Advanced Mathematical Economics – Part II

Some exercises correction

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1.1.

(a) $T = \mathbb{R}_0^+, \quad \Omega = \mathbb{R}_0^+.$

(b) $T = \mathbb{R}_0^+, \quad \Omega = \{\text{position i}, \ \text{position ii}, \ \text{position iii}\}.$

(c) $T = \mathbb{N}, \quad \Omega = \{\text{Printer On}, \ \text{Printer Off}\}.$

1.3
(a)

Regarding X_t:

- $E[X_t] = E[\varepsilon_t] = 0.$
- $E[X_t^2] = Var(X_t) + \big(E[X_t]\big)^2 \iff Var(X_t) = \sigma^2 < +\infty, \quad \forall\, t \in \mathbb{N}.$
- As $\text{Cov}(X_t, X_s) = 0$ by the definition of white noise, the process X_t is weakly stationary.

Regarding Y_t:

- $E[Y_t] = E\big[(-1)^t \varepsilon_t\big] = (-1)^t \, E[\varepsilon_t] = 0.$
- $E[Y_t^2] = Var(Y_t) + \big(E[Y_t]\big)^2 = Var\big((-1)^t \varepsilon_t\big) = (-1)^{2t} Var(\varepsilon_t) = \sigma^2 < +\infty.$
- The covariance is $Cov(Y_t, Y_s) = Cov\big((-1)^t \varepsilon_t, \, (-1)^s \varepsilon_s\big) = (-1)^{t+s} Cov(\varepsilon_t, \varepsilon_s) = 0, \quad \text{for } t \neq s.$
- Therefore, Y_t is a weakly stationary process.

(b)

- $E[Z_t] = E[\varepsilon_t + (-1)^t \varepsilon_t] = E[\varepsilon_t] + (-1)^t \, E[\varepsilon_t] = 0.$
- Moreover,

$$\begin{aligned}(Z_t) &= (\varepsilon_t + (-1)^t \varepsilon_t) \\ &= (\varepsilon_t) + \big((-1)^t \varepsilon_t\big) + 2 \ \text{Cov}\left(\varepsilon_t, (-1)^t \varepsilon_t\right) \\ &= \sigma^2 + (-1)^{2t}\sigma^2 + 2(-1)^t \ \text{Cov}(\varepsilon_t, \varepsilon_t) \\ &= 2\sigma^2 + 2(-1)^t \sigma^2.\end{aligned}$$

- Hence,

$$(Z_t) = \begin{cases} 4\sigma^2, & \text{if } t \text{ is even}, \\ 0, & \text{if } t \text{ is odd}. \end{cases}$$

- Since the variance depends on t and is not constant, the process is not weakly stationary.

1.4
(a) By definition, white noises indexed at different times have zero covariance, i.e. they are independent. It is also well known that any linear combination of independent Gaussian random variables is Gaussian. Thus Y_t is always a Gaussian process.

(b)

- $E[Y_t] = E\big[\varepsilon_t - \theta\,\varepsilon_{t-1}\big] = E[\varepsilon_t] - \theta\,E[\varepsilon_{t-1}] = 0.$
- $Var(Y_t) = Var(\varepsilon_t - \theta\,\varepsilon_{t-1}) = Var(\varepsilon_t) + Var(\theta\,\varepsilon_{t-1}) - 2Cov(\varepsilon_t,\,\theta\,\varepsilon_{t-1}) = \sigma^2 + \theta^2\sigma^2 - 2\times 0 = \sigma^2(1+\theta^2).$
- Therefore, $Y_t \sim \mathcal{N}\big(0,\ \sigma^2(1+\theta^2)\big), \quad \forall\, t \in \mathbb{N}.$

(c)

$$\gamma(1) = Cov(Y_t, Y_{t+1}) = Cov\big(\varepsilon_t - \theta\,\varepsilon_{t-1},\ \varepsilon_{t+1} - \theta\,\varepsilon_t\big) = -\theta\,Cov(\varepsilon_t, \varepsilon_t) = -\theta\,\sigma^2.$$

Hence,

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{-\theta\,\sigma^2}{(1+\theta^2)\sigma^2} = \frac{-\theta}{1+\theta^2}.$$

For $|h| > 1$, because the ε_t are independent, $\rho(h) = 0$. Thus,

$$\rho(h) = \begin{cases} 1, & \text{if } h = 0, \\ \dfrac{-\theta}{1+\theta^2}, & \text{if } |h| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(d) The process Y_t is weakly stationary since:

- The mean is constant in time: $E[Y_t] = 0$;
- The variance is constant: $Var(Y_t) = \sigma^2(1+\theta^2)$;
- The second moment is finite: $E[Y_t^2] = Var(Y_t) + \big(E[Y_t]\big)^2 = \sigma^2(1+\theta^2) < +\infty$;
- The covariance depends only on the lag h and not on the time t.
- Moreover, since Y_t is a linear combination of Gaussian white noise ε_t, it is Gaussian. Gaussian weakly stationary processes enjoy the property of strict stationarity.

1.7

(a) A process is Gaussian if, for all $n \in \mathbb{N}$ and all $t_1, \ldots, t_n \in T$, the vector

$$(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}) \sim \mathcal{N}_n(\mu, \Sigma).$$

Hence, Y_t is Gaussian because it is a linear combination of independent Gaussian random variables.

(b) We consider three relevant cases.

1. If $h = 0$:

$$\begin{aligned} Var(Y_t) &= Var(\varepsilon_t - \theta\varepsilon_{t-1} - \tfrac{\theta}{2}\varepsilon_{t-2}) \\ &= Var(\varepsilon_t) + (-\theta)^2 Var(\varepsilon_{t-1}) + \left(-\tfrac{\theta}{2}\right)^2 Var(\varepsilon_{t-2}) \\ &= \sigma^2\left(1 + \tfrac{5}{4}\theta^2\right) \end{aligned}$$

2. If $h = 1$:

$$\begin{aligned} \gamma(1) &= \mathrm{Cov}(Y_t, Y_{t+1}) \\ &= \mathrm{Cov}(\varepsilon_t - \theta\varepsilon_{t-1} - \tfrac{\theta}{2}\varepsilon_{t-2},\ \varepsilon_{t+1} - \theta\varepsilon_t - \tfrac{\theta}{2}\varepsilon_{t-1}) \\ &= \mathrm{Cov}(\varepsilon_t, -\theta\varepsilon_t) + \mathrm{Cov}(-\theta\varepsilon_{t-1}, -\tfrac{\theta}{2}\varepsilon_{t-1}) \\ &= -\theta\sigma^2 + \frac{\theta^2}{2}\sigma^2 = \sigma^2\left(-\theta + \frac{\theta^2}{2}\right). \end{aligned}$$

Therefore,

$$\rho(1) = \frac{\sigma^2(-\theta + \frac{\theta^2}{2})}{\sigma^2(1 + \frac{5}{4}\theta^2)} = \frac{-\theta + \frac{\theta^2}{2}}{1 + \frac{5}{4}\theta^2}.$$

3. If $h = 2$:

$$\gamma(2) = \mathrm{Cov}(Y_t, Y_{t+2}) = -\tfrac{\theta}{2}(\varepsilon_t) = -\tfrac{\theta}{2}\sigma^2.$$

Thus,

$$\rho(2) = \frac{-\frac{\theta}{2}\sigma^2}{\sigma^2(1+\frac{5}{4}\theta^2)} = \frac{-\frac{\theta}{2}}{1+\frac{5}{4}\theta^2}.$$

For all other $|h| > 2$, the independence of the white noises implies the autocorrelation is zero. Hence,

$$\rho(h) = \begin{cases} 1, & \text{if } h = 0, \\ \dfrac{-\theta + \frac{\theta^2}{2}}{1+\frac{5}{4}\theta^2}, & \text{if } h = \pm 1, \\ \dfrac{-\frac{\theta}{2}}{1+\frac{5}{4}\theta^2}, & \text{if } h = \pm 2, \\ 0, & \text{otherwise.} \end{cases}$$

1.10 From the exercise statement we have:

- $E[X_i] = 1, \quad \forall\, i \in \mathbb{N}$,
- $E[|X_i|] < +\infty$.

Thus,

$$E[|Z_n|] = E\left[\left|\prod_{i=1}^{n} X_i\right|\right] = \prod_{i=1}^{n} E[|X_i|] < +\infty.$$

Moreover,

$$\begin{aligned} E[Z_{n+1} \mid Z_n, \dots, Z_1] &= E\left[\prod_{i=1}^{n+1} X_i \,\middle|\, Z_n, \dots, Z_1\right] \\ &= E\left[X_{n+1}\prod_{i=1}^{n} X_i \,\middle|\, Z_n, \dots, Z_1\right] \\ &= E[X_{n+1} \mid Z_n, \dots, Z_1]\, E[Z_n \mid Z_n, \dots, Z_1] \\ &= 1 \times Z_n = Z_n. \end{aligned}$$

Therefore, (Z_n) is a martingale.

1.11

(a) A process X is said to have independent increments if for every $n \in \mathbb{N}$ and indices $t_1 < \cdots < t_n$, the increments

$$X_{t_2} - X_{t_1},\ X_{t_3} - X_{t_2},\ \dots,\ X_{t_n} - X_{t_{n-1}}$$

are mutually independent random variables.

(b) From the exercise statement we have $E[X_n] = 1$ for all n. Also,

$$1 = E[X_n] = |E[X_n]| \le E[|X_n|] < +\infty.$$

In addition,

$$\begin{aligned} E[X_{n+1} \mid X_n, \dots, X_1] &= E[X_n + (X_{n+1} - X_n) \mid X_n, \dots, X_1] \\ &= X_n + E[X_{n+1} - X_n \mid X_n, \dots, X_1] \\ &= X_n + E[X_{n+1}] - E[X_n] = X_n. \end{aligned}$$

Hence, $(X_n,\ n = 0, 1, 2, \dots)$ is a martingale.

(c) We are given $\mathrm{Var}(X_n) = 1$, hence $E[X_n^2] = \mathrm{Var}(X_n) + E[X_n]^2 = 1 + 1 = 2 < +\infty$.
Compute the covariance for $h \ge 0$:

$$\begin{aligned} \mathrm{Cov}(X_n, X_{n+h}) &= E[X_n X_{n+h}] - E[X_n]\, E[X_{n+h}] \\ &= E[X_n(X_{n+h} - X_n + X_n)] - 1 \\ &= E[X_n(X_{n+h} - X_n)] + E[X_n^2] - 1 \\ &= E[X_n(X_{n+h} - X_n)] + 2 - 1. \end{aligned}$$

Using independence of the increments and the given expectations we obtain

$$E[X_n(X_{n+h} - X_n)] = E[X_n]\, E[X_{n+h} - X_n] = 1 \cdot (E[X_{n+h}] - E[X_n]) = 0.$$

Therefore $\mathrm{Cov}(X_n, X_{n+h}) = 1$ for all n, h, so the process is weakly stationary.

2.1

1.

$$\mathbb{P}(W(2.7) > 1.5) = 1 - \mathbb{P}(W(2.7) \leq 1.5) = 1 - \mathbb{P}\left(\frac{W(2.7) - 0}{\sqrt{2.7}} \leq \frac{1.5 - 0}{\sqrt{2.7}}\right), \quad \frac{W(2.7)}{\sqrt{2.7}} \sim \mathcal{N}(0, 1).$$

$$\frac{1.5}{\sqrt{2.7}} \approx 0.91 \quad \Rightarrow \quad 1 - \Phi(0.91) = 1 - 0.8186 = 0.1814.$$

2.

$$\mathbb{P}(-1.5 < W(2.7) < 1.5) = \mathbb{P}\left(-0.91 < \frac{W(2.7)}{\sqrt{2.7}} < 0.91\right) = \Phi(0.91) - \Phi(-0.91) = 2\Phi(0.91) - 1 = 0.6372.$$

3.

$$\mathbb{P}(W(2.7) < 1.5 \mid W(1.8) = 1) = \mathbb{P}(W(2.7) - W(1.8) < 0.5) = \mathbb{P}\left(\frac{W(2.7) - W(1.8)}{\sqrt{0.9}} < \frac{0.5}{\sqrt{0.9}}\right) = \Phi(0.52) = 0.6985.$$

4. $\mathbb{P}(-1.5 < W(2.7) < 1.5 \mid W(1.8) = 1) = \mathbb{P}(-2.5 < W(2.7) - W(1.8) < 0.5) = \mathbb{P}(-2.64 < Z < 0.52) = \Phi(0.52) - \Phi(-2.64) = \Phi(0.52) + \Phi(2.64) - 1 = 0.6985 + 0.9959 - 1 = 0.6944.$

5.

$$\mathbb{E}[W(t) \mid W(s), W(u)] = \mathbb{E}[W(t) \mid W(s)] = W(s),$$

by the Markov property of Brownian motion and the fact that it is a martingale.

6.

$$\mathrm{Var}(W(t) \mid W(s), W(u)) = \mathrm{Var}(W(t) \mid W(s)) = t - s.$$

7.

$$\mathbb{P}(W(2.7) > 1.5 \mid W(1.8) = 1) = \mathbb{P}(W(2.7) - W(1.8) > 0.5) = 1 - \Phi(0.52) = 1 - 0.6985 = 0.3015.$$

8.

$$\mathbb{E}[W(2.7) \mid W(1.8) = 1,\, W(0.5) = -2] = \mathbb{E}[W(2.7) \mid W(1.8) = 1] = 1.$$

9. We know that for jointelly normally distributed variables X, Y:

$$X \mid Y = y \sim \mathcal{N}\left(\mu_X + \frac{\mathrm{Cov}(X,Y)}{\sigma_Y^2}(y - \mu_Y),\, \sigma_X^2\left(1 - \frac{\mathrm{Cov}(X,Y)^2}{\sigma_X^2\sigma_Y^2}\right)\right).$$

Thus:

$$W(1.8) \mid W(2.7) = 1.5 \sim \mathcal{N}\left(0 + \frac{1.8}{2.7}(1.5 - 0),\, 1.8\left(1 - \frac{1.8}{2.7}\right)\right) = \mathcal{N}(1,\, 0.6).$$

Therefore:

$$\mathbb{P}(W(1.8) < 1 \mid W(2.7) = 1.5) = \mathbb{P}\left(\frac{W(1.8) - 1}{\sqrt{0.6}} < 0\right) = \Phi(0) = 0.5.$$

10.

$$\mathbb{P}(W(1.8) = 1,\, W(2.7) < 1.5) = \frac{\mathbb{P}(W(1.8) = 1,\, W(2.7) < 1.5)}{\mathbb{P}(W(2.7) < 1.5)} \leq \frac{\mathbb{P}(W(1.8) = 1)}{\mathbb{P}(W(2.7) < 1.5)} = 0.$$

11.

$$\mathbb{P}(W(2.7) = 1.5,\, W(1.8) = 1) \leq \mathbb{P}(W(2.7) = 1) = 0.$$

12.

$$\mathbb{P}(W(2.7) < 1.5,\, W(1.8) = 1) \leq \mathbb{P}(W(1.8) = 1) = 0.$$

13.

$\mathbb{P}(-1 < W(2.7) - W(1.8) < 1.4,\ 0.5 < W(1.6) - W(0.9) < 1.5)$

Using the independence of increments:

$$= \mathbb{P}\left(-\frac{1}{\sqrt{0.9}} < Z < \frac{1.4}{\sqrt{0.9}}\right)\mathbb{P}\left(\frac{0.5}{\sqrt{0.7}} < Z < \frac{1.5}{\sqrt{0.7}}\right), \quad Z \sim \mathcal{N}(0,1).$$

$= (\Phi(1.47) - \Phi(-1.05))(\Phi(1.79) - \Phi(0.60)) = (0.9292 - 0.1469)(0.9633 - 0.7257) = 0.7823 \times 0.2376 = 0.1859.$

14.

$\mathbb{P}(-1 < W(2.7) - W(1.8) < 1.4 \mid W(1.6) - W(0.9) < 1.5) = \mathbb{P}(-1 < W(2.7) - W(1.8) < 1.4) = 0.7823.$

2.2

We need to write the process in terms of increments of the Wiener process, so that we can take advantage of the fact that these increments are independent.

Thus, let:

$$\begin{cases} X = B(u), \\ Y = B(u+v) - B(u), \\ Z = B(u+v+w) - B(u+v). \end{cases}$$

Here X, Y, Z are independent by definition.

Therefore:

$\mathbb{E}\big[B(u)\,B(u+v)\,B(u+v+w)\big] = \mathbb{E}\big[X\,(X+Y)\,(X+Y+Z)\big].$

Expanding the product:

$X\,(X+Y)\,(X+Y+Z) = (X^2 + XY)(X+Y+Z) = X^3 + X^2Y + X^2Z + XY^2 + XYZ.$

Hence:

$\mathbb{E}[X(X+Y)(X+Y+Z)] = \mathbb{E}[X^3] + \mathbb{E}[X^2Y] + \mathbb{E}[X^2Z] + \mathbb{E}[XY^2] + \mathbb{E}[XYZ].$

Since X, Y, Z are independent and have mean zero:

$\mathbb{E}[X^3] = 0, \quad \mathbb{E}[X^2Y] = \mathbb{E}[X^2]\,\mathbb{E}[Y] = 0, \quad \mathbb{E}[X^2Z] = \mathbb{E}[X^2]\,\mathbb{E}[Z] = 0,$

$\mathbb{E}[XY^2] = \mathbb{E}[X]\,\mathbb{E}[Y^2] = 0, \quad \mathbb{E}[XYZ] = \mathbb{E}[X]\,\mathbb{E}[Y]\,\mathbb{E}[Z] = 0.$

Therefore:

$\boxed{\mathbb{E}\big[B(u)\,B(u+v)\,B(u+v+w)\big] = 0.}$

2.3

According to the properties of Brownian motion and the normal distribution, one way to formalise the process is

$$B(t) = 3 + \sigma W_t,$$

where W_t is a standard Brownian motion.

Then,

$\mathrm{Cov}(B(t), B(s)) = \mathrm{Cov}(3 + \sigma W_t,\ 3 + \sigma W_s) = \sigma^2\,\mathrm{Cov}(W_t, W_s) = \sigma^2 \min(t, s).$

2.4

a)

$\mathrm{Cov}(U(t), U(s)) = \mathrm{Cov}\big(e^{-t}B(e^{2t}),\ e^{-s}B(e^{2s})\big) = e^{-(t+s)}\,\mathrm{Cov}\big(B(e^{2t}), B(e^{2s})\big)\,.$

Since $\text{Cov}(B(t), B(s)) = \min(t, s)$, we have

$\text{Cov}(U(t), U(s)) = e^{-(t+s)} \min(e^{2t}, e^{2s}) = e^{-(t+s)} e^{2\min(t,s)} = e^{-|t-s|}$.

b)

$$\text{Cov}(V(t), V(s)) = \text{Cov}\left((1-t)B\left(\frac{t}{1-t}\right), (1-s)B\left(\frac{s}{1-s}\right)\right) = (1-t)(1-s)\,\text{Cov}\left(B\left(\frac{t}{1-t}\right), B\left(\frac{s}{1-s}\right)\right).$$

Since $\text{Cov}(B(t), B(s)) = \min(t, s)$, it follows that

$$\text{Cov}(V(t), V(s)) = (1-t)(1-s)\min\left(\frac{t}{1-t}, \frac{s}{1-s}\right).$$

Noting that

$$\left(\frac{t}{1-t}\right)' = \frac{1}{(1-t)^2} \Rightarrow \text{an increasing function on } t \in (0,1),$$

Thus,

$$\text{Cov}(V(t), V(s)) = (1-t)(1-s)\frac{\min(t,s)}{1-\min(t,s)} = \min(t,s)\,[1-\max(t,s)].$$

c)

$$\text{Cov}(W(t), W(s)) = \text{Cov}\left(tB\left(\frac{1}{t}\right), sB\left(\frac{1}{s}\right)\right) = ts\,\text{Cov}\left(B\left(\frac{1}{t}\right), B\left(\frac{1}{s}\right)\right).$$

Since $\text{Cov}(B(t), B(s)) = \min(t, s)$, we obtain

$$\text{Cov}(W(t), W(s)) = ts\min\left(\frac{1}{t}, \frac{1}{s}\right) = \frac{ts}{\max(t,s)} = \min(t,s).$$

2.5 (hard exercise!)

Let $B_1(t)$ and $B_2(t)$ be two independent Brownian motions such that

$$B_1(t),\ B_2(t) \sim \mathcal{N}(0, t).$$

Since they are independent, the joint density is

$$f_{B_1,B_2}(x,y) = \frac{1}{2\pi t}\exp\left(-\frac{x^2+y^2}{2t}\right).$$

We wish to compute

$$\mathbb{E}[R(t)] = \iint_{\mathbb{R}^2} \sqrt{x^2+y^2}\, f_{B_1,B_2}(x,y)\,dx\,dy.$$

Using polar coordinates:

$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad J = r,$

we obtain

$$\mathbb{E}[R(t)] = \int_0^{2\pi}\int_0^{+\infty} r \cdot \frac{1}{2\pi t}\exp\left(-\frac{r^2}{2t}\right) r\,dr\,d\theta.$$

Integrating with respect to θ:

$$\mathbb{E}[R(t)] = \frac{1}{t}\int_0^{+\infty} r^2 e^{-r^2/2t}\,dr = \sqrt{\frac{\pi t}{2}}.$$

2.7

To compute the differential of the product tW_t, we introduce the auxiliary process

$$Y_t = h(t, X_t) = tX_t, \qquad \text{where } X_t = W_t.$$

We now apply Itô's formula to the function $h(t, x) = tx$. Its partial derivatives are:

$$\frac{\partial h}{\partial t} = X_t, \qquad \frac{\partial h}{\partial x} = t, \qquad \frac{\partial^2 h}{\partial x^2} = 0.$$

Since $X_t = W_t$ satisfies

$$dX_t = dW_t,$$

Itô's formula yields:

$$dY_t = \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}F_t + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}G_t^2\right) dt + \frac{\partial h}{\partial x}G_t\, dW_t.$$

In our case we have $F_t = 0$ and $G_t = 1$, so the expression becomes:

$$d(tX_t) = \left(X_t + t \cdot 0 + \tfrac{1}{2} \cdot 0 \cdot 1\right) dt + t \cdot 1\, dW_t,$$

and therefore,

$$d(tX_t) = X_t\, dt + t\, dW_t, \qquad d(tW_t) = W_t\, dt + t\, dW_t.$$

Integrating from 0 to t, we obtain:

$$\int_0^t d(sW_s) = \int_0^t W_s\, ds + \int_0^t s\, dW_s,$$

and hence,

$$tW_t = \int_0^t W_s\, ds + \int_0^t s\, dW_s.$$

Rearranging the identity leads to the classical relationship:

$$\int_0^t s\, dW_s = tW_t - \int_0^t W_s\, ds.$$

2.8

Consider the stochastic differential equation

$$dY_t = Y_t\, dW_t, \qquad Y_0 = 1.$$

To solve it, we apply Itô's formula to the transformation

$$\boxed{X_t = h(t, Y_t) = \ln Y_t}.$$

The derivatives of $h(t, y) = \ln y$ are

$$\frac{\partial h}{\partial t} = 0, \qquad \frac{\partial h}{\partial y} = \frac{1}{y}, \qquad \frac{\partial^2 h}{\partial y^2} = -\frac{1}{y^2}.$$

Since the diffusion satisfies $dY_t = 0 \cdot dt + Y_t\, dW_t$, Itô's formula yields

$$dX_t = \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y}F_t + \frac{1}{2}\frac{\partial^2 h}{\partial y^2}G_t^2\right) dt + \frac{\partial h}{\partial y}G_t\, dW_t.$$

Here $F_t = 0$ and $G_t = Y_t$, giving

$$d(\ln Y_t) = \left(0 + \frac{1}{Y_t} \cdot 0 + \frac{1}{2}\left(-\frac{1}{Y_t^2}\right) Y_t^2\right) dt + \frac{1}{Y_t}Y_t\, dW_t.$$

Thus,

$$\boxed{d(\ln Y_t) = -\frac{1}{2}\, dt + dW_t}.$$

Integrating from 0 to t,

$$\ln Y_t - \ln Y_0 = -\frac{1}{2}t + W_t - W_0.$$

Since $Y_0 = 1$ and $W_0 = 0$, we obtain the explicit solution

$$\ln Y_t = -\frac{1}{2}t + W_t.$$

Consequently,

$$\boxed{Y_t = \exp\left(-\frac{1}{2}t + W_t\right)}.$$

2.9

$$dY(t) = \mu\, dt + \sigma\, dW(t), \quad Y_0 = y_0.$$

Integrating from 0 to t yields:

$$\int_0^t dY_s = \int_0^t \mu ds + \int_0^t \sigma dW_s \iff Y(t) - Y(0) = \mu(t-0) + \sigma(W(t) - W(0)).$$

Hence,

$$Y(t) = y_0 + \mu t + \sigma W(t).$$

2.10

To solve this SDE, we apply the transformation:

$$Y_t = h(t, X_t) = e^{\theta t} X_t.$$

The derivatives needed for Itô's formula are:

$$\frac{\partial h}{\partial t} = \theta e^{\theta t} x, \qquad \frac{\partial h}{\partial x} = e^{\theta t}, \qquad \frac{\partial^2 h}{\partial x^2} = 0.$$

Applying Itô's formula gives

$$\begin{aligned}
dY_t &= \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}(-\theta X_t) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}\sigma^2\right) dt + \frac{\partial h}{\partial x}\,\sigma\, dW_t && (1)\\
&= \left(\theta e^{\theta t} X_t - \theta e^{\theta t} X_t + 0\right) dt + e^{\theta t}\sigma\, dW_t && (2)\\
&= \sigma e^{\theta t} dW_t. && (3)
\end{aligned}$$

Thus,

$$d\left(e^{\theta t} X_t\right) = \sigma e^{\theta t} dW_t.$$

Integrating from 0 to t,

$$\int_0^t d\left(e^{\theta s} X_s\right) = \int_0^t \sigma e^{\theta s} dW_s,$$

which yields

$$e^{\theta t} X_t - x_0 = \sigma \int_0^t e^{\theta s} dW_s.$$

Hence,

$$e^{\theta t} X_t = x_0 + \sigma \int_0^t e^{\theta s} dW_s,$$

and multiplying through by $e^{-\theta t}$ gives the explicit solution:

$$\boxed{X_t = x_0 e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-s)}\, dW_s}. \tag{4}$$

2.11

Define

$$X_t = h(t, Y_t) := Y_t - A, \qquad \Rightarrow \qquad X_0 = y_0 - A.$$

Since

$$\frac{\partial h}{\partial t} = 0, \qquad \frac{\partial h}{\partial y} = 1, \qquad \frac{\partial^2 h}{\partial y^2} = 0,$$

Itô's formula yields

$$dX_t = \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} b(A - Y_t) + \frac{1}{2}\frac{\partial^2 h}{\partial y^2}\sigma^2\right) dt + \frac{\partial h}{\partial y}\sigma\, dW_t.$$

Replacing $Y_t = X_t + A$,

$$dX_t = -bX_t\, dt + \sigma\, dW_t.$$

The SDE

$$dX_t = -bX_t\, dt + \sigma\, dW_t$$

is linear. Its solution is

$$X_t = X_0 e^{-bt} + \sigma \int_0^t e^{-b(t-s)}\, dW_s.$$

Returning to the original process:
Since $Y_t = X_t + A$,

$$Y_t = A + (y_0 - A)e^{-bt} + \sigma \int_0^t e^{-b(t-s)}\, dW_s.$$

Final expression:

$$\boxed{Y_t = A + (y_0 - A)e^{-bt} + \sigma \int_0^t e^{-b(t-s)}\, dW_s}$$

2.12

Introduce the transformation

$$Y_t = h(t, X_t) := e^{rt}\ln X_t.$$

Derivatives of h:

$$\frac{\partial h}{\partial t} = re^{rt}\ln X_t, \qquad \frac{\partial h}{\partial x} = \frac{e^{rt}}{x}, \qquad \frac{\partial^2 h}{\partial x^2} = -\frac{e^{rt}}{x^2}.$$

Applying Itô's formula:

$$dY_t = \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} f_t + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} g_t^2\right) dt + \frac{\partial h}{\partial x} g_t\, dW_t,$$

where $f_t = rX_t \ln(k/X_t)$ and $g_t = \sigma X_t$. Substituting,

$$dY_t = \left(re^{rt}\ln X_t + \frac{e^{rt}}{X_t} rX_t \ln\Big(\frac{k}{X_t}\Big) - \frac{1}{2}\frac{e^{rt}}{X_t^2}\sigma^2 X_t^2\right) dt + e^{rt}\sigma\, dW_t$$

$$= e^{rt}\left[r\ln k - \tfrac{1}{2}\sigma^2\right] dt + e^{rt}\sigma\, dW_t.$$

Thus

$$dY_t = e^{rt}\left(r\ln k - \frac{\sigma^2}{2}\right) dt + e^{rt}\sigma\, dW_t.$$

Integrating from 0 to t,

$$Y_t - Y_0 = \int_0^t e^{rs}\left(r\ln k - \tfrac{\sigma^2}{2}\right) ds + \sigma \int_0^t e^{rs}\, dW_s.$$

Since,

$$Y_0 = \ln x_0, \qquad \int_0^t e^{rs} ds = \frac{e^{rt} - 1}{r},$$

we obtain

$$Y_t = \ln x_0 + \frac{e^{rt} - 1}{r}\left(r \ln k - \frac{\sigma^2}{2}\right) + \sigma \int_0^t e^{rs}\, dW_s.$$

Recalling that $Y_t = e^{rt} \ln X_t$, we divide by e^{rt}:

$$\ln X_t = e^{-rt} \ln x_0 + \left(1 - e^{-rt}\right)\left(\ln k - \frac{\sigma^2}{2r}\right) + \sigma \int_0^t e^{-r(t-s)}\, dW_s.$$

Exponentiating yields the explicit strong solution:

$$\boxed{X_t = \exp\left\{ e^{-rt} \ln x_0 + \left(1 - e^{-rt}\right)\left(\ln k - \frac{\sigma^2}{2r}\right) + \sigma \int_0^t e^{-r(t-s)}\, dW_s \right\}.}$$

2.13

Introduce the transformation

$$Y_t = h(t, X_t) := \ln\left(\frac{X_t}{x_0}\right).$$

Derivatives of h:

$$\frac{\partial h}{\partial t} = 0, \qquad \frac{\partial h}{\partial x} = \frac{1}{X_t}, \qquad \frac{\partial^2 h}{\partial x^2} = -\frac{1}{X_t^2}.$$

Itô's formula gives

$$dY_t = \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} f_t + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} g_t^2\right) dt + \frac{\partial h}{\partial x} g_t\, dW_t,$$

where $f_t = rX_t$ and $g_t = \sigma X_t$. Substituting,

$$\begin{aligned} dY_t &= \left(0 + \frac{1}{X_t} r X_t + \frac{1}{2}\left(-\frac{1}{X_t^2}\right) \sigma^2 X_t^2\right) dt + \frac{1}{X_t} \sigma X_t\, dW_t \\ &= \left(r - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t. \end{aligned}$$

Thus,

$$dY_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t.$$

Integrating from 0 to t:

$$Y_t - Y_0 = \int_0^t \left(r - \frac{\sigma^2}{2}\right) ds + \sigma \int_0^t dW_s.$$

Since $Y_0 = 0$ (because $Y_0 = \ln(X_0/x_0) = 0$),

$$Y_t = \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t.$$

Recalling that $Y_t = \ln(X_t/x_0)$, we obtain

$$\ln\left(\frac{X_t}{x_0}\right) = \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t.$$

Exponentiating,

$$\frac{X_t}{x_0} = \exp\left\{\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t\right\},$$

and therefore

$$\boxed{X_t = x_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}.}$$

2.14

$$dY_t = -\frac{\sigma^2}{2} Y_t \, dt + \sigma Y_t \, dW_t, \qquad Y_0 = y_0, \qquad Z(t) = h(t, Y_t) = \ln Y_t$$

$$\frac{\partial h}{\partial t} = 0, \qquad \frac{\partial h}{\partial y} = \frac{1}{y}, \qquad \frac{\partial^2 h}{\partial y^2} = -\frac{1}{y^2}$$

$$dZ_t = \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} F_t + \frac{1}{2}\frac{\partial^2 h}{\partial y^2} G_t^2\right) dt + \frac{\partial h}{\partial y} G_t \, dW_t$$

$$= \left(0 + \frac{1}{Y_t}\left(-\frac{\sigma^2}{2} Y_t\right) + \frac{1}{2}\left(-\frac{1}{Y_t^2}\right)\sigma^2 Y_t^2\right) dt + \frac{1}{Y_t}\sigma Y_t \, dW_t$$

$$= \left(-\frac{\sigma^2}{2} - \frac{\sigma^2}{2}\right) dt + \sigma \, dW_t$$

$$dZ_t = -\sigma^2 \, dt + \sigma \, dW_t$$

$$\int_0^t dZ_s = \int_0^t -\sigma^2 \, ds + \int_0^t \sigma \, dW_s$$

$$Z_t - Z_0 = -\sigma^2 t + \sigma W_t \qquad \Longrightarrow \qquad Z_t = Z_0 - \sigma^2 t + \sigma W_t$$

$$\ln Y_t = \ln y_0 - \sigma^2 t + \sigma W_t$$

$$Y_t = y_0 \exp\left(-\sigma^2 t + \sigma W_t\right)$$

2.15

$$dX_t = (\beta - \gamma)(X_t - \gamma)\left(\alpha - \beta \ln(X_t - \gamma)\right) dt + \sigma(X_t - \gamma) dW_t, \qquad X_0 = x_0$$

$$Y_t = h(t, X_t) = \ln(X_t - \gamma)$$

$$\frac{\partial h}{\partial t} = 0, \qquad \frac{\partial h}{\partial x} = \frac{1}{X_t - \gamma}, \qquad \frac{\partial^2 h}{\partial x^2} = -\frac{1}{(X_t - \gamma)^2}$$

$$dY_t = \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} F_t + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} G_t^2\right) dt + \frac{\partial h}{\partial x} G_t \, dW_t$$

$$= \left[\frac{1}{X_t - \gamma}(X_t - \gamma)(\alpha - \beta \ln(X_t - \gamma)) + \frac{1}{2}\left(-\frac{1}{(X_t - \gamma)^2}\right)\sigma^2 (X_t - \gamma)^2\right] dt + \frac{1}{X_t - \gamma}\sigma(X_t - \gamma) \, dW_t$$

$$= (\alpha - \beta Y_t) dt - \frac{\sigma^2}{2} dt + \sigma dW_t$$

$$dY_t = \big(\alpha - \beta Y_t - \frac{\sigma^2}{2}\big)dt + \sigma dW_t$$

$$e^{\beta t} dY_t = e^{\beta t}\left(\alpha - \frac{\sigma^2}{2} - \beta Y_t\right)dt + \sigma e^{\beta t} dW_t$$

$$d\left(e^{\beta t} Y_t\right) = e^{\beta t}\left(\alpha - \frac{\sigma^2}{2}\right)dt + \sigma e^{\beta t} dW_t$$

$$e^{\beta t} Y_t - Y_0 = \int_0^t e^{\beta s}\left(\alpha - \frac{\sigma^2}{2}\right)ds + \sigma \int_0^t e^{\beta s} dW_s$$

$$Y_t = e^{-\beta t} Y_0 + \left(\alpha - \frac{\sigma^2}{2}\right)e^{-\beta t}\int_0^t e^{\beta s} ds + \sigma e^{-\beta t}\int_0^t e^{\beta s} dW_s$$

$$Y_t = e^{-\beta t} Y_0 + \left(\alpha - \frac{\sigma^2}{2}\right)\left(\frac{1 - e^{-\beta t}}{\beta}\right) + \sigma e^{-\beta t}\int_0^t e^{\beta s} dW_s$$

$$\ln(X_t - \gamma) = e^{-\beta t}\ln(x_0 - \gamma) + \left(\alpha - \frac{\sigma^2}{2}\right)\frac{1 - e^{-\beta t}}{\beta} + \sigma e^{-\beta t}\int_0^t e^{\beta s} dW_s$$

$$X_t - \gamma = \exp\left[e^{-\beta t}\ln(x_0 - \gamma) + \left(\alpha - \frac{\sigma^2}{2}\right)\frac{1 - e^{-\beta t}}{\beta} + \sigma e^{-\beta t}\int_0^t e^{\beta s} dW_s\right]$$

$$X_t = \gamma + \exp\left[e^{-\beta t}\ln(x_0 - \gamma) + \left(\alpha - \frac{\sigma^2}{2}\right)\frac{1 - e^{-\beta t}}{\beta} + \sigma e^{-\beta t}\int_0^t e^{\beta s} dW_s\right]$$