepsilon. \end{aligned}$$

Notice that because $\varepsilon < 1/5$, the best-response of country 2 to $(K^a K^w)$ is DS (i.e., $EU_2(K^a K^w, S) < EU_2(K^a K^w, DS)$). Notice also that for any $\varepsilon \in [0, 1]$, $EU_2(K^a D^w, S) < EU_2(K^a D^w, DS)$, so the best-response would be not to spy again.

	Spy	Don't Spy
$K^a K^w$	$10\varepsilon - (1 - \varepsilon), 1 - 10\varepsilon$	$\frac{5\varepsilon + (1 - \varepsilon), -1}{5\varepsilon, 2 - 3\varepsilon}$
$K^a D^w$	$\underline{10\varepsilon}, 2 - 11\varepsilon$	

Thus, we have a Bayesian Nash equilibrium with $(K^a K^w, DS)$.

[2, MWG 8.D.5] Consumers are uniformly distributed along a boardwalk that is 1 mile long. Ice-cream prices are regulated, so consumers go to the nearest vendor because they dislike walking (assume that at the regulated prices all consumers will purchase an ice cream even if they have to walk a full mile). If more than one vendor is at the same location, they split the business evenly.

- (a) Consider a game in which two ice-cream vendors pick their locations simultaneously. Show that there exists a unique pure strategy Nash equilibrium and that it involves both vendors locating at the midpoint of the boardwalk.
- (b) Show that with three vendors, no pure strategy Nash equilibrium exists.

Solution

In this game, each player (vendor 1 and vendor 2) chooses a location along the boardwalk, where the possible locations are represented by $x_i \in [0, 1]$. Since prices are regulated, their profits depend solely on the size of the market they serve.

Given that consumers are uniformly distributed along the boardwalk, the demand at any location x_i is represented by the cumulative distribution function (CDF) at that point. In this case, the CDF is simply x_i , as the distribution's support is defined over $[0, 1]$. Thus, each vendor's payoff is a function of their location strategy.

For any chosen pair of locations (x_1, x_2) , each vendor serves the consumers on their side of the boardwalk, with the dividing line set at $\frac{x_1 + x_2}{2}$. This midpoint allocation arises because, when the vendors do not choose the same location, half of the consumers in the segment between them will go to each shop. This allows us to compute the payoffs based on each vendor's chosen location. Thus, we have:

- If $x_i < x_j$ for $i, j = 1, 2$ and $i \neq j$:

$$u_i(x_i, x_j) = \frac{x_i + x_j}{2},$$

$$u_j(x_i, x_j) = 1 - \frac{x_i + x_j}{2}.$$

We first prove that the location choice $x_1^* = x_2^* = \frac{1}{2}$ constitutes a Nash Equilibrium by showing that there are no profitable unilateral deviations. Then, we demonstrate that this equilibrium is unique.

In this context, a profitable unilateral deviation would involve a vendor shifting their location by a very small amount, ϵ , to either side. Due to the symmetry of the game, it is sufficient to show that one vendor has no incentive to deviate, as the same reasoning will apply to the other.

Suppose, then, without loss of generality, that Vendor 1 considers shifting ϵ to the right of $x_2^* = \frac{1}{2}$. This means Vendor 1's new location is $x'_1 = \frac{1}{2} + \epsilon$. Now positioned to the right of Vendor 2, Vendor 1's new payoff would be determined by the consumers located between the midpoint of x_2^* and x'_1 and the end of the boardwalk. Specifically:

$$u_1(x'_1, x_2^*) = 1 - \frac{\frac{1}{2} + \frac{1}{2} + \epsilon}{2} = \frac{1}{2} - \frac{\epsilon}{2} < \frac{1}{2} = u_1(x_1^*, x_2^*).$$

The same reasoning applies if Vendor 1 shifts ϵ to the left of x_2^* , such that $x'_1 = \frac{1}{2} - \epsilon$. In this case, the payoff would be:

$$u_1(x'_1, x_2^*) = \frac{\frac{1}{2} + \frac{1}{2} - \epsilon}{2} = \frac{1}{2} - \frac{\epsilon}{2} < \frac{1}{2} = u_1(x_1^*, x_2^*).$$

Thus, $x_1 = x_2 = \frac{1}{2}$ is a Nash Equilibrium. Next, we will demonstrate that this equilibrium is unique. We start by showing that no equilibrium can exist

if $x_i \neq x_j$, and then we prove that the only value for which $x_i = x_j$ constitutes an equilibrium is $x_i = x_j = \frac{1}{2}$.

- **Case $x_i < x_j$:** Here, x_i would benefit by shifting ϵ to the right, and x_j would similarly benefit by shifting ϵ to the left, as both would capture a larger market segment.
- **Case $x_i = x_j < \frac{1}{2}$:** Either vendor would gain by shifting ϵ to the right. In this configuration, the profits from not deviating are smaller than $\frac{1}{2}$. However, if either vendor moves ϵ to the right of the other, they would capture a market share of $1 - x_i - \frac{\epsilon}{2} > 0.5$, since $x_i < 0.5$.
- **Case $x_i = x_j > \frac{1}{2}$:** By similar reasoning, either vendor would benefit by shifting ϵ to the left of the other, thereby increasing their market share.

This reasoning confirms that $x_1 = x_2 = \frac{1}{2}$ is the unique Nash Equilibrium in this game.

(b) Suppose an equilibrium (x_1, x_2, x_3) exists. First, consider the case where $x_1 = x_2 = x_3$. In this scenario, each firm would serve $\frac{1}{3}$ of the market. However, any firm can increase its sales by moving ϵ to the right (if $x_1 = x_2 = x_3 < \frac{2}{3}$) or to the left (if $x_1 = x_2 = x_3 > \frac{1}{3}$), which contradicts the existence of an equilibrium.

To understand the thresholds for profitable unilateral deviations, consider the following points:

Moving ϵ to the right is a profitable unilateral deviation if it results in profits greater than $\frac{1}{3}$. This profit arises from serving the additional market between the midpoint of the previous location ($x_1 = x_2 = x_3 = x$) and the new location ($x + \epsilon$). Mathematically, this condition can be expressed as:

$$1 - \frac{x + (x + \epsilon)}{2} > \frac{1}{3}.$$

Simplifying this inequality, we find that for small ϵ , the condition implies:

$$x < \frac{2}{3}.$$

Similarly, moving ϵ to the left is a profitable deviation if it also results in profits greater than $\frac{1}{3}$. The profit for the firm moving left can be expressed as:

$$\frac{x + (x - \epsilon)}{2} > \frac{1}{3}.$$

For small ϵ , this leads to the condition:

$$x > \frac{1}{3}.$$

Now, suppose two firms are located at the same point, say $x_1 = x_2$. If $x_1 = x_2 < x_3$, Firm 3 could increase its sales by moving slightly left to $x_1 + \epsilon$.

Alternatively, if $x_1 = x_2 > x_3$, Firm 3 would benefit by moving slightly right to $x_1 - \epsilon$, again contradicting the existence of an equilibrium.

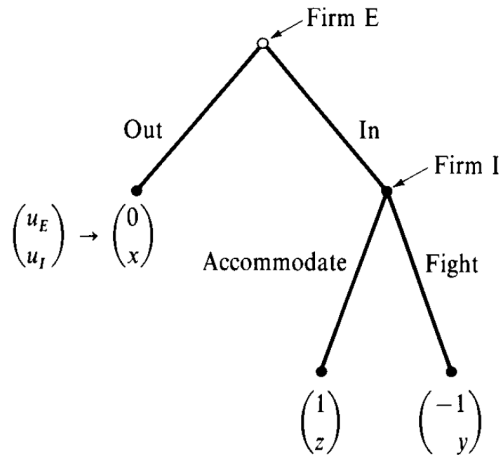
Finally, suppose all three firms are located at different points. In this case, the firm positioned farthest to the right could increase its sales by moving slightly right to position itself just beyond its closest competitor. This is again a contradiction, as similar adjustments could be made by the leftmost firm or even the middle firm leaping to the right or left of a competitor.

Thus, no pure strategy Nash Equilibrium exists in this game.

[3, MWG 9.B.14] At time 0, an incumbent firm (firm I) is already in the widget market, and a potential entrant (firm E) is considering entry. In order to enter, firm E must incur a cost of $K > 0$. Firm E's only opportunity to enter is at time 0. There are three production periods. In any period in which both firms are active in the market, the game in the figure is played. Firm E moves first, deciding whether to stay in or exit the market. If it stays in, firm I decides whether to fight (the upper payoff is for firm E). Once firm E plays "out," it is out of the game.

Assume that:

- (A.1) $x > z > y$,
- (A.2) $y + \delta x > (1 + \delta)z$,
- (A.3) $1 + \delta > K$.



(a) What is the (unique) subgame perfect Nash equilibrium of this game?

(b) Suppose now that firm E faces a financial constraint. In particular, if firm I fights *once* against firm E (in any period), firm E will be forced out of the market from that point on. Now what is the (unique) subgame perfect Nash equilibrium of this game? (If the answer depends on the values of parameters beyond the three assumptions, indicate how.)

Solution (a) It is evident that, due to the nature of the stage game and condition (A.1), Firm I has an incentive to accommodate the entrant if it decides to enter. The central question is whether the incumbent can induce the entrant to exit in some periods, thereby securing higher payoffs in the long run.

To analyze this, we will employ backward induction and examine the scenario at $t = 3$. At this point, the incumbent is at the final decision node, faced with the choice to either accommodate or fight. If the game reaches $t = 3$ and Firm I is making this decision, it implies that the entrant has produced during all preceding periods.

We can categorize the decision nodes into two broad types:

1. **Nodes where Firm I always accommodates the entrant.** The payoff for this type of node can be expressed as:

$$z + \delta z + \delta^2 z,$$

where z represents the payoff from accommodating, and δ is the discount factor.

2. **Nodes where Firm I chooses to fight at some point.** In this case, the payoff would be y , which is typically less favorable than the accommodating scenario.

Given condition (A.1), and assuming that all $x > z > y > 0$ and $\delta \in (0, 1)$, we can conclude that the payoff from consistently accommodating will always be greater than the payoffs from any scenario in which Firm I opts to fight at any stage.

Anticipating the incumbent's behavior, the entrant will always choose to produce in any period t when it must decide whether to stay in the market or exit, assuming that the incumbent will accommodate its entry. Specifically, the entrant compares the payoff from entering and staying in the market, which is given by $1 + \delta + \delta^2 - k$, with the payoff from exiting at any moment, which would reduce the payoff by 1 or δ , depending on the stage at which it opts out. Notably, by Assumption 3, $1 + \delta > k$, ensuring that production at $t = 1$ and $t = 2$ yields positive profits. Consequently, the entrant would never opt out at $t = 3$.

Assumption 2 indicates that it is profitable for the incumbent to fight at least once to eliminate the entrant. However, this represents a non-credible threat. If Firm E enters and chooses to remain in the market (which is its best response), it is not in Firm I's best interest to fight because $y < z$.

Thus, the Subgame Perfect Nash Equilibrium (SPNE) dictates that Firm E enters at $t = 0$ and consistently chooses to stay in, while Firm I accommodates at all stages.

Furthermore, as discussed in class, the stage dynamic game has a unique Nash Equilibrium (Enter, In, Accommodate). Consequently, the repeated game with $t = 3$ will also possess a unique Nash Equilibrium, reflecting the same strategy from the stage game across all three periods. This guarantees the uniqueness of our equilibrium.

(b) The critical difference now is that the entrant E cannot leverage the non-credible threat posed by the incumbent I ; thus, if I decides to fight, the entrant will be forced out of the market. It is important to note that at $t = 3$, the incumbent will not choose to fight if it has accommodated the entrant up to that point. We will then consider the scenario at $t = 2$.

For the incumbent to find it optimal to fight at $t = 2$, the profit it would gain from fighting must exceed the profit from accommodating the entrant. This condition can be expressed as:

$$z + \delta y + \delta^2 x > z + \delta z + \delta^2 z \Leftrightarrow \delta(y + \delta x) > \delta(z + \delta z),$$

which holds true due to Assumption 2. Thus, at $t = 2$, it becomes profitable for the incumbent to fight.

Consequently, the entrant must now weigh the payoff from producing for just one period and opting out against the potential costs of being forced to fight. Specifically, the entrant compares the following payoffs:

- Opting out yields $1 - k$ (no production at $t = 2$ and receiving δ due to opting out).
- Engaging in a fight results in a payoff of $-1 - k$.

Given this comparison, the entrant would choose to stay out at $t = 2$.

Now, let's return to $t = 1$. At this stage, the incumbent chooses to accommodate while the entrant chooses to stay in (Compares 0 from not-entering with $1 - k$, from entering, producing and leaving due to the fight) However, the decision made by Firm E at $t = 0$ hinges on the value of k :

- If $k > 1$, then Firm E will opt not to enter the market.
- If $k < 1$, then Firm E will decide to enter.

In the case where $k = 1$, both Enter and Not enter participate in the (unique) continuation of the Subgame Perfect Nash Equilibrium (SPNE), leading to the possibility of up to two SPNE in this scenario.

[4, Collusion in a Cournot Duopoly] Consider a Cournot duopoly where two firms produce a homogeneous product. Let q_1 and q_2 denote the quantities produced by Firm 1 and Firm 2, respectively. The market price $P(Q)$ is determined by the total quantity produced, given by the formula:

$$P(Q) = a - Q = a - q_1 - q_2 \quad \text{for } Q < a,$$

where Q is the total quantity in the market. If the total quantity produced by both firms is greater than or equal to a , the market price becomes zero, i.e., $P(Q) = 0$ for $Q \geq a$.

Each firm incurs a production cost. The total cost of producing q_i units for Firm i is:

$$C_i(q_i) = cq_i,$$

where c is the marginal cost of production, assumed to be less than a to ensure positive profits for both firms.

In this setup, both firms choose their production quantities q_1 and q_2 simultaneously, aiming to maximize their respective profits.

(a) Compute the unique Nash Equilibrium of the game.

Now consider an infinitely repeated game based on the Cournot stage game. Both firms have the discount factor δ

(b) Compute the values of δ , δ^* , for which the following trigger strategy is a subgame-perfect Nash Equilibrium:

Produce half the monopoly quantity, $q_m/2$ in the first period. In the t^{th} period, produce $q_m/2$ if both firms have produced $q_m/2$ in each of the $t - 1$ previous periods; otherwise produce the Cournot quantity, q_c (i.e., what you found on (a)).

(c) Show, for a given value of $\delta < \delta^*$, the most-profitable quantity, q^* , that can be produced if both play trigger strategies that switch to Cournot after any deviation. You might consider the following strategy:

Produce q^ in the first period. In the t^{th} period, produce q^* , if both firms have produced q^* in each of the $t - 1$ previous periods; otherwise, produce the Cournot quantity, q_c .*

(d) Consider the following carrot-and-stick strategy to show that the monopoly outcome can be achieved when $\delta = 1/2$:

Produce half the monopoly quantity, $q_m/2$, in the first period. In the t^{th} period, produce $q_m/2$ if both firms produced $q_m/2$ in period $t - 1$, produce $q_m/2$ if both firms produced x in period $t - 1$, and otherwise produce x .

Solution

(a) Following from the class notes, in the normal form representation of this game, we have two players, each with a strategy space $S_i = [0, a]$, as producing $q_i \geq a$ would yield zero revenue due to the price constraint. The payoff for each firm is its profit, given by

$$u_i(s_i, s_j) = \pi(q_i, q_j) = q_i[a - q_i - q_j - c].$$

The Cournot-Nash equilibrium occurs when an output vector $\bar{\mathbf{q}}$ maximizes each firm's profit, given the competitor's quantity choice. Thus, we seek (q_1^*, q_2^*) that satisfies the first-order conditions for profit maximization:

$$q_1^* = \frac{1}{2}(a - q_2^* - c), \quad q_2^* = \frac{1}{2}(a - q_1^* - c).$$

Solving this system of equations yields the equilibrium quantities: $q_1 = q_2 = \frac{a-c}{3}$.

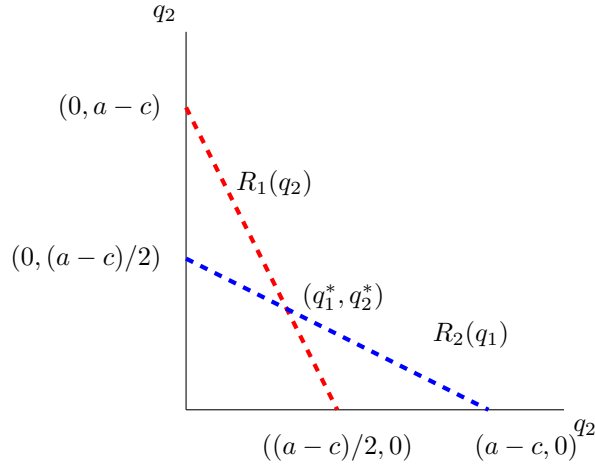
The intuition behind this result is as follows: If each firm were a monopolist, it would set q_i to maximize its profit as if the competitor's output were zero, yielding a monopoly quantity of $q_m = \frac{a-c}{2}$. In this scenario, each firm's profit would be $\frac{(a-c)^2}{4}$.

If the firms were to cooperate, they would jointly set their combined output to the monopoly level, for example, each producing $q_i = q_m/2$. However, this arrangement cannot be a Nash equilibrium, as each firm would have an incentive to increase its output, given that the high price encourages an expansion in quantity. This response drives down the price until the incentive to increase quantity is reduced.

In the Cournot-Nash equilibrium with $q_1 = q_2 = \frac{a-c}{3}$, the aggregate output is higher, reducing the market price, and diminishing the incentive for either firm to unilaterally increase its production.

Graphically, we can analyze the solution using the reaction functions:

$$R_2(q_1) = \frac{1}{2}(a - q_1 - c), \quad R_1(q_2) = \frac{1}{2}(a - q_2 - c).$$



Assuming $q_i < a - c$, the interior solution ensures that the first-order conditions are both necessary and sufficient. Plotting R_1 and R_2 yields an intersection at the Cournot-Nash equilibrium point $(q_1^*, q_2^*) = (\frac{a-c}{3}, \frac{a-c}{3})$.

(b) To find the value of δ that prevents any firm from wanting to deviate from the agreed strategy, we need to compare the profits under both cooperation and deviation scenarios.

First, consider the profit of each firm if they both produce half the monopoly quantity, $q_m/2$. This yields a profit of

$$\pi_m/2 = \frac{(a-c)^2}{8}.$$

Under Cournot competition, however, the profit for each firm is

$$\pi_c = \frac{(a-c)^2}{9}.$$

To identify the *most profitable deviation* for one of the firms, suppose that firm i adheres to the strategy $q_i = q_m/2$ while firm j deviates. Firm j would choose q_j to maximize its profit given $q_i = q_m/2$:

$$\max_{q_j} \pi_j = \left(a - q_j - \frac{1}{2}q_m - c \right) q_j.$$

The optimal deviation quantity for firm j is therefore $q_j^d = \frac{3(a-c)}{4}$, yielding a profit of

$$\pi_d = \frac{9(a-c)^2}{64}.$$

Now, to ensure that the firm prefers to stick with the agreed strategy over deviating, we compare the present value of profits from cooperation with the profits from deviating. The condition is:

$$\frac{1}{1-\delta} \frac{\pi_m}{2} \geq \pi_d + \frac{\delta}{1-\delta} \pi_c.$$

Solving this inequality for δ , we find that

$$\delta \geq \frac{9}{17}.$$

This threshold ensures that the firm values the future stream of cooperative profits more than the short-term gain from deviation.

(c) For values of $\delta < 9/17$, trigger strategies cannot sustain quantities as low as half the monopoly quantity, $q_m/2$, as a Nash equilibrium; however, reverting to Cournot quantities indefinitely can be an equilibrium. Thus, the most profitable quantity that can be sustained lies somewhere between $q_m/2$ and q_c . To determine this optimal sustainable quantity, we proceed as before by comparing profits from following the strategy versus deviating.

Let's denote this intermediate quantity as q^* . If both firms produce q^* , their profits are

$$\pi^* = (a - 2q^* - c)q^*.$$

We now identify the best deviation for firm j , assuming firm i sticks to producing q^* :

$$\max_{q_j} \pi_j = q_j(a - q_j - q^* - c).$$

Solving, we find that the optimal deviation quantity for firm j is

$$q_j = \frac{a - q^* - c}{2}$$

with corresponding deviation profits of

$$\pi_d = \frac{(a - q^* - c)^2}{4}.$$

Given these quantities, the condition for sustaining the strategy as a Nash Equilibrium for any δ becomes:

$$\frac{1}{1 - \delta} \pi^* \geq \pi_d + \frac{\delta}{1 - \delta} \pi_c.$$

Unlike the previous case, here we are not solving for δ but rather taking δ as given and solving for q^* , the cooperative quantity that maintains the strategy as an equilibrium. Solving for q^* , we find the lowest quantity for which the trigger strategy forms a Nash equilibrium:

$$q^* = \frac{9 - 5\delta}{3(9 - \delta)}(a - c).$$

Observe that this q^* lies between $q_m/2$ and q_c , depending on the value of δ .

(d) Let us define the profit a firm earns when they are in the punishment phase as $\pi(x) = x(a - 2x - c)$. Notice that if firm i is producing x , firm j could deviate from the punishment strategy by choosing a quantity that maximizes its profits, knowing that they remain in the punishment phase and are not returning directly to the collusive quantity $q_m/2$. Thus, firm j would choose:

$$q_j = \frac{a - x - c}{2}$$

resulting in a deviation profit of

$$\pi_d(x) = \frac{(a - x - c)^2}{4}.$$

There are two types of subgames to consider: (i) collusive subgames, where both firms produced either $q_m/2$ or x in the previous period, and (ii) non-collusive subgames, where the outcome in the previous period was neither of these quantities.

In collusive subgames, both firms must prefer the stream of half-monopoly profits to the profits from deviating and then entering punishment (where both remain collusive during punishment) before eventually returning to the monopoly output. This requires:

$$\frac{1}{1 - \delta} \cdot \frac{1}{2} \pi_m \geq \pi_d + \delta \left(\pi(x) + \frac{\delta}{1 - \delta} \cdot \frac{1}{2} \pi_m \right).$$

This inequality can be rearranged as:

$$\delta \left(\frac{1}{2} \pi_m - \pi(x) \right) \geq \pi_d - \frac{1}{2} \pi_m.$$

In the punishment subgames, for the punishment strategy to be an equilibrium, each firm must prefer continuing with the punishment output x over deviating to achieve $\pi_d(x)$ and then re-entering the punishment phase. This requires:

$$\pi(x) + \frac{\delta}{1-\delta} \cdot \frac{1}{2} \pi_m \geq \pi_d(x) + \delta \left(\pi(x) + \frac{\delta}{1-\delta} \cdot \frac{1}{2} \pi_m \right).$$

This condition can be rewritten as:

$$\delta \left(\frac{1}{2} \pi_m - \pi(x) \right) \geq \pi_d(x) - \pi(x).$$

Based on these inequalities, if $\delta = \frac{1}{2}$, the strategy sustains the monopoly outcome if:

$$\frac{3}{8} \leq \frac{x}{a-c} \leq \frac{1}{2}.$$