Bachelor's Degrees in Economics and Management
Mathematics II (1st semester, 2026/2027)

EXERCISES 1*

Further Topics in Linear Algebra

Eigenvalues and eigenvectors

1. Determine whether 5 is an eigenvalue of
$$A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}.$$

2. For each of the following matrices, determine whether the given λ is an eigenvalue, justifying your answer. If so, determine a corresponding eigenvector.

a) $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}; \quad \lambda = 2$

b) $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}; \quad \lambda = -2$

c) $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}; \quad \lambda = 4$

d) $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}; \quad \lambda = 3$

3. Determine whether the given vector is an eigenvector of the corresponding matrix. If so, determine the corresponding eigenvalue.

a) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}; \quad \begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$

b) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}; \quad \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

c) $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}; \quad \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}; \quad \begin{bmatrix} 2 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$

4. Determine a basis† for each eigenspace of the following matrices:

a) $A = \begin{bmatrix} 9 & 0 \\ 2 & 3 \end{bmatrix}$

b) $A = \begin{bmatrix} 14 & -4 \\ 16 & -2 \end{bmatrix}$

c) $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

e) $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

f) $A = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 3 & 3 \\ 6 & -6 & 2 \end{bmatrix}$

g) $A = \begin{bmatrix} 8 & 3 & -4 \\ -1 & 4 & 4 \\ 2 & 6 & -1 \end{bmatrix}$

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†A *basis* of a vector space V is any subset $\mathcal{B}$ of V that satisfies the following two conditions: $\mathcal{B}$ spans V and $\mathcal{B}$ is linearly independent. In this case, we want to find a set of eigenvectors associated with the eigenvalues of the matrix.

h) $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

5. Determine the eigenvalues of $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$.

6. If $\mathbf{x}$ is an eigenvector of A associated with the eigenvalue λ, determine $A^3\mathbf{x}$.

7. For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, determine, without performing any calculations, an eigenvalue of A. Justify your answer.

8. Determine, without performing any calculations, an eigenvalue and two linearly independent eigenvectors of

$$A = \begin{bmatrix} 4 & 4 & -4 \\ 4 & 4 & -4 \\ 4 & 4 & -4 \end{bmatrix}.$$

Justify your answer.

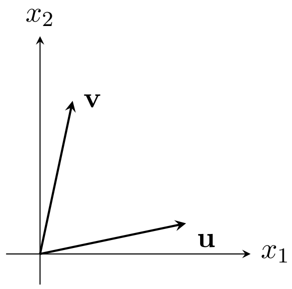
9. State, with justification, whether the following statements are true or false:
   a) If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue of A.
   b) If $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, then $\mathbf{x}$ is an eigenvector of A.
   c) A matrix A is invertible if and only if 0 is an eigenvalue of A.
   d) A number c is an eigenvalue of A if and only if the equation $(A - cI)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.
   e) To determine the eigenvalues of A, it is enough to reduce A to row echelon form.
   f) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.
   g) The eigenvalues of a matrix are found on its main diagonal.
   h) If $\mathbf{v}$ is an eigenvector associated with the eigenvalue 2, then $2\mathbf{v}$ is an eigenvector associated with the eigenvalue 4.

10. Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}.

11. Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

12. Show that λ is an eigenvalue of A if and only if it is an eigenvalue of A^T.

13. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors shown in the figure. Suppose they are eigenvectors of a 2×2 matrix A, associated with the eigenvalues 2 and 3, respectively. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ and let $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Carefully represent, on the same coordinate system, the vectors $T(\mathbf{u})$, $T(\mathbf{v})$ and $T(\mathbf{w})$.



14. Repeat Exercise **??**, now assuming that $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A associated with the eigenvalues -1 and 3, respectively.

15. The algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the corresponding eigenspace. Determine h so that the eigenspace associated with $\lambda = 6$ is two-dimensional, for

$$A = \begin{bmatrix} 6 & 3 & 9 & -5 \\ 0 & 9 & h & 2 \\ 0 & 0 & 6 & 8 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

16. Let A be an $n \times n$ matrix with n real eigenvalues $\lambda_1, \ldots, \lambda_n$, repeated according to their respective multiplicities, such that

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda).$$

Explain why $\det A$ is equal to the product of the n eigenvalues of A. The result remains valid for any square matrix when complex eigenvalues are considered.

17. (Computer) Consider

$$A = \begin{bmatrix} -6 & 28 & 21 \\ 4 & -15 & -12 \\ -8 & a & 25 \end{bmatrix}.$$

For each $a \in \{32, 31.9, 31.8, 32.1, 32.2\}$, determine the characteristic polynomial of A and its eigenvalues. In each case, plot the characteristic polynomial

$$p(t) = \det(A - tI), \qquad 0 \le t \le 3.$$

Whenever possible, plot all the graphs on the same coordinate system and describe how they reveal the changes in the eigenvalues as a varies.

Symmetric matrices and quadratic forms

18. Show that if A is a symmetric matrix, then A^2 is also symmetric.

19. Determine whether the following matrices are symmetric:

a) $\begin{bmatrix} 4 & 3 \\ 3 & -8 \end{bmatrix}$

b) $\begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 5 \\ 3 & 7 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 1 & -6 \\ 5 & 4 & 1 \end{bmatrix}$

e) $\begin{bmatrix} -2 & 4 & 5 \\ 4 & -2 & 3 \\ 5 & 3 & -2 \end{bmatrix}$

f) $\begin{bmatrix} 2 & 1 & 1 & 2 \\ 3 & 3 & 3 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix}$

20. Describe a symmetric positive semidefinite matrix in terms of its eigenvalues.

21. Consider

$$A = \begin{bmatrix} 3 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{bmatrix}.$$

Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$ for each of the following vectors:

a) $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; b) $\mathbf{x} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$; c) $\mathbf{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

22. Consider

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & 3 & 0 \end{bmatrix}.$$

Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$ for each of the following vectors:

a) $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; b) $\mathbf{x} = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix}$; c) $\mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$.

23. Determine the matrix associated with each of the following quadratic forms, assuming that $\mathbf{x} \in \mathbb{R}^2$:
    a) $4x_1^2 - 6x_1x_2 + 5x_2^2$;
    b) $5x_1^2 + 4x_1x_2$.

24. Determine the matrix associated with each of the following quadratic forms, assuming that $\mathbf{x} \in \mathbb{R}^3$:
    a) $5x_1^2 + 3x_2^2 - 7x_3^2 - 4x_1x_2 + 6x_1x_3 - 2x_2x_3$;
    b) $8x_1x_2 + 10x_1x_3 - 6x_2x_3$.

25. Classify the following quadratic forms:

    a) $6x_1^2 - 4x_1x_2 + 3x_2^2$
    b) $3x_1^2 + 8x_1x_2 - 3x_2^2$
    c) $4x_1^2 - 8x_1x_2 - 2x_2^2$
    d) $-2x_1^2 - 4x_1x_2 - 2x_2^2$
    e) $x_1^2 - 4x_1x_2 + 4x_2^2$
    f) $5x_1^2 + 12x_1x_2$
    g) $-3x_1^2 - 7x_2^2 - 10x_3^2 - 10x_4^2 + 4x_1x_2 + 4x_1x_3 + 4x_1x_4 + 6x_3x_4$
    h) $4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_4^2 + 8x_1x_2 + 8x_3x_4 - 6x_1x_4 + 6x_2x_3$

26. Determine the maximum possible value of the quadratic form

$$4x_1^2 + 9x_2^2$$

subject to the condition $\mathbf{x}^T\mathbf{x} = 1$, that is, $x_1^2 + x_2^2 = 1$.

27. State, with justification, whether the following statement is true or false: if A is symmetric and the quadratic form $\mathbf{x}^T A\mathbf{x}$ takes only negative values for $\mathbf{x} \neq \mathbf{0}$, then all eigenvalues of A are positive.

28. Let

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}, \qquad \det A \neq 0,$$

and consider the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. Verify that:

a) Q is positive definite if $\det A > 0$ and $a > 0$;

b) Q is negative definite if $\det A > 0$ and $a < 0$;

c) Q is indefinite if $\det A < 0$.