# Stochastic differential equations harvesting models: simulation and numerical solution


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# Outline

# 1. Introduction

## 1.1. Stochastic Gompertz model with harvesting

$$dX_t = rX_t \ln\left(\frac{K}{X_t}\right) dt - qE_tX_t dt + \sigma_1 X_t dW_t^X, \quad X(0) = x > 0.$$

| | |
|---|---|
| $X_t$ | **stock size at time** $t$ |
| $r$ | **intrinsic growth rate** |
| $K$ | **carrying capacity** |
| $q$ | **catchability parameter** |
| $E_t$ | **fishing effort at time** $t$ |
| $\sigma_1$ | **noise intensity** |
| $W_t^X$ | **standard Wiener process** |
| $x$ | **initial population size (known)** |
| $Y_t := qE_tX_t$ | **yield** |

# 2. Problem Definition

## 2.1. Variables

- Fish price, $P_t$, follows a Geometric Brownian Motion:

$$dP_t = \mu P_t dt + \sigma_2 P_t dW_t^P, \quad P(0) = p > 0.$$

- Costs are defined as:

$$C_t = c_1 E_t + c_2 E_t^2.$$

- Profit is the difference between revenues and costs:

$$\begin{aligned} \Pi_t &= P_t . Y_t - C_t \\ &= \left(P_t q X_t - c_1 - c_2 E_t\right) E_t. \end{aligned}$$

- The total expected discounted profit over the interval $(t, T)$ is given by the functional:

$$J = \mathbb{E}\left[\int_t^T e^{-\delta\tau}\Pi(\tau)d\tau \middle| X(t) = x_t, P(t) = p_t\right] := \mathbb{E}_{x_t, p_t}\left[\int_t^T e^{-\delta\tau}\Pi(\tau)d\tau\right].$$

# 2. Problem Definition

## 2.2. Problem Formalization

**Goal: Using $E(t)$ as control, solve the Stochastic Optimal Control Problem (SOCP):**

$$J^* := \max_{\substack{E(\tau) \\ 0 \leq \tau \leq T}} J = \max_{\substack{E(\tau) \\ 0 \leq \tau \leq T}} \mathbb{E}_{x,p} \left[ \int_0^T e^{-\delta \tau} \Pi(\tau) d\tau \right]$$

**s.t.**

**growth equation:** $dX_t = rX_t \ln\left(\frac{K}{X_t}\right) dt - qE_t X_t dt + \sigma_1 X_t dW_t^X,$

**price dynamics:** $dP_t = \mu P_t dt + \sigma_2 P_t dW_t^P,$

**effort restrictions:** $0 \leq E_{min} \leq E(t) \leq E_{max} < \infty, \quad \forall t \in [0, T],$

**terminal condition:** $J(X_T, P_T, T) = 0,$

**initial conditions:** $X(0) = x, \ P(0) = p.$

# 2. Problem Definition

## 2.3. HJB and Control Function

Using dynamic programming, the solution to the SOCP is obtained via the HJB equation:

$$\begin{aligned}
-\frac{\partial J^*(X_t, P_t, t)}{\partial t} &= \left(P_t q X_t - c_1 - c_2 E_t^*\right) E_t^* - \delta J^*(X_t, P_t, t) \\
&+ \frac{\partial J^*(X_t, P_t, t)}{\partial X_t}\left(r X_t \ln\left(\frac{K}{X_t}\right) - q E_t^* X_t\right) + \frac{\partial J^*(X_t, P_t, t)}{\partial P_t}\mu P_t \\
&+ \frac{1}{2}\frac{\partial^2 J^*(X_t, P_t, t)}{\partial X_t^2}\sigma_1^2 X_t^2 + \frac{1}{2}\frac{\partial^2 J^*(X_t, P_t, t)}{\partial P_t^2}\sigma_2^2 P_t^2 \\
&+ \frac{\partial^2 J^*(X_t, P_t, t)}{\partial X_t \partial P_t}\rho\sigma_1 X_t \sigma_2 P_t,
\end{aligned}$$

The unconstrained effort is: $E_{free}^*(t) = \frac{qX_t}{2c_2}\left(P_t - \frac{\partial J^*(X_t, P_t, t)}{\partial X_t}\right) - \frac{c_1}{2c_2}$, which turns to be the optimal variable effort if boundaries are satisfied.

$$E^*(t) = \begin{cases} E_{min}, & if \quad E_{free}^*(t) < E_{min} \\ E_{free}^*(t), & if \quad E_{min} \leq E_{free}^*(t) \leq E_{max}, \\ E_{max}, & if \quad E_{free}^*(t) > E_{max} \end{cases}$$

- The time derivative is approximated by a forward difference quotient.

$$\frac{\partial J^*_{i,l,j}}{\partial t} \approx \frac{J^*_{i,l,j+1} - J^*_{i,l,j}}{\Delta t}, \quad 0 \le i \le m, \quad 0 \le p \le k, \quad 0 \le j \le n-1.$$

- Population derivatives are approximated by the following schemes.

$$\begin{aligned}
\frac{\partial J^*_{i,l,j}}{\partial x} &\approx \frac{J^*_{i+1,l,j} - J^*_{i-1,l,j}}{2\Delta x}, \quad 1 \le i \le m-1, \\
\frac{\partial^2 J^*_{i,l,j}}{\partial x^2} &\approx \frac{J^*_{i+1,l,j} - 2J^*_{i,l,j} + J^*_{i-1,l,j}}{\Delta x^2}, \quad 1 \le i \le m-1, \\
\frac{\partial J^*_{m,l,j}}{\partial x} &\approx \frac{J^*_{m,l,j} - J^*_{m-1,l,j}}{\Delta x}, \quad i = m, \\
\frac{\partial^2 J^*_{m,l,j}}{\partial x^2} &\approx \frac{J^*_{m,l,j} - 2J^*_{m-1,l,j} + J^*_{m-2,l,j}}{\Delta x^2} \quad i = m.
\end{aligned}$$

- Partial derivatives w.r.t. price follow the same reasoning as population ones.

$$
\begin{aligned}
\frac{\partial J^*_{i,l,j}}{\partial p} &\approx \frac{J^*_{i,l+1,j} - J^*_{i,l-1,j}}{2\Delta p}, \quad 1 \leq l \leq k-1, \\
\frac{\partial^2 J^*_{i,l,j}}{\partial p^2} &\approx \frac{J^*_{i,l+1,j} - 2J^*_{i,l,j} + J^*_{i,l-1,j}}{\Delta p^2}, \quad 1 \leq l \leq k-1, \\
\frac{\partial J^*_{i,k,j}}{\partial p} &\approx \frac{J^*_{i,k,j} - J^*_{i,k-1,j}}{\Delta p}, \quad l = k, \\
\frac{\partial^2 J^*_{i,k,j}}{\partial p^2} &\approx \frac{J^*_{i,k,j} - 2J^*_{i,k-1,j} + J^*_{i,k-2,j}}{\Delta p^2} \quad l = k, \\
\frac{\partial J^*_{i,0,j}}{\partial p} &\approx \frac{J^*_{i,1,j} - J^*_{i,0,j}}{\Delta p}, \quad l = 0, \\
\frac{\partial^2 J^*_{i,0,j}}{\partial p^2} &\approx \frac{J^*_{i,2,j} - 2J^*_{i,1,j} + J^*_{i,0,j}}{\Delta p^2} \quad l = 0.
\end{aligned}
$$

# 2. Problem Definition

## 2.4. Discretization

- The cross partial derivatives w.r.t. population and price schemes are presented below.

$$
\begin{aligned}
\frac{\partial J^*_{i,l,j}}{\partial x \partial p} &\approx \frac{J^*_{i+1,l+1,j} - J^*_{i-1,l+1,j} - J^*_{i+1,l-1,j} + J^*_{i-1,l-1,j}}{4\Delta x \Delta p}, \quad 1 \le l \le k-1, \quad 1 \le i \le m-1 \\
\frac{\partial J^*_{m,l,j}}{\partial x \partial p} &\approx \frac{J^*_{m,l+1,j} - J^*_{m-1,l+1,j} - J^*_{m,l-1,j} + J^*_{m-1,l-1,j}}{2\Delta x \Delta p}, \quad 1 \le l \le k-1, \quad i = m \\
\frac{\partial J^*_{i,k,j}}{\partial x \partial p} &\approx \frac{J^*_{i+1,k,j} - J^*_{i-1,k,j} - J^*_{i+1,k-1,j} + J^*_{i-1,k-1,j}}{2\Delta x \Delta p}, \quad l = k, \quad 1 \le i \le m-1 \\
\frac{\partial J^*_{m,k,j}}{\partial x \partial p} &\approx \frac{J^*_{m,k,j} - J^*_{m-1,k,j} - J^*_{m,k-1,j} + J^*_{m-1,k-1,j}}{\Delta x \Delta p}, \quad l = k, \quad i = m \\
\frac{\partial J^*_{i,0,j}}{\partial x \partial p} &\approx \frac{J^*_{i+1,1,j} - J^*_{i-1,1,j} - J^*_{i+1,0,j} + J^*_{i-1,0,j}}{2\Delta x \Delta p}, \quad l = 0, \quad 1 \le i \le m-1 \\
\frac{\partial J^*_{m,0,j}}{\partial x \partial p} &\approx \frac{J^*_{m,1,j} - J^*_{m-1,1,j} - J^*_{m,0,j} + J^*_{m-1,0,j}}{\Delta x \Delta p}, \quad l = 0, \quad i = m
\end{aligned}
$$

# 3. Solution

- When discretizations are applied and some simplifications performed, it is possible to write the system with matrices $A$, $B$ and $C$ such that

$$A\boldsymbol{J}_{-}^{*} = B\,\boldsymbol{J}_{+}^{*} + C,$$

with

$$\boldsymbol{J}_{-}^{*} = \left[\boldsymbol{J}_{0}^{*} \quad | \quad \boldsymbol{J}_{1}^{*} \quad | \quad \cdots \quad | \quad \boldsymbol{J}_{n-1}^{*}\right], \quad \boldsymbol{J}_{+}^{*} = \left[\boldsymbol{J}_{1}^{*} \quad | \quad \boldsymbol{J}_{2}^{*} \quad | \quad \cdots \quad | \quad \boldsymbol{J}_{n}^{*}\right],$$

and

$$\boldsymbol{J}_{j}^{*} = \left[J_{0,0,j}^{*} \quad \cdots \quad J_{i,l,j}^{*} \quad \cdots \quad J_{m,k,j}^{*}\right]', \quad 0 \leq i \leq m, \quad 0 \leq l \leq k, \quad 0 \leq j \leq n.$$

$'$ denotes the transpose operator.

# 4. Application

## 4.1. Scenarios

We will present and compare 3 scenarios:

S1 Profit based on stochastic prices: $P_t$ is described by the SDE $dP_t = \mu P_t dt + \sigma_2 P_t dW_t^P$, and the profit is given by

$$\Pi_t = P_t Y_t - C_t = \left(P_t q X_t - c_1 - c_2 E_t\right) E_t.$$

S2 Profit based on deterministic Prices: $P_t$ is defined as $P_t = p_1 - p_2 Y_t$, and the profit is given by

$$\Pi_t = P_t Y_t - C_t = \left(p_1 q X_t - c_1\right) E_t - \left(p_2 q^2 X_t^2 + c_2\right) E_t^2.$$

S3 Profit based on a penalised effort:

$$\Pi_t = P_t Y_t - \left(C_t + C_t^{pen}(\varepsilon)\right) = \left(p_1 q X_t - c_1\right) E_t - \left(p_2 q^2 X_t^2 + c_2\right) E_t^2 - \varepsilon (E_t - E_{ref})^2,$$

- where the artificial cost component penalizes profit values when the effort takes abrupt changes from a reference effort value, say $E_{ref}$.
- *The higher (lower) the value of $\varepsilon$, the higher (lower) the impact on effort and, consequently, the lower (higher) the profit.*

# 4. Application

## 4.2. Data

| Parameter | Value | Units |
|---|---|---|
| Population growth rate: $r$ | 1.331 | $\text{year}^{-1}$ |
| Population carrying capacity: $K$ | 11400 | tonnes |
| Initial population size: $x$ | $0.5K$ | tonnes |
| Maximum population size: $x_{max}$ | $2K$ | tonnes |
| Population volatility: $\sigma_1$ | 0.2 | $\text{year}^{-1/2}$ |
| Catchability: $q$ | $9.77 \cdot 10^{-5}$ | $\text{SFU}^{-1} \cdot \text{year}^{-1}$ |
| Maximum allowed effort: $E_{max}$ | $r/q$ | SFU |
| Minimum allowed effort: $E_{min}$ | 0 | SFU |
| Linear cost coefficient: $c_1$ | 1156.8 | $\text{BDT} \cdot \text{SFU}^{-1} \cdot \text{year}^{-1}$ |
| Quadratic cost coefficient: $c_2$ | 0.01 | $\text{BDT} \cdot \text{SFU}^{-2} \cdot \text{year}^{-1}$ |
| Discount factor: $\delta$ | 0.05 | $\text{year}^{-1}$ |
| Time horizon: $T$ | 50 | year |

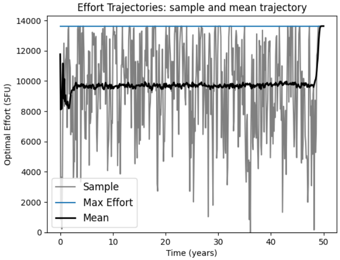
BDT is an abbreviation for Bangladesh Taka.

# 4. Application

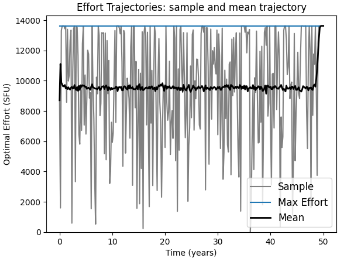
## 4.2. Data

| Parameters | Value | Units |
|---|---|---|
| Constant coefficient: $p_1$ | 8362.3 | $\text{BDT} \cdot \text{tonnes}^{-1}$ |
| Linear coefficient: $p_2$ | 0 | |
| Reference Effort: $E_{ref}$ | $0.5 \cdot r/q$ | SFU |
| Magnitude of Penalisation: $\varepsilon$ | 0.4 | |
| Price growth rate: $\mu$ | 0.001 | $\text{year}^{-1}$ |
| Price volatility: $\sigma_2$ | 0.01 | $\text{year}^{-1/2}$ |
| Initial price value: $p_0$ | 8362.3 | $\text{BDT} \cdot \text{year}^{-1}$ |
| Wiener processes correlation: $\rho$ | 0 | |

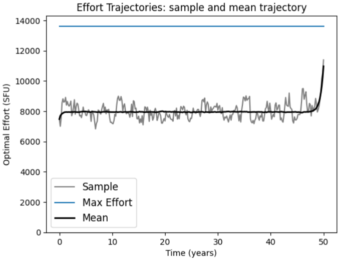
Effort trajectories: mean and sample trajectory for each scenario




(a) S1 - stochastic prices




(b) S2 - deterministic prices



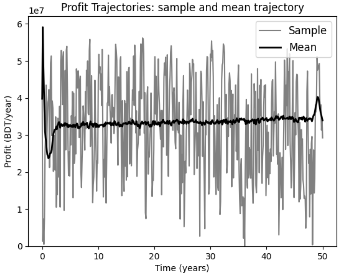

(c) S3 - penalised effort

- Besides being optimal, effort strategies (a) and (b) cannot be implemented due to social and logistic issues arising from the frequent and significant adjustments on effort.
- Regarding (c), the penalization parameter reduces the amplitude of oscillations, benefiting social issues, but logistical problems remain, albeit to a lesser extent.

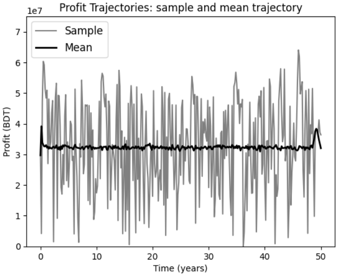
# 4. Application

## 4.4. Profit

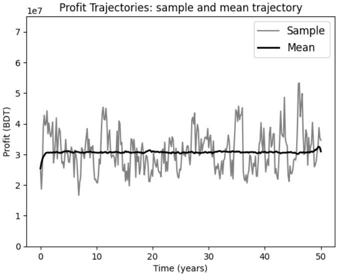
Profit trajectories: mean and sample trajectory for each scenario




(a) S1 - stochastic prices




(b) S2 - deterministic prices



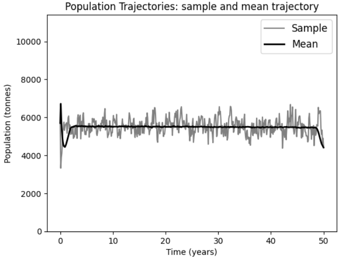

(c) S3 - penalised effort

- Effort influences profit by affecting both revenues and costs. More aggressive policies raise costs and reduce profit.
- In case (c), profit never reaches zero because there are no inactive periods; in (a), profit shows a slight upward trend due to the drift in price dynamics.

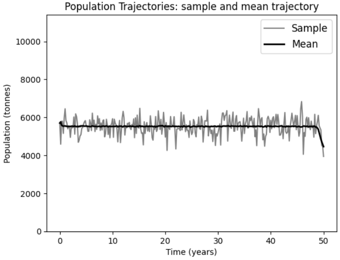
# 4. Application

## 4.5. Population

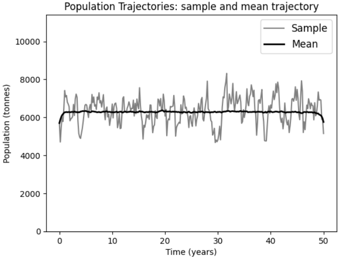
Population trajectories: mean and sample trajectory for each scenario




(a) S1 - stochastic prices




(b) S2 - deterministic prices




(c) S3 - penalised effort

- Due to softer effort strategy, (c) presents a higher population value.
- In general, population decreases whenever effort increases to maximum allowed levels.
- All trajectories present a stable population size, guaranteeing the sustainability of the activity in the long run.

| Model | $J^*$ (x $10^6$ BDT) | $\Delta$ (%) |
|---|---|---|
| S1 - stochastic prices | 605.813 | +2.19 |
| S2 - deterministic prices | 592.840 | (reference) |
| S3 - penalised effort | 561.756 | -5.24 |

# Main bibliography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**Thank you!**

**Moitas grazas!**

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