# Mathematical Programming I

## BSc in Applied Mathematics for Economics and Management (MAEG)

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2025-2026

# The Simplex Method

# Extreme points of an LP

## Matricial Form

Consider the following LP

$$\begin{array}{ll} \max & z = c^t x \\ \text{s. t.:} & Ax = b \\ & x \geq 0 \end{array}$$

with

- $x \in \mathbb{R}^n$, $n$ variables in total (decision variables plus slack variables)
- $m$ constraints (technological)
- $c \in \mathbb{R}^n, \; b \in \mathbb{R}^m$
- $A_{m \times n}$ matrix such that $n > m$ and $rank(A, b) = rank(A) = m$

The non-empty set $X = \{x \in \mathbb{R}^{\bar{n}} : Ax \leq b, \; x \geq 0\}$ of feasible solutions of a linear programming problem is a polyhedron, and has a finite number of extreme points and extreme rays.

# Example of the RM model

The standard form of the RM model is

$$\begin{aligned}
\max \quad & z = 4x_1 + 5x_2, \\
\text{s. t.:} \quad & 4x_1 + 6x_2 + x_3 = 24, \\
& 2x_1 + x_2 + x_4 = 6, \\
& x_1 - x_2 + x_5 = 1, \\
& x_1 + x_6 = 2, \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}$$

- $x \in \mathbb{R}^6$, 6 variables in total (decision variables plus slack variables)
- 4 constraints (technological)
- $c \in \mathbb{R}^6, \ b \in \mathbb{R}^4$
- $A_{4\times 6}$ matrix such that $6 > 4$ and $rank(A, b) = rank(A) = 4$

## Extreme points

A non-empty bounded feasible region has a finite number of extreme points.

Consider the RM example. There are:

$m = 4$ technological constraints ($\rightarrow 4$ slack variables)
$n = 6$ variables (2 decision plus 4 slack)
$\bar{n} = n - m = 2$ decision variables

There are at most $C^n_{n-m}$ extreme points

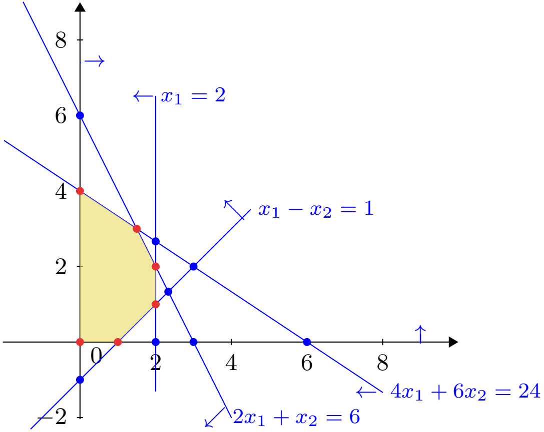
The RM model has at most $C^n_{n-m} = C^6_2 = \frac{6!}{(6-2)!\ 2!} = \frac{6\times5}{2} = 15$ extreme points

**Property**

An extreme point is a feasible solution at which at least $n - m$ constraints are binding.

# Extreme points of the RM model

Extreme points are: $(0, 0)$; $(1, 0)$; $(2, 1)$; $(2, 2)$; $(1.5, 3)$; $(0, 4)$



# How to find extreme points? for the RM model

solutions at which there are 2 binding constraints:

| | | |
|---|---|---|
| $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \Leftrightarrow (0,0)$ | $\begin{cases} x_1 = 0 \\ 4x_1 + 6x_2 = 24 \end{cases} \Leftrightarrow (0,4)$ | $\begin{cases} x_1 = 0 \\ 2x_1 + x_2 = 6 \end{cases} \Leftrightarrow (0,6)$ |
| $\begin{cases} x_1 = 0 \\ x_1 - x_2 = 1 \end{cases} \Leftrightarrow (0,-1)$ | $\begin{cases} x_2 = 0 \\ 4x_1 + 6x_2 = 24 \end{cases} \Leftrightarrow (6,0)$ | $\begin{cases} x_2 = 0 \\ 2x_1 + x_2 = 6 \end{cases} \Leftrightarrow (3,0)$ |
| $\begin{cases} x_2 = 0 \\ x_1 - x_2 = 1 \end{cases} \Leftrightarrow (1,0)$ | $\begin{cases} x_1 = 2 \\ x_2 = 0 \end{cases} \Leftrightarrow (2,0)$ | $\begin{cases} 4x_1 + 6x_2 = 24 \\ 2x_1 + x_2 = 6 \end{cases} \Leftrightarrow (\frac{3}{2},3)$ |
| $\begin{cases} 4x_1 + 6x_2 = 24 \\ x_1 - x_2 = 1 \end{cases} \Leftrightarrow (3,2)$ | $\begin{cases} 4x_1 + 6x_2 = 24 \\ x_1 = 2 \end{cases} \Leftrightarrow (2,\frac{8}{3})$ | $\begin{cases} 2x_1 + x_2 = 6 \\ x_1 - x_2 = 1 \end{cases} \Leftrightarrow (\frac{7}{3},\frac{4}{3})$ |
| $\begin{cases} 2x_1 + x_2 = 6 \\ x_1 = 2 \end{cases} \Leftrightarrow (2,2)$ | $\begin{cases} x_1 - x_2 = 1 \\ x_1 = 2 \end{cases} \Leftrightarrow (2,1)$ | $C_2^6 = \frac{6!}{(6-2)!\ 2!} = \frac{6\times5}{2} = 15$ |

# How to find extreme points? for the RM model

equivalently we have:

| | | |
|---|---|---|
| $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \Leftrightarrow (0,0)$ | $\begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases} \Leftrightarrow (0,4)$ | $\begin{cases} x_1 = 0 \\ x_4 = 0 \end{cases} \Leftrightarrow (0,6)$ |
| $\begin{cases} x_1 = 0 \\ x_5 = 0 \end{cases} \Leftrightarrow (0,-1)$ | $\begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases} \Leftrightarrow (6,0)$ | $\begin{cases} x_2 = 0 \\ x_4 = 0 \end{cases} \Leftrightarrow (3,0)$ |
| $\begin{cases} x_2 = 0 \\ x_5 = 0 \end{cases} \Leftrightarrow (1,0)$ | $\begin{cases} x_2 = 0 \\ x_6 = 0 \end{cases} \Leftrightarrow (2,0)$ | $\begin{cases} x_3 = 0 \\ x_4 = 0 \end{cases} \Leftrightarrow (\frac{3}{2},3)$ |
| $\begin{cases} x_3 = 0 \\ x_5 = 0 \end{cases} \Leftrightarrow (3,2)$ | $\begin{cases} x_3 = 0 \\ x_6 = 0 \end{cases} \Leftrightarrow (2,\frac{8}{3})$ | $\begin{cases} x_4 = 0 \\ x_5 = 0 \end{cases} \Leftrightarrow (\frac{7}{3},\frac{4}{3})$ |
| $\begin{cases} x_4 = 0 \\ x_6 = 0 \end{cases} \Leftrightarrow (2,2)$ | $\begin{cases} x_5 = 0 \\ x_6 = 0 \end{cases} \Leftrightarrow (2,1)$ | $C_2^6 = \frac{6!}{(6-2)!\ 2!} = \frac{6\times5}{2} = 15$ |

# Extreme points

solutions at which there are 2 binding constraints:

$$(0,0),(0,4),(0,6),(0,-1),(6,0),(3,0),(1,0),$$

$$(2,0),(\frac{3}{2},3),(3,2),(2,\frac{8}{3}),(\frac{7}{3},\frac{4}{3}),(2,2),(2,1)$$

the feasible solutions at which there are 2 binding constraints, which are the extreme points, are

$$(0,0),(0,4),(1,0),(\frac{3}{2},3),(2,2),(2,1)$$

the unfeasible solutions at which there are 2 binding constraints:

$$(0,6),(0,-1),(6,0),(3,0),(2,0),(3,2),(2,\frac{8}{3}),(\frac{7}{3},\frac{4}{3}),$$

## Example 2:

Identify the extreme points of the following LP.

$$
\begin{aligned}
&\max\ z = x_1 + 3x_2 \\
&\text{s. to} \quad x_1 - 3x_2 \leq 3 \\
&\qquad\quad -2x_1 + x_2 \leq 2 \\
&\qquad\quad -3x_1 + 4x_2 \leq 12 \\
&\qquad\quad 3x_1 + x_2 \leq 9 \\
&\qquad\quad x_1,\ x_2 \geq 0
\end{aligned}
$$

# Example 2: LP in the standard form

$$
\begin{aligned}
\max\ z &= x_1 + 3x_2 \\
\text{s. a} \quad & x_1 - 3x_2 + x_3 = 3 \\
& -2x_1 + x_2 + x_4 = 2 \\
& -3x_1 + 4x_2 + x_5 = 12 \\
& 3x_1 + x_2 + x_6 = 9 \\
& x_1,\ x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

Let

$$
A = \begin{bmatrix} 1 & -3 & 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 & 0 \\ -3 & 4 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 2 \\ 12 \\ 9 \end{bmatrix}
$$

# Example 2: the matrix of the linear equations system

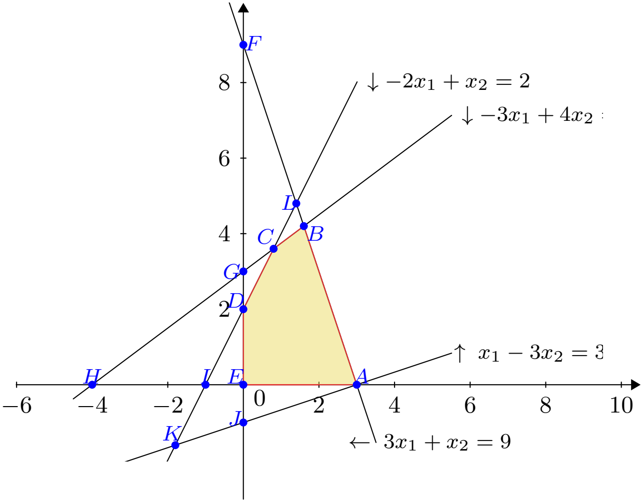
The matrix of the linear equations system

$$A = \begin{bmatrix} 1 & -3 & 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 & 0 \\ -3 & 4 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- is $4 \times 6$ ($m \times n$, with $m$ equations and $n$ variable)
- has $rank(A) = m = 4$ (matrix $A$ has the identity matrix $I_m$ as submatrix)
- has $C_4^6 = \frac{6!}{4!\ 2!}$ possible square submatrix $4 \times 4$

The BASIS is the square submatrix $B$ of $A$ such that $rank(B) = rank(A) = m$; hence, it is an invertible matrix whose rows and columns are linearly independent.

# Example 2: extreme points are $A, B, C, D, E$



| extreme point | $\rightarrow$ | solution | $\rightarrow$ | associated basis |
|---|---|---|---|---|
| $E = (0,0)$ | $\rightarrow$ | $x = (0,0,3,2,12,9)$ | $\rightarrow$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| $D = (0,2)$ | $\rightarrow$ | $x = (0,2,9,0,4,7)$ | $\rightarrow$ | $\begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ |
| $C = (\frac{4}{5}, \frac{18}{5})$ | $\rightarrow$ | $x = (\frac{4}{5}, \frac{18}{5}, 13, 0, 0, 3)$ | $\rightarrow$ | $\begin{bmatrix} 1 & -3 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 4 & 0 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ |

| extreme point | → | solution | → | associated basis |
|---|---|---|---|---|
| $B = (\frac{8}{5}, \frac{21}{5})$ | → | $x = (\frac{8}{5}, \frac{21}{5}, 14, 1, 0, 0)$ | → | $\begin{bmatrix} 1 & -3 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ -3 & 4 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix}$ |
| $A = (3, 0)$ | → | $x = (3, 0, 0, 8, 21, 0)$ | → | $\begin{bmatrix} 1 & -3 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -3 & 4 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{bmatrix},$ |

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

# Example 3

Consider the following LP

$$
\begin{array}{lrrl}
\max z = 8x_1 + 5x_2 & & & \\
\text{s. a} & x_1 + & x_2 & \leq 8 \\
& -3x_1 + & x_2 & \leq 0 \\
& x_1 & & \geq 1 \\
& & x_2 & \geq 2 \\
& x_1, & x_2 & \geq 0
\end{array}
$$

in the standard form

$$
\begin{array}{llllllll}
\max Z = 8x_1 + 5x_2 & & & & & & & \\
\text{s. a} & x_1 & +x_2 & +x_3 & & & & = 8 \\
& -3x_1 & +x_2 & & +x_4 & & & = 0 \\
& -x_1 & & & & +x_5 & & = -1 \\
& & -x_2 & & & & +x_6 & = -2 \\
& x_1, & x_2, & x_3, & x_4, & x_5, & x_6 & \geq 0
\end{array}
$$

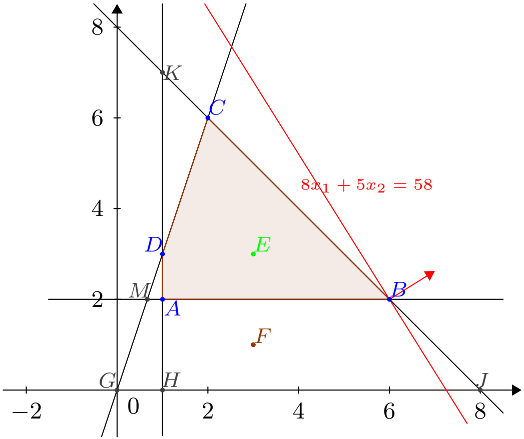
# Example 3

Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$, $b = \begin{bmatrix} 8 \\ 0 \\ -1 \\ -2 \end{bmatrix}$

and $c^T = \begin{bmatrix} 8 & 5 & 0 & 0 & 0 & 0 \end{bmatrix}$

The matrix $A$ has dimension $4 \times 6$.
The system has $4$ equations and $6$ variables, of which $4$ are slack variables.

# Example 3: Feasible region



# Points and solutions

| points | solutions | classification |
|---|---|---|
| A(1,2) | (1,2,5,1,0,0) | BFS |
| B(6,2) | (6,2,0,16,5,0) | BFS |
| C(2,6) | (2,6,0,0,1,4) | BFS |
| D(1,3) | (1,3,4,0,0,1) | BFS |
| E(3,3) | (3,3,2,6,2,1) | NBFS |
| F(3,1) | (3,1,4,8,2,-1) | NBIS |
| G(0,0) | (0,0,8,0,-1,-2) | BIS degenerada |
| H(1,0) | (1,0,7,3,0,-2) | BIS |
| J(8,0) | (8,0,0,24,7,-2) | BIS |
| K(1,7) | (1,7,0,-4,0,5) | BIS |
| M($\frac{2}{3}$,2) | ($\frac{2}{3}$,2,$\frac{16}{3}$,0,$-\frac{1}{3}$,0) | SBNA |

BFS = basic feasible solution;

NBFS = nonbasic feasible solution;

NBIS = nonbasic and infeasible solution;

BIS = basic, infeasible solution.

# Basic Feasible Solutions

$$\begin{array}{lll} \max & z = c^T x \\ \text{s. t.:} & Ax = b \\ & x \geq 0 \end{array}$$

$$\begin{array}{l} A_{m\times n} \quad (n > m) \\ \text{rank}(A, b) = \text{rank}(A) = m \\ n \leftarrow \text{num. var.} \\ m \leftarrow \text{num. constr.} \end{array}$$

We call BASIS to a square submatrix $B$ of $A$ such that $\text{rank}(B) = \text{rank}(A) = m$;

$B$ it is therefore an invertible matrix whose rows and columns are linearly independent

$$A = [B \quad N]$$

$$B_{m\times m} \leftarrow \text{basis}$$

$$N_{m\times(n-m)}$$

$$x = [x_B \quad x_N]^T$$

$$\begin{aligned} Ax = b \quad &\Leftrightarrow [B \quad N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = b \\ &\Leftrightarrow Bx_B + Nx_N = b \\ &\Leftrightarrow x_B = B^{-1}b - B^{-1}Nx_N \end{aligned}$$

# Basic Feasible Solutions

$B$ is the basic matrix (formed by the basic columns of $A$)
$N$ is the non-basic matrix (formed by the non-basic columns of $A$)

the components of $x_B$ are the basic variables
the components of $x_N$ are the non-basic variables

The solution $x = [x_B \quad x_N]^T$ of the system $Ax = b$ is called

a basic solution when $\begin{cases} x_B = B^{-1}b \\ x_N = 0 \end{cases}$

a basic feasible solution when additionally $x_B \geq 0$

if at least one component of $x_B$ is equal to zero, then $x$ is a degenerate basic solution

# Extreme points = Basic Feasible Solutions

The basic feasible solutions correspond to extreme points.

A solution $x$ of an LP is basic feasible
if and only if
it is an extreme point of the feasible region.

## Solutions of the LP in standard form

The solutions of the LP in standard form are the solutions of the system of linear equations $Ax = b$ to which we can associate the points in the space of decision variables, in the example in the plane $\mathbb{R}^2$ of the variables $x_1$ and $x_2$

The solutions can be classified as

- feasible solutions
  corresponding to all points in the feasible region
- infeasible solutions

The solutions can also be classified as

- basic solutions
  to which we can associate a basis and which are the unique solution of the system whose matrix is that basis $(Bx = b)$
- non-basic solutions (no basis can be associated)

# Basic Feasible Solutions and Optimality

When an optimal solution of an LP exists,
an optimal extreme point also exists.

# Properties of LP models

# Properties

## Property 1

The feasible region of an LP problem is either an empty set or a convex set.

## Property 2

If the feasible region of an LP problem is nonempty and bounded, then there exists an optimal solution.

## Property 3

If an LP problem has an optimal solution, then at least one of the extreme points of the feasible region is an optimal solution.

## Property 4

Given an LP problem with an optimal solution, if an extreme point of the feasible region has no adjacent extreme points with a better value for the objective function, then that extreme point is an optimal solution.

# The Simplex method

The key to the Simplex method lies in recognizing the optimality of a given extreme-point solution through local optimality conditions, without the need to globally enumerate all extreme points.

Let $x$ be a BFS, $x_B$ the set of basic variables, and $x_N$ the set of non-basic variables for the given associated basis $B$, hence $x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$

Then $z_0 = c^T x = c^T \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} = [c_B c_N] = c_B B^{-1} b$.

Feasibility requires that $Ax = b$, $x_B \geq 0$ and $x_N \geq 0$.

We have

$$Ax = b \quad \Leftrightarrow \quad x_B = B^{-1}b - B^{-1}Nx_N \quad \Leftrightarrow \quad x_B = B^{-1}b - \sum_{j\in J} B^{-1}A_j x_j$$

where $J$ is the current set of the indices of nonbasic variables, and $A_j$ is the column in matrix $A$ corresponding to variable $x_j$,

$$z = cx = c_B B^{-1}b - \sum_{j\in J}(c_B B^{-1}A_j - c_j)x_j = z_0 - \sum_{j\in J}(c_B B^{-1}A_j - c_j)x_j$$

# The Simplex method: optimality condition

$$\begin{aligned} \max \quad & z = c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \max \quad & z = z_0 - \textstyle\sum_{j \in J} (c_B B^{-1} A_j - c_j) x_j \\ \text{s.t.} \quad & \textstyle\sum_{j \in J} B^{-1} A_j x_j + x_B = B^{-1} b \\ & x \geq 0 \end{aligned}$$

reduced cost coefficient: $(z_j - c_j) = (c_B B^{-1} A_j - c_j)$

if $(z_j - c_j) \geq 0$ for all $j \in J$, then the current basic feasible solution is optimal:

$x_N = 0$ and $x_B = B^{-1} b$.

# The Simplex method

if $(z_j - c_j) < 0$ for $k \in J$ then fix the remaining nonbasic variables to zero

we have $z = z_0 - (c_B B^{-1} A_k - c_k) x_k = z_0 - (z_k - c_k) x_k$

and $x_B = B^{-1} b - B^{-1} A_k x_k = \begin{bmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \end{bmatrix} - \begin{bmatrix} \bar{a}_{1k} \\ \vdots \\ \bar{a}_{mk} \end{bmatrix} x_k$

if $\bar{a}_{ik} \leq 0$ then $(x_b)_i$ increases as $x_k$ increases and so contiues to be non-negative, $(x_b)_i \geq 0$

if $\bar{a}_{ik} > 0$ then $(x_b)_i$ decreases as $x_k$ increases and so to satisfy non-negativity, $(x_b)_i \geq 0$, $x_k$ is increased until the first point at which some basic variable $(x_b)_r$ drops to zero

# The Simplex method: feasibility condition

if $\bar{a}_{ik} > 0$ then $(x_b)_i$ decreases as $x_k$ increases and so to satisfy non-negativity, $(x_b)_i \geq 0$, $x_k$ is increased until the first point at which some basic variable $(x_b)_r$ drops to zero

the first basic variable dropping to zero corresponds to the minimum of $\frac{\bar{b}_i}{\bar{a}_{ik}}$ for positive $\bar{a}_{ik}$

we can increase $x_k$ until

$$x_k = \frac{\bar{b}_r}{\bar{a}_{rk}} \quad = \quad \min_{i=1,\ldots,m} \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} \ : \ \bar{a}_{ik} > 0 \right\}$$

in the absence of degeneracy, $\bar{b}_r > 0$, and so

$$x_k = \frac{\bar{b}_r}{\bar{a}_{rk}} > 0$$

therefore from $z = z_0 - (z_k - c_k)x_k$, and $(z_k - c_k) < 0$ and $x_k > 0$

it follows that $z > z_0$ and the objective function strictly improves

as $x_k$ increases from $0$ to $\frac{\bar{b}_r}{\bar{a}_{rk}}$ a new feasible solution is obtained

# The Simplex method: new basic feasible solution

a new feasible solution is obtained:

$$x_k = \frac{\bar{b}_r}{\bar{a}_{rk}}$$

$$(x_B)_i = \bar{b}_i - \frac{\bar{a}_{ik}}{\bar{a}_{rk}}\bar{b}_r, \ i = 1, \dots, m$$

at most $m$ variables are positive

the corresponding columns in $A$ are linearly independent since $\bar{a}_{rk} \neq 0$

therefore we obtain a new basic feasible solution

## The Simplex method

The simplex algorithm examines the extreme points until it finds the one that optimises the total value of the objective function or until it determines that the optimal solution occurs along an extreme direction

1. Consider an initial feasible solution corresponding to an extreme point: identify its non-binding constraints (basic variables) and build the Simplex table
2. Evaluate its adjacent extreme points and determine if there is one with a better objective function value.
3. If there is none: STOP the current extreme point is an optimal solution
4. Otherwise, identify a basic variable and a non-basic variable to exchange: obtain the new extreme point and go to Step 2.

# Simplex method: algebraic form

The Simplex Algorithm in its algebraic form, for a minimization problem:

0. Choose an initial basic feasible solution and let $B$ be the associated basis.
1. Solve the system $Bx_B = b$ and let $x_N = 0, \quad x_B = B^{-1}b = \bar{b}$ be its unique solution, with $z = c_B x_B$.
2. Solve the system $y_B B = c_B$ and let $y_B = c_B B^{-1}$ be its unique solution.

   Compute $z_j - c_j = y_B A_j - c_j$ for all nonbasic variables.

   Let $z_k - c_k = \max_{j \in I_N} \{z_j - c_j\}$, where $I_N$ denotes the set of indices of the nonbasic variables.

   If $z_k - c_k < 0$, $STOP$, the current feasible basic solution is an optimal solution.

   If $z_k - c_k = 0$, the current feasible basic solution is an optimal solution, but it is not unique. Continue to Step 3, considering variable $x_k$ as the entering variable in the basis, in order to find an alternative optimal solution.

   Otherwise, continue to Step 3 considering variable $x_k$ as the entering variable in the basis.

# Simplex method: algebraic form (cont.)

3. Solve the system $B\overline{A}_k = A_k$ and let $\overline{A}_k = B^{-1}A_k$ be its unique solution.
   If $\overline{A}_k \leq 0$, $STOP$, the optimal solution is unbounded along the ray
   $$\left\{ \begin{bmatrix} \bar{b} \\ 0 \end{bmatrix} + x_k \begin{bmatrix} -\overline{A}_k \\ e_k \end{bmatrix} : x_k \geq 0 \right\}$$
   where $e_k$ is a vector of dimension $|I_N|$.
   Otherwise, continue to Step 4.
4. Let $x_k$ be the entering variable in the basis.
   The index $r$ of the variable $x_{B_r}$ leaving the basis is determined by the following ratio
   $$\frac{\bar{b}_r}{\bar{a}_{rk}} = \min_{i \in I_B} \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} : \bar{a}_{ik} > 0 \right\},$$
   where $I_B$ denotes the set of indices of the basic variables.
   Update the basis $B$ by replacing column $A_{B_r}$ with column $A_k$.
   Update the index sets $I_B$ and $I_N$.
   Return to Step 1.

# The Simplex method: the table of the simplex

$$\begin{array}{ll} \max & z = c^T x \\ \text{s. t.:} & Ax = b \\ & x \geq 0 \end{array}$$

- $x \in \mathbb{R}^n$, $A_{m\times n}$ such that $n > m$ and $rank(A, b) = rank(A) = m$
- $B_{m\times m}$ invertible submatrix of matrix $A$ ($rank(B) = m$)

The Simplex table is always associated to an extreme point and is

| $\bar{z}$ and basic var. $x_B$ | $x$ | RHS |
|---|---|---|
| $x_B$ | $B^{-1}A$ | $B^{-1}b$ |
| $\bar{z}$ | $c_B B^{-1}A - c$ | $c_B B^{-1}b$ |

# The Simplex method: initial table

Step 0. Obtain an initial extreme point (basic solution) and build the Simplex table

Step 1. Use as an initial feasible basic solution (extreme point) the one obtained by setting the decision variables to zero: $x = (0, 0, 24, 6, 1, 2)$, with value $z = 0$, at which corresponds the identity basis $B = \mathcal{I}$

| $\bar{z}$ and basic var | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| $x_3$ | 4 | 6 | 1 | 0 | 0 | 0 | 24 |
| $x_4$ | 2 | 1 | 0 | 1 | 0 | 0 | 6 |
| $x_5$ | 1 | $-1$ | 0 | 0 | 1 | 0 | 1 |
| $x_6$ | 1 | 0 | 0 | 0 | 0 | 1 | 2 |
| $\bar{z}$ | $-4$ | $-5$ | 0 | 0 | 0 | 0 | 0 |

## is the optimality condition satisfied?

Evaluate the current solution: if all values in the row $\bar{z}$ satisfy the optimality condition, this is the optimal solution:

for a maximization problem, the optimality condition is $\bar{z}_j \geq 0$ for all $j$

for a minimization problem, the optimality condition is $\bar{z}_j \leq 0$ for all $j$

Otherwise, there are adjacent extreme points with a better objective function value

| $\bar{z}$ and basic var | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| $x_3$ | 4 | 6 | 1 | 0 | 0 | 0 | 24 |
| $x_4$ | 2 | 1 | 0 | 1 | 0 | 0 | 6 |
| $x_5$ | 1 | $-1$ | 0 | 0 | 1 | 0 | 1 |
| $x_6$ | 1 | 0 | 0 | 0 | 0 | 1 | 2 |
| $\bar{z}$ | $-4$ | $-5$ | 0 | 0 | 0 | 0 | 0 |

In this example, the current solution does not satisfy the optimality condition.

# identify an entering variable in the basis

- Select, from the reduced cost row $\bar{z}$, a variable (column) that violates the optimality condition, namely the one farthest from optimality:
  - **for a maximization problem:** choose the most negative reduced cost, $\min_j\{\bar{z}_j\}$;
  - **for a minimization problem:** choose the most positive reduced cost, $\max_j\{\bar{z}_j\}$.

in this example, select column of $x_2$, thus $j = 2$

| $\bar{z}$ and basic var | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| $x_3$ | 4 | 6 | 1 | 0 | 0 | 0 | 24 |
| $x_4$ | 2 | 1 | 0 | 1 | 0 | 0 | 6 |
| $x_5$ | 1 | $-1$ | 0 | 0 | 1 | 0 | 1 |
| $x_6$ | 1 | 0 | 0 | 0 | 0 | 1 | 2 |
| $\bar{z}$ | $-4$ | $-5$ | 0 | 0 | 0 | 0 | 0 |

## identify a leaving variable from the basis

- Select, from the column associated with the entering variable $x_j$ (in this example, the column of $x_2$), the variable that leaves the basis by applying the minimum ratio test: $\min_i \left\{ \frac{b_i}{a_{ij}} \ : \ a_{ij} > 0 \right\}$.

| $\bar{z}$ and basic var | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| $x_3$ | 4 | 6 | 1 | 0 | 0 | 0 | 24 |
| $x_4$ | 2 | 1 | 0 | 1 | 0 | 0 | 6 |
| $x_5$ | 1 | −1 | 0 | 0 | 1 | 0 | 1 |
| $x_6$ | 1 | 0 | 0 | 0 | 0 | 1 | 2 |
| $\bar{z}$ | −4 | −5 | 0 | 0 | 0 | 0 | 0 |

in this example $\min\{\frac{24}{6}, \frac{6}{1}\} = \min\{4, 6\} = 4$ associate to the row of $x_3$
thus, basic variable $x_3$ will leave the basis

# The Simplex method: elementary row operations

Use elementary row operations (Gaussian elimination) to update the Simplex table and replace the leaving variable $x_3$ with the entering variable $x_2$.

| $\bar{z}$ and basic var | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| $x_3$ | 4 | 6 | 1 | 0 | 0 | 0 | 24 |
| $x_4$ | 2 | 1 | 0 | 1 | 0 | 0 | 6 |
| $x_5$ | 1 | −1 | 0 | 0 | 1 | 0 | 1 |
| $x_6$ | 1 | 0 | 0 | 0 | 0 | 1 | 2 |
| $\bar{z}$ | −4 | −5 | 0 | 0 | 0 | 0 | 0 |

The circled entry (in the row of $x_3$ and the column of $x_2$) is the *pivot element*.

- Divide the pivot row by the pivot element.
- Use elementary row operations to make all other entries in the pivot column equal to zero.

# The Simplex method: the updated Simplex table

The following table is obtained

| $\bar{z}$ and basic var | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| $x_2$ | $\frac{2}{3}$ | $1$ | $\frac{1}{6}$ | $0$ | $0$ | $0$ | $4$ |
| $x_4$ | $\frac{4}{3}$ | $0$ | $-\frac{1}{6}$ | $1$ | $0$ | $0$ | $2$ |
| $x_5$ | $\frac{5}{3}$ | $0$ | $\frac{1}{6}$ | $0$ | $1$ | $0$ | $5$ |
| $x_6$ | $1$ | $0$ | $0$ | $0$ | $0$ | $1$ | $2$ |
| $\bar{z}$ | $-\frac{2}{3}$ | $0$ | $\frac{5}{6}$ | $0$ | $0$ | $0$ | $20$ |

This table corresponds to the extreme point / basic solution $x = (0, 4, 0, 2, 5, 2)$ with value $z = 20$

Evaluate this solution and repeat!

# The Simplex method: optimal table

$x_1$ enters the base and $x_4$ leaves the base
Obtain the table

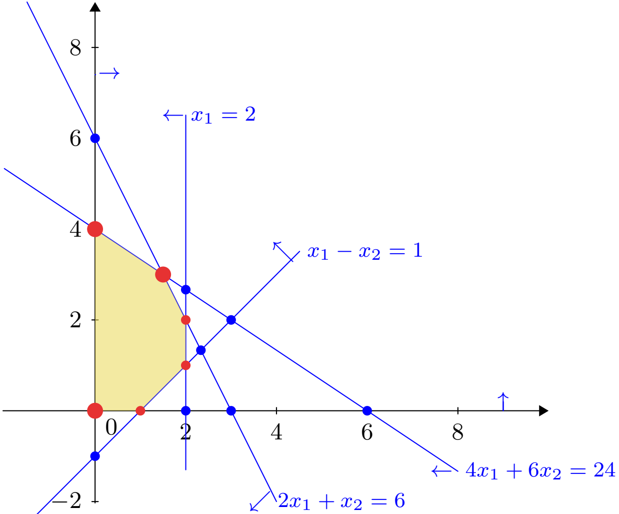
| $\bar{z}$ and basic var | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| $x_2$ | $0$ | $1$ | $\frac{1}{12}$ | $-\frac{1}{2}$ | $0$ | $0$ | $3$ |
| $x_1$ | $1$ | $0$ | $-\frac{1}{8}$ | $\frac{3}{4}$ | $0$ | $0$ | $\frac{3}{2}$ |
| $x_5$ | $0$ | $0$ | $\frac{3}{8}$ | $-\frac{5}{4}$ | $1$ | $0$ | $\frac{5}{2}$ |
| $x_6$ | $0$ | $0$ | $\frac{1}{8}$ | $-\frac{3}{4}$ | $0$ | $1$ | $\frac{1}{2}$ |
| $\bar{z}$ | $0$ | $0$ | $\frac{3}{4}$ | $\frac{1}{2}$ | $0$ | $0$ | $21$ |

This table corresponds to the extreme point / basic solution $x = (\frac{3}{2}, 3, 0, 0, \frac{5}{2}, \frac{1}{2})$ with value $z = 21$

Evaluate this solution! This is the optimal solution.

# Path of the Simplex method

the Simplex method evaluates the extreme points $(0, 0)$; $(0, 4)$; $(1.5, 3)$;



# Example: unique optimal solution

$$\begin{aligned} \max \quad & z = -2x_1 + 4x_2 - 6x_3 \\ \text{s. t.:} \quad & 3x_1 - 2x_2 - 4x_3 \leq 4 \\ & 2x_1 + x_2 + x_3 \leq 10 \\ & x_1 + 3x_2 - 2x_3 \leq 5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

| $x_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $\bar{b}$ |
|---|---|---|---|---|---|---|---|
| $x_4$ | $11/3$ | $0$ | $-16/3$ | $1$ | $0$ | $2/3$ | $22/3$ |
| $x_5$ | $5/3$ | $0$ | $5/3$ | $0$ | $1$ | $-1/3$ | $25/3$ |
| $x_2$ | $1/3$ | $1$ | $-2/3$ | $0$ | $0$ | $1/3$ | $5/3$ |
| $z_j - c_j$ | $10/3$ | $0$ | $10/3$ | $0$ | $0$ | $4/3$ | $20/3$ |

Unique optimal solution!

$x^* = (0, 5/3, 0, 22/3, 25/3, 0),\ z^* = 20/3$

# Example: alternative optimal solutions

Case of convex combination of two extreme points

$$\begin{array}{ll} \max & z = x_1 - 2x_2 + x_3 \\ \text{s. t.:} & x_1 + 2x_2 + x_3 \leq 12 \\ & 2x_1 + x_2 - x_3 \leq 6 \\ & -x_1 + 3x_2 \leq 9 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

after some iterations we obtain the following optimal table

| $x_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $\bar{b}$ |
|---|---|---|---|---|---|---|---|
| $x_3$ | 0 | 1 | 1 | 2/3 | −1/3 | 0 | 6 |
| $x_1$ | 1 | 1 | 0 | 1/3 | 1/3 | 0 | 6 |
| $x_6$ | 0 | 4 | 0 | 1/3 | 1/3 | 1 | 15 |
| $z_j - c_j$ | 0 | 4 | 0 | 1 | 0 | 0 | 12 |

$$x^{1*} = (6, 0, 6, 0, 0, 15),\ z^* = 12$$

however, there is an indication of an alternative optimum

# Example: alternative optimal solutions

one more iteration

| $x_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $\bar{b}$ |
|---|---|---|---|---|---|---|---|
| $x_3$ | 1 | 2 | 1 | 1 | 0 | 0 | 12 |
| $x_5$ | 3 | 3 | 0 | 1 | 1 | 0 | 18 |
| $x_6$ | $-1$ | 3 | 0 | 0 | 0 | 1 | 9 |
| $z_j - c_j$ | 0 | 4 | 0 | 1 | 0 | 0 | 12 |

$$x^{2*} = (0, 0, 12, 0, 18, 9),\ z^* = 12$$

$$x^* = \alpha(6, 0, 6, 0, 0, 15) + (1 - \alpha)(0, 0, 12, 0, 18, 9), \alpha \in [0, 1],\ z^* = 12$$

## Example: alternative optimal solutions

Case of optimal solutions along an extreme direction

$$\begin{array}{ll} \max & z = -4x_1 + 10x_2 \\ \text{s. t.:} & -3x_1 + 2x_2 \leq 3 \\ & -2x_1 + 5x_2 \leq 20 \\ & x_1, x_2 \geq 0 \end{array}$$

after some iterations we obtain the optimal table

| $x_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $\bar{b}$ |
|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | $-4/22$ | $3/11$ | $108/22$ |
| $x_1$ | 1 | 0 | $-5/11$ | $2/11$ | $25/11$ |
| $z_j - c_j$ | 0 | 0 | 0 | 2 | 40 |

$$\overline{x}^* = (25/11, 108/22, 0, 0),\ z^* = 40$$

however, there is an indication of an alternative optimum, but this time it's along the direction

$$d^* = (5/11, 4/22, 1, 0)$$

$$x^* = (25/11, 108/22, 0, 0) + \beta(5/11, 4/22, 1, 0), \beta \geq 0,\ z^* = 40$$

# Example: unbounded solution

$$\begin{array}{ll} \max & z = 2x_1 + 3x_2 \\ \text{s. t.:} & 2x_1 + 2x_2 \geq 6 \\ & -x_1 + x_2 \leq 1 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

after some iterations we obtain the table

| $x_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $\bar{b}$ |
|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 0 | $-1$ | 1 | 2 |
| $x_2$ | 0 | 1 | 0 | 0 | 1 | 3 |
| $x_3$ | 0 | 0 | 1 | $-2$ | 4 | 4 |
| $z_j - c_j$ | 0 | 0 | 0 | $-2$ | 5 | 13 |

problem unbounded along the direction

$$d^* = (1, 0, 2, 1, 0)$$

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# The two-phases method

# Determination of an Initial Basic Feasible Solution

$$
\begin{array}{lll}
\text{max} & z = \sum_{j \in J} c_j x_j & \\
\text{s.t.} & \sum_{j \in J} a_{ij} x_j \leq b_i, & i \in I_1 \\
 & \sum_{j \in J} a_{ij} x_j \geq b_i, & i \in I_2 \\
 & \sum_{j \in J} a_{ij} x_j = b_i, & i \in I_3 \\
 & x_j \geq 0, & j \in \bar{J}
\end{array}
$$

Auxiliary Problem

$$
\begin{aligned}
\text{min} \quad & z^a = \sum_{i \in I_2} a_i + \sum_{i \in I_3} x_{ai} & \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j + y_i = b_i, & i \in I_1 \\
& \sum_{j \in J} a_{ij} x_j - y_i + x_{ai} = b_i, & i \in I_2 \\
& \sum_{j \in J} a_{ij} x_j + x_{ai} = b_i, & i \in I_3 \\
& x_j \geq 0, & j \in \bar{J} \\
& y_i \geq 0, & i \in I_1 \cup I_2 \\
& x_{ai} \geq 0, & i \in I_2 \cup I_3
\end{aligned}
$$

# Determination of an Initial Basic Feasible Solution

The auxiliary problem is always a minimization problem.

If $z^a > 0$, the original problem has no feasible solutions.

If $z^a = 0$, the optimal solution of the auxiliary problem is an initial basic feasible solution of the original problem, and two cases may occur:

1. All artificial variables are nonbasic: we may proceed without any difficulty.
2. Some artificial variables are still in the basis.

## Determination of an Initial Basic Feasible Solution

2. Some artificial variables are still in the basis:

   The first step is to remove these variables from the basis.

   To do so, select as pivot a nonzero element (possibly negative) located at the intersection of:

   - a non-artificial column (corresponding to the entering variable), and
   - a row associated with an artificial variable (the leaving variable).

   If all the elements in the column of the entering non-artificial variable, corresponding to rows of artificial variables, are zero, this means that the corresponding constraint in the original problem is redundant and can therefore be removed.

# Example 1

Determine an Initial Basic Feasible Solution of the LP:

$$
\begin{aligned}
\text{min} \quad & z = x_1 - 2x_2 + 3x_3 \\
\text{s.t.} \quad & \\
& x_1 + x_2 + x_3 \leq 7 \\
& -x_1 + x_2 - x_3 \geq 2 \\
& 3x_1 + 2x_3 = 5 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

# Example 1

Auxiliary Problem

$$
\begin{aligned}
\text{min} \quad & z^a = x_{a2} + x_{a3} \\
\text{s.t.} \quad & \\
& x_1 + x_2 + x_3 + x_4 = 7 \\
& -x_1 + x_2 - x_3 - x_5 + x_{a1} = 2 \\
& 3x_1 + 2x_3 + x_{a2} = 5 \\
& x_1, x_2, x_3 \geq 0 \\
& x_4, x_5, x_{a1}, x_{a2} \geq 0
\end{aligned}
$$

# Example 1

initial table

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|---|
| $x_4$ | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 7 |
| $x_{a1}$ | $-1$ | 1 | $-1$ | 0 | $-1$ | 1 | 0 | 2 |
| $x_{a2}$ | (3) | 0 | 2 | 0 | 0 | 0 | 1 | 5 |
| $z_j - c_j$ | $-1$ | 2 | $-3$ | 0 | 0 | 0 | 0 | 0 |
| $z_j^a - c_j^a$ | 2 | 1 | 1 | 0 | $-1$ | 0 | 0 | 7 |

$$x = (x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 7, 0), \quad z = 0$$

$$x^a = (x_1, x_2, x_3, x_4, x_5, x_{a1}, x_{a2}) = (0, 0, 0, 7, 0, 2, 5), \quad z^a = 7$$

## Example 1

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_{a1}$ | $x_{a2}$ | |
|---|---|---|---|---|---|---|---|---|
| $x_4$ | 0 | 1 | $1/3$ | 1 | 0 | 0 | $-1/3$ | $16/3$ |
| $x_{a1}$ | 0 | 1 | $-1/3$ | 0 | $-1$ | 1 | $1/3$ | $11/3$ |
| $x_1$ | 1 | 0 | $2/3$ | 0 | 0 | 0 | $1/3$ | $5/3$ |
| $z_j - c_j$ | 0 | 2 | $-7/3$ | 0 | 0 | 0 | $1/3$ | $5/3$ |
| $z_j^a - c_j^a$ | 0 | 1 | $-1/3$ | 0 | $-1$ | 0 | $-2/3$ | $11/3$ |

$$x = (x_1, x_2, x_3, x_4, x_5) = (\tfrac{5}{3}, 0, 0, \tfrac{16}{3}, 0), \quad z = \tfrac{5}{3}$$

$$x^a = (x_1, x_2, x_3, x_4, x_5, x_{a1}, x_{a2}) = (\tfrac{5}{3}, 0, 0, \tfrac{16}{3}, 0, \tfrac{11}{3}, 0), \quad z^a = \tfrac{11}{3}$$

# Example 1

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_{a1}$ | $x_{a2}$ | |
|---|---|---|---|---|---|---|---|---|
| $x_4$ | 0 | 0 | $2/3$ | 1 | 1 | $-1$ | $-2/3$ | $5/3$ |
| $x_2$ | 0 | 1 | $-1/3$ | 0 | $-1$ | 1 | $1/3$ | $11/3$ |
| $x_1$ | 1 | 0 | $2/3$ | 0 | 0 | 0 | $1/3$ | $5/3$ |
| $z_j - c_j$ | 0 | 0 | $-5/3$ | 0 | 2 | $-2$ | $-1/3$ | $-17/3$ |
| $z_j^a - c_j^a$ | 0 | 0 | 0 | 0 | 0 | $-1$ | $-1$ | 0 |

$x = (x_1, x_2, x_3, x_4, x_5) = (\frac{5}{3}, \frac{11}{3}, 0, \frac{5}{3}, 0), \quad z = -\frac{17}{3}$

$x^a = (x_1, x_2, x_3, x_4, x_5, x_{a1}, x_{a2}) = (\frac{5}{3}, \frac{11}{3}, 0, \frac{5}{3}, 0, 0, 0), \quad z^a = 0$

$z^a = 0$ and no artificial variables are in the base — end of Phase I

the initial basic feasible solution is $x = (\frac{5}{3}, \frac{11}{3}, 0, \frac{5}{3}, 0), \quad z = -\frac{17}{3}$

# Example 1

continue to phase II to obtain the optimal solution

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_{a1}$ | $x_{a2}$ | |
|---|---|---|---|---|---|---|---|---|
| $x_5$ | 0 | 0 | 2/3 | 1 | 1 | | | 5/3 |
| $x_2$ | 0 | 1 | 1/3 | 1 | 0 | | | 16/3 |
| $x_1$ | 1 | 0 | 2/3 | 0 | 0 | | | 5/3 |
| $z_j - c_j$ | 0 | 0 | $-3$ | $-2$ | 0 | | | $-9$ |

$$x = (x_1, x_2, x_3, x_4, x_5) = (\tfrac{5}{3}, \tfrac{16}{3}, 0, 0, \tfrac{5}{3}), \quad z = -9$$

optimal solution