Bachelor's in Economics, in Finance and in Management
Mathematics II
2nd Semester - 2025/2026

I - Linear Algebra

1. Complements of Linear Algebra

Exercise 1.

Determine which of the following vectors are eigenvectors of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and compute the associated eigenvalue:

(a) $u = (0, 0, 5)$, (b) $u = (1, 2, 3)$, (c) $u = (0, 0, 4)$, (d) $u = (2, -2, 0)$.

Solution

(a) $Au = (0, 0, 10) = 2u$, hence u is an eigenvector with eigenvalue $\lambda = 2$.

(b) $Au = (3, 3, 6)$ is not a scalar multiple of u, hence u is not an eigenvector.

(c) $Au = (0, 0, 8) = 2u$, hence u is an eigenvector with eigenvalue $\lambda = 2$.

(d) $Au = (0, 0, 0) = 0u$, hence u is an eigenvector with eigenvalue $\lambda = 0$.

Exercise 2.

Determine the real eigenvalues of each matrix and the corresponding eigenvectors, together with the algebraic multiplicity (a.m.) and geometric multiplicity (g.m.):

(a) $\begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 4 \\ -2 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & -1 & -2 \\ 0 & 3 & 0 \\ -2 & 5 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(i) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Solution

(a) $\lambda_1 = -5$ (a.m.=1), eigenvectors $u = (c, c)$, $c \neq 0$, g.m.=1; $\lambda_2 = -1$ (a.m.=1), eigenvectors $u = (\frac{7}{3}c, c)$, $c \neq 0$, g.m.=1.

(b) The characteristic polynomial has no real roots ($\lambda = 4 \pm 2i$).

(c) $\lambda_1 = 1 + \sqrt{2}$, $\lambda_2 = 1 - \sqrt{2}$, both with a.m.=g.m.=1, eigenvectors $u = ((1 \pm \sqrt{2})c, c)$.

(d) $\lambda = 0$, a.m.=2, eigenvectors $u = (c, 0)$, g.m.=1.

(e) Eigenvalues $2, 3, 4$, each with a.m.=g.m.=1 and canonical basis eigenvectors.

(f) $\lambda_1 = 3$ (a.m.=2), eigenvectors $u = (-c, 0, c)$, g.m.=1; $\lambda_2 = -1$ (a.m.=1), eigenvectors $u = (c, 0, c)$.

(g) $\lambda_1 = 0$ (a.m.=2), eigenvectors $u = (-c_1 - c_2, c_1, c_2)$, g.m.=2; $\lambda_2 = 3$ (a.m.=1), eigenvectors $u = (c, c, c)$.

(h) $\lambda = 1$, a.m.=3, eigenvectors $u = (c, 0, 0)$, g.m.=1.

(i) $\lambda_1 = 2$ (a.m.=2), eigenvectors $u = (c_1, c_1, c_2)$, g.m.=2; $\lambda_2 = 0$ (a.m.=1), eigenvectors $u = (-c, c, 0)$.

Exercise 3.
Prove that:

(a) Every eigenvalue of A is also an eigenvalue of A^T.

(b) If λ is an eigenvalue of A and $\det A \neq 0$, then $1/\lambda$ is an eigenvalue of A^{-1}.

(c) If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k, $k \in \mathbb{N}$.

Solution

Each statement follows directly from the invariance of the characteristic polynomial under transposition, and from applying the eigenvalue equation $Ax = \lambda x$ to A^{-1} and A^k, respectively.

Exercise 4.
Let $A = \begin{pmatrix} 1 & 4 \\ 6 & -1 \end{pmatrix}$.

(a) Determine the eigenvalues and eigenvectors of A.

(b) Determine the eigenvalues and eigenvectors of A^{100}.

Solution

(a) Eigenvalues $\lambda_1 = -5$, eigenvectors $u = (-\frac{2}{3}c, c)$; $\lambda_2 = 5$, eigenvectors $u = (c, c)$.

(b) Eigenvalues $(-5)^{100}$ and 5^{100}, with the same eigenvectors.

Exercise 5.
Consider

$$A = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ 0 & 0 & b \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

(a) Write the characteristic polynomial.

(b) Determine the eigenvalues and eigenvectors of A.

(c) Compute $\mathbf{x}^T A\mathbf{x}$.

(d) Assume $a, b \geq 0$. Without performing calculations show that there exists $\mathbf{x} \neq 0$ such that $\mathbf{x}^T A\mathbf{x} = 0$.

(e) Classify the quadratic form $\mathbf{x}^T A\mathbf{x}$ for all possible values of a, b.

Solution

(a) $p(\lambda) = (b-\lambda)(-\lambda)(2a-\lambda)$.

(b) Eigenvalues $b, 0, 2a$ with corresponding eigenspaces obtained by direct computation.

(c) $\mathbf{x}^T A\mathbf{x} = ax^2 + 2axy + ay^2 + bz^2$.

(d) Since the form is positive semidefinite, there exists a non-zero vector in the kernel.

(e) PSD if $a \geq 0$ and $b \geq 0$; NSD if $a \leq 0$ and $b \leq 0$; indefinite otherwise.

Exercise 6.

Classify the following quadratic forms:

(a) $q(x,y) = x^2 + 2xy + y^2$

(b) $q(x,y) = x^2 - 2xy + y^2$

(c) $q(x,y) = x^2 - y^2$

(d) $q(x,y,z) = x^2 + 4xy - 2xz + 7y^2 - 3z^2$

(e) $q(x,y,z) = x^2 - 4xy + 4xz - z^2$

(f) $q(x,y) = 6x^2 + 4xy + 3y^2$

(g) $q(x,y) = x^2 + 4xy + y^2$

(h) $q(x,y) = 2x^2 + 6xy + 4y^2$

(i) $q(x,y,z) = 3y^2 + 4xz$

(j) $q(x,y) = x^2 + 4xy + ay^2, \quad a \in \mathbb{R}$.

Solution

(a) PSD; (b) PSD; (c) IND; (d) IND; (e) IND; (f) PD; (g) IND; (h) IND; (i) IND; (j) IND if $a < 4$, PSD if $a = 4$, PD if $a > 4$.

Exercise 7.

Classify the following symmetric matrices (with respect to being PD, PSD, ND, NSD, IND):

(a) $\begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} -5 & 1 \\ 1 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} -5 & 1 \\ 1 & -5 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 \\ 2 & a \end{pmatrix}, \quad a \in \mathbb{R}$

(e) $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 7 & 0 \\ -1 & 0 & -3 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 7 & -5 \\ -1 & -5 & 4 \end{pmatrix}$

(g) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$

(h) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & a & 2 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}, \quad a \in \mathbb{R}$

(i) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -3 & 0 \\ 1 & 0 & -2 \end{pmatrix}$

(j) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Solution

(a) PD; (b) Indefinite; (c) ND; (d) Indefinite if $a < 4$, PSD if $a = 4$, ND if $a > 4$; (e) IND; (f) PSD; (g) IND; (h) IND ($|a| > \sqrt{2}$), PD ($|a| < \sqrt{2}$), PSD ($|a| = \sqrt{2}$); (i) ND; (j) IND.

2. Functions of Several Variables and Topology in $\mathbb{R}^n$

Exercise 8.

Determine the domain of the following functions and represent it graphically.

(a) $f(x,y) = \frac{\sqrt{9-x^2-y^2}}{1-\ln x}$

(b) $f(x,y) = \frac{\sqrt{e-e^x}}{\ln(4-x^2-y^2)}$

(c) $f(x,y) = \ln(x-y)^2$

(d) $f(x,y) = \frac{\sqrt[3]{x+y}}{\ln x^2 - \ln(3-x)^2}$

(e) $f(x,y) = \ln(x-y)\sqrt{(y-x)(x^2+y^2-1)}$

Solution

(a) $\{(x,y) \in \mathbb{R}^2 : x > 0 \wedge x \neq e \wedge x^2+y^2 \leq 9\}$

(b) $\{(x,y) \in \mathbb{R}^2 : x \leq 1 \wedge x^2+y^2 < 4 \wedge x^2+y^2 \neq 3\}$

(c) $\{(x,y) \in \mathbb{R}^2 : x \neq y\}$

(d) $\{(x,y) \in \mathbb{R}^2 : x \neq 0 \wedge x \neq 3 \wedge x \neq 3/2\}$

(e) $\{(x,y) \in \mathbb{R}^2 : x-y > 0 \wedge x^2+y^2-1 \leq 0\}$

Exercise 9.

Determine the interior, exterior and boundary of the following subsets of $\mathbb{R}^2$. Classify them with respect to being open, closed and bounded.

$$\mathrm{A} = \{(x,y) \in \mathbb{R}^2 : (x-1)^2+(y+2)^2 < 4\}$$

$$\mathrm{B} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \geq 9\}$$

$$\mathrm{C} = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9}+\frac{y^2}{16} \leq 1\right\}$$

$$\mathrm{D} = \left\{(x,y) \in \mathbb{R}^2 : \frac{(x-2)^2}{4}+\frac{(y+1)^2}{9} = 1\right\}$$

Solution

$\mathrm{Int}(A) = A,\ \mathrm{Bdy}(A) = \{(x,y) \in \mathbb{R}^2 : (x-1)^2+(y+2)^2 = 4\}$

$\mathrm{Ext}(A) = \{(x,y) \in \mathbb{R}^2 : (x-1)^2+(y+2)^2 > 4\}$, A is open and bounded.

$\mathrm{Int}(B) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 9\}$, $\mathrm{Bdy}(B) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 9\}$

$\mathrm{Ext}(B) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 9\}$, B is closed and not bounded.

$\mathrm{Int}(C) = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9}+\frac{y^2}{16} < 1\right\}$, $\mathrm{Bdy}(C) = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9}+\frac{y^2}{16} = 1\right\}$

$\mathrm{Ext}(C) = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9}+\frac{y^2}{16} > 1\right\}$, C is closed and bounded.

$\mathrm{Int}(D) = \emptyset,\ \mathrm{Bdy}(D) = D,\ \mathrm{Ext}(D) = \left\{(x,y) \in \mathbb{R}^2 : \frac{(x-2)^2}{4}+\frac{(y+1)^2}{9} \neq 1\right\}$, D is closed and bounded.

Exercise 10.

Represent graphically and analytically the domain D_f, as well as $\text{Int}(D_f)$, $\text{Bdy}(D_f)$ and D'_f. State in each case if D_f is open, closed, bounded and compact.

(a) $f(x,y) = \frac{|x|-4}{\ln(4-x^2-y^2)}$

(b) $f(x,y) = \sqrt{x(1-x)} + \frac{\ln(x^2-y)}{\sqrt{y+x}}$

(c) $f(x,y) = \sqrt{y+x-1} \cdot \ln(4-(x+1)^2-(y-1)^2)$

(d) $f(x,y) = \sqrt{x-|y|} + \sqrt{2-y^2-x}$

(e) $f(x,y) = x\sqrt{y^2-4} + \sqrt[4]{16-x^2-y^2}$

Solution

(a) $D_f = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 4 \wedge x^2+y^2 \neq 3\}$, $\text{Int}(D_f) = D_f$, $\text{Bdy}(D_f) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 4 \vee x^2+y^2 = 3\}$, $D'_f = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 4\}$. D_f is open, not closed, bounded, not compact.

(b) $D_f = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \wedge -x < y < x^2\}$, $\text{Int}(D_f) = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1 \wedge -x < y < x^2\}$, $\text{Bdy}(D_f) = \{(x,y) \in \mathbb{R}^2 : (y = x^2 \ \wedge \ 0 \leq x \leq 1) \vee (y = -x \ \wedge \ 0 \leq x \leq 1) \vee (x = 1 \ \wedge \ -x \leq y \leq x^2)\}$, $D'_f = \{(x,y) \in \mathbb{R}^2 : (0 \leq x \leq 1) \wedge (-x \leq y \leq x^2)\}$. D_f is not open nor closed, bounded, not compact.

(c) $D_f = \{(x,y) \in \mathbb{R}^2 : y \geq 1-x \wedge (x+1)^2+(y-1)^2 < 4\}$, $\text{Int}(D_f) = \{(x,y) \in \mathbb{R}^2 : y > 1-x \wedge (x+1)^2+(y-1)^2 < 4\}$, $\text{Bdy}(D_f) = \{(x,y) \in \mathbb{R}^2 : y = 1-x \wedge (x+1)^2+(y-1)^2 \leq 4\} \cup \{(x,y) \in \mathbb{R}^2 : y \geq 1-x \wedge (x+1)^2+(y-1)^2 = 4\}$, $D'_f = \{(x,y) \in \mathbb{R}^2 : y \geq 1-x \wedge (x+1)^2+(y-1)^2 \leq 4\}$. D_f is not open nor closed, bounded, not compact.

(d) $D_f = \{(x,y) \in \mathbb{R}^2 : x-|y| \geq 0 \wedge 2-y^2-x \geq 0\}$, $\text{Int}(D_f) = \{(x,y) \in \mathbb{R}^2 : x-|y| > 0 \wedge 2-y^2-x > 0\}$, $\text{Bdy}(D_f) = \{(x,y) \in \mathbb{R}^2 : x-|y| = 0 \wedge 2-y^2-x \geq 0\} \cup \{(x,y) \in \mathbb{R}^2 : x-|y| \geq 0 \wedge 2-y^2-x = 0\}$, $D'_f = D_f$. D_f is not open, is closed, bounded, compact.

(e) $D_f = \{(x,y) \in \mathbb{R}^2 : (y \geq 2 \vee y \leq -2) \wedge x^2+y^2 \leq 16\}$, $\text{Int}(D_f) = \{(x,y) \in \mathbb{R}^2 : (y > 2 \vee y < -2) \wedge x^2+y^2 < 16\}$, $\text{Bdy}(D_f) = \{(x,y) \in \mathbb{R}^2 : (y = 2 \vee y = -2) \wedge x^2+y^2 \leq 16\} \cup \{(x,y) \in \mathbb{R}^2 : (y \geq 2 \vee y \leq -2) \wedge x^2+y^2 = 16\}$, $D'_f = D_f$. D_f is not open, is closed, bounded, compact.

Exercise 11.

Determine the domain D_f of the function defined by

$$f(x,y) = \frac{\ln(x^2+y^2-4)}{|x|-4}.$$

Represent D_f graphically and show that it is open and unbounded. Check if $\text{Ext}(D_f)$ is also open and unbounded.

Solution

$D_f = \{(x,y) \in \mathbb{R}^2 : x^2+y^2-4 > 0 \wedge |x| \neq 4\}$. $\text{Ext}(D_f) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 4\}$ is open and bounded.

Exercise 12.

Determine the interior, the boundary and the limit points of the following subsets of $\mathbb{R}^2$.

A $= \{(0,0),(0,1),(1,0),(1,1)\}$

B $= \left\{(x,y) \in \mathbb{R}^2 : (x,y) = \left(\frac{1}{n},\frac{1}{n}\right),\ n \in \mathbb{N}\right\}$

C $= \left\{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y = (-1)^n \frac{1}{n},\ n \in \mathbb{N}\right\}$

Solution

$$\mathrm{Int}(A) = \emptyset,\ \mathrm{Bdy}(A) = A,\ A' = \emptyset.$$

$$\mathrm{Int}(B) = \emptyset,\ \mathrm{Bdy}(B) = B \cup \{(0,0)\},\ B' = \{(0,0)\}.$$

$$\mathrm{Int}(C) = \emptyset,\ \mathrm{Bdy}(C) = C \cup \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y = 0\},\ C' = \mathrm{Bdy}(C).$$

3. Limits and Continuity

Exercise 13.

Compute the limits of the following sequences, or show that they do not exist.

(a) $\lim \left(\left(\frac{2n^2+3}{1+2n^2}\right)^{n^2},\ \ln \left(\frac{2n}{2n+1}\right)^{n+\frac{1}{2}} \right)$

(b) $\lim \left(n^3 + n - n^2 - 1,\ \sqrt{n} \cdot \frac{\sqrt{n}+3}{\left(\sqrt{n}+1\right)^2} \right)$

(c) $\lim \left(n\left(e^{\frac{1}{n}} - 1\right),\ \sin \frac{n\pi}{2} \right)$

Solution

(a) $\left(e, -\frac{1}{2}\right)$ (b) $(+\infty, 1)$ (c) does not exist.

Exercise 14.

Compute the following limits:

(a) $\lim \overline{x}_n$ where $\overline{x}_n = \left[\frac{n}{2n+1},\ \left(1+\frac{2}{n}\right)^n,\ \ln\left(1+\frac{1}{n}\right)^n\right]$

(b) $\lim \overline{y}_n$ where $\overline{x}_n = \left[\sqrt{n} - \sqrt{n-1},\ \left(\sqrt[n]{e} - 1\right)n,\ n \ln \frac{n+2}{n},\ \left(1 - \frac{n^2}{n^2+1}\right)^{\frac{1}{3}} \left(\frac{n^2+1}{2n^2}\right)^{\frac{1}{3}}\right]$

Solution

(a) $\left(\frac{1}{2}, e^2, 1\right)$ (b) $(0, 1, 2, 0)$

Exercise 15.

For each of the following functions, investigate the existence of the limit at the point $(0,0)$:

(a) $f(x,y) = \frac{x^2-y^2}{x(x+y)}$

(b) $f(x,y) = \frac{x^3-y^3}{x^2+y^2}$

(c) $f(x,y) = \frac{x^2+y}{\sqrt{x^2+y^2}}$

(d) $f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$

(e) $f(x,y) = \frac{xy}{x^2+y^2}$

(f) $f(x,y) = \frac{x^2y}{x^4+y^2}$

(g) $f(x,y) = \frac{x^2(x+y)}{x^2+y^2}$

(h) $f(x,y) = \frac{x^2-y^2+2x^3}{x^2+y^2}$

(i) $f(x,y) = \begin{cases} \dfrac{y}{x}\sqrt{x^2+y^2}, & x > 0 \wedge y > 0 \\ 0, & x < 0 \vee y < 0 \end{cases}$

Solution

(a) does not exist
(b) 0
(c) does not exist
(d) 0
(e) does not exist
(f) does not exist
(g) 0
(h) does not exist
(i) does not exist.

Exercise 16.

Determine, if possible, a continuous extension of each function in the previous exercise to the point $(0,0)$.

Solution

When the limit does not exist it is not possible to define such an extension. In the other cases simply define

$$f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y).$$

Exercise 17.

Investigate the existence of the following limits:

(a) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$

(b) $\displaystyle\lim_{(x,y)\to(1,-3)} \frac{\sqrt{(x-1)(y+3)}+\sin{(x-1)(y+3)}}{\sqrt{(x-1)(y+3)}}$

(c) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{3(x-2)^2(y-1)}{(x-2)^2+(y-1)^2}$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x}$

(e) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{x^2-y^2-1}{x-1}$

(f) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y+x^2+y^2}{x^2+y^2}$

(g) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x^2\sqrt{|y-1|}}{x^2+(y-1)^2}$

(h) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$

Solution

(a) does not exist
(c) 0
(e) does not exist
(g) 0
(b) 1
(d) does not exist
(f) 1
(h) 0

Exercise 18.

Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by $f(x,y)=\dfrac{x^3-y^3}{x-y}$.

(a) Determine the domain D_f.

(b) Show that $x^3-y^3=(x-y)p(x,y)$, where $p(x,y)$ is a polynomial.

(c) Can f be extended by continuity to the line $y=x$?

Solution

(a) $D_f=\mathbb{R}^2\setminus\{(x,x):x\in\mathbb{R}\}$.
(b) $x^3-y^3=(x-y)(x^2+xy+y^2)$.
(c) Yes, just define $f(x,x)=3x^2$.

Exercise 19.

Consider the function $f(x,y)=\dfrac{x^2y}{x^2-y^2}$.

(a) Compute $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=x+x^2}} f(x,y)$. What can you conclude about $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$?

(b) Compute $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y)$, $|m|\neq 1$. What can you conclude about $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$?

Solution

(a) $-\dfrac{1}{2}$. If $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exists, it must be $-\frac{1}{2}$.

(b) 0. The limit does not exist, since $0\neq-\frac{1}{2}$.

4. Differential Calculus in $\mathbb{R}^n$

Exercise 20.

Determine the first order partial derivatives (in the largest possible domain) of the following functions:

(a) $f(x,y,z) = 3xy + x^2 - zy + z^2$;

(b) $f(x,y) = \begin{cases} x^2 - yx, & y \neq x, \\ x, & y = x. \end{cases}$

Solution

(a) $f'_x = 3y + 2x$; $f'_y = 3x - z$; $f'_z = -y + 2z$; $D_{f'_x} = D_{f'_y} = D_{f'_z} = \mathbb{R}^3$.

(b) $f'_x = 2x - y$ if $x \neq y$ and $f'_x = 0$ if $x = y = 0$; $f'_y = -x$ if $x \neq y$ and $f'_y = 0$ if $x = y = 0$; $D_{f'_x} = D_{f'_y} = \mathbb{R}^2 \setminus \{(a,a) : a \neq 0\}$.

Exercise 21.

Show that $f(x,y) = \dfrac{x-y+1}{x+y}$ is a solution of the equation

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = \frac{2}{x+y}$$

in any of the sets defined by $x + y > 0$ or $x + y < 0$.

Exercise 22.

Consider the function $f(x,y) = \begin{cases} \dfrac{e^{x-y} - (x-y+1)}{x-y}, & x \neq y, \\ 0, & x = y. \end{cases}$

(a) Discuss the continuity of $f(x,y)$ at $(1,1)$.

(b) Check that $f'_x(a,a) + f'_y(a,a) = 0$, $\forall a \in \mathbb{R}$.

Solution

(a) f is continuous at $(1,1)$.

Exercise 23.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0, \\ 0, & \text{otherwise.} \end{cases}$ Compute the directional derivatives at $(0,0)$, whenever they exist.

Solution

$\partial_{\overrightarrow{v}} f(0,0)$ exists for $\overrightarrow{v} = (\alpha, \alpha)$ and $\overrightarrow{v} = (\alpha, -\alpha)$, $\alpha \in \mathbb{R} \setminus \{0\}$. In that case, $\partial_{\overrightarrow{v}} f(0,0) = 0$.

Exercise 24.

Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

(a) Show that f admits a directional derivative at $(0,0)$ along any direction and compute it.

(b) Show that f is not continuous at $(0,0)$.

(c) Without performing any calculations, state the value of $\dfrac{\partial f}{\partial x}(0,0)$ and of $\dfrac{\partial f}{\partial y}(0,0)$.

Solution

(a) $\partial_{(\alpha,\beta)}f(0,0) = \begin{cases} \dfrac{\beta^2}{\alpha}, & \alpha \neq 0, \\ 0, & \alpha = 0, \end{cases} \quad (\alpha,\beta) \in \mathbb{R}^2 \setminus \{(0,0)\}.$

(b) $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = 0.$

Exercise 25.

Study the differentiability of the following functions at the proposed points and obtain the expression of the first order differentials (in case they are differentiable):

(a) $f(x,y) = x^2 + y^2$, at $(0,0)$,

(b) $f(x,y) = \begin{cases} x+y, & x \neq y, \\ x+1, & x = y, \end{cases}$ at $(1,1)$,

(c) $f(x,y) = \begin{cases} xy - 2y + 3x, & x \neq y, \\ x^2y^2 + 3x - 2y, & x = y, \end{cases}$ at $(0,0)$,

(d) $y = (x^2+1, x)$, at $x = 1$.

Solution

(a) f differentiable at $(0,0)$; $Df(0,0)(\mathbf{h}) = 0$,

(b) f not differentiable at $(1,1)$,

(c) f differentiable at $(0,0)$; $Df(0,0)(\mathbf{h}) = 3h_1 - 2h_2$,

(d) y differentiable at $x = 1$; $Df(1)(\mathbf{h}) = (2h, h)$.

Exercise 26.

Write down the expressions of the first order differentials of each given function, at the proposed points (admit differentiability):

(a) $f(x,y) = y^x$, at a generic point (a,b), with $b > 0$;

(b) $f(x_1,x_2,x_3) = \dfrac{x_1 - x_2 + x_3}{\sqrt{x_3 - 1}}$, at $(1,-3,2)$.

Solution

(a) $Df(a,b)(\mathbf{h}) = b^a \log b \cdot h_1 + ab^{a-1} \cdot h_2$;

(b) $Df(1,-3,2)(\mathbf{h}) = h_1 - h_2 - 2h_3$.

Exercise 27.

Show that the following functions are continuous but not differentiable at the given points:

(a) $f(x,y) = \begin{cases} \dfrac{-3x(y-2)^2 + x^3}{x^2 + (y-2)^2}, & \text{if } (x,y) \neq (0,2), \\ 0, & \text{if } (x,y) = (0,2), \end{cases}$

(b) $g(x,y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}}, & \text{if } x^2 + y^2 \neq 0, \\ 0, & \text{if } x = y = 0, \end{cases}$

(c) $h(x,y) = \sqrt{|x|}\cos y$, at $(0,0)$.

Exercise 28.

Use the chain rule to compute:

(a) $\dfrac{df}{dt}$, where $f = x^2y^3$, $x = te^t$ and $y = t^2+1$;

(b) $\dfrac{df}{dt}$, where $f = u^2+v^3$, $u = \dfrac{x}{y}$, $v = (x+2y)^3$, $x = \dfrac{1}{t}$, $y = \tan t$;

(c) $\dfrac{dz}{dt}$, where $z = \dfrac{2xy}{x^2+y^2}$, $x = \cos t$, $y = \sin t$;

(d) $\nabla f(1,1)$, where $f(x,y) = \sin(2u - v^3 + w)$, $u = e^{x^2-y}$, $v = xy^2$, $w = x^3y^2$;

(e) $\dfrac{\partial f}{\partial y}(0,1,1)$, where $f(x,y,z) = (u^2-3v)^5$, $u = e^{\frac{xy}{z}}$, $v = \ln(y^2z^3)$;

(f) $\nabla f(1,2,3)$, where $f(x,y,z) = g(u,v,w)$, $u = 5x+3z$, $v = 8x+2y$, $w = -y+z$ and $\nabla g(14,12,1) = (4,5,6)$.

Solution

(a) $\dfrac{df}{dt} = 2te^{2t}(t+1)(t^2+1)^3 + 6t^3e^{2t}(t^2+1)^2$

(b) $\dfrac{df}{dt} = -2\dfrac{1}{t^3}\dfrac{1}{\tan^2 t} - 2\dfrac{1}{t^2}\dfrac{\sec^2 t}{\tan^3 t} + 9\left(-\dfrac{1}{t^2} + 2\sec^2 t\right)\left(\dfrac{1}{t} + 2\tan t\right)^8$

(c) $\dfrac{dz}{dt} = 2 - 4\sin^2 t$ (d) $\nabla f(1,1) = (4\cos 2,\ -6\cos 2)$

(e) $\dfrac{\partial f}{\partial y}(0,1,1) = -30$ (f) $\nabla f(1,2,3) = (60,4,18)$.

Exercise 29.

If a function $f(u,v,w)$ is differentiable at $u = x-y$, $v = y-z$ and $w = z-x$, show that setting $F(x,y,z) = f(x-y, y-z, z-x)$ we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = 0.$$

Exercise 30.

Consider the function $g(x,y) = \begin{cases} \dfrac{(x-1)^2y^2}{(x-1)^2+y^2}, & (x,y) \neq (1,0), \\ 0, & (x,y) = (1,0). \end{cases}$

(a) Determine the partial derivatives $g'_x(x,y)$ and $g'_y(x,y)$, as well as their domain of definition.

(b) Show that $g'_x(x,y)$ and $g'_y(x,y)$ are continuous over $\mathbb{R}^2$.

(c) Study the differentiability of g at $(1,0)$.

(d) Discuss the continuity of g at $(1,0)$.

Solution

(a) $g'_x(x,y) = \begin{cases} \dfrac{2(x-1)y^4}{((x-1)^2+y^2)^2}, & (x,y) \neq (1,0), \\ 0, & (x,y) = (1,0), \end{cases}$ $g'_y(x,y) = \begin{cases} \dfrac{2(x-1)^4y}{((x-1)^2+y^2)^2}, & (x,y) \neq (1,0), \\ 0, & (x,y) = (1,0). \end{cases}$

Thus $D_{g'_x} = D_{g'_y} = \mathbb{R}^2$.

(b) g is differentiable at $(1,0)$. (c) g is continuous at $(1,0)$.

Exercise 31.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

(a) Compute $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

(b) Determine $\dfrac{\partial f}{\partial y}(x,y)$ and show that it is discontinuous at $(0,0)$.

(c) Check that f is differentiable at $(0,0)$.

(d) Compute $\partial_{(\frac{3}{5},\frac{4}{9})} f(0,0)$.

(e) Discuss the continuity of f at $(0,0)$.

Solution

(a) $f'_x(0,0) = f'_y(0,0) = 0$ (b) $f'_y(x,y) = \begin{cases} 2y\sin\dfrac{1}{\sqrt{x^2+y^2}} - \dfrac{y}{\sqrt{x^2+y^2}}\cos\dfrac{1}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0); \end{cases}$

(c) 0 (d) f is continuous at $(0,0)$.

Exercise 32.

Use the function

$$g(x,y) = \begin{cases} \dfrac{\sin x}{y}, & y \neq 0, \\ 0, & y = 0, \end{cases}$$

and the point $(0,0)$ to show that a function with finite partial derivatives at a given point is not necessarily continuous at that point. Is the given function differentiable at $(0,0)$? Why?

Solution

The function is not continuous at $(0,0)$, and so it is also not differentiable.

Exercise 33.

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \begin{cases} \dfrac{xy}{|x|+|y|}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

(a) Show that f is continuous at $(0,0)$.

(b) Determine $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

(c) Show that f is not differentiable at $(0,0)$. Without performing any calculations, what can you conclude about the continuity of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(0,0)$?

Solution

(a) $f'_x(0,0) = f'_y(0,0) = 0$;

(b) Since f is not differentiable at $(0,0)$, at least one of the functions f'_x or f'_y is not continuous at $(0,0)$.

Exercise 34.

Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=\begin{cases}\dfrac{x(x-y)}{x+y}, & \text{if } x+y\neq 0,\\ 0, & \text{if } x+y=0.\end{cases}$

(a) Study the continuity of f at $(0,0)$.

(b) Compute the partial derivative $\dfrac{\partial f}{\partial x}$ and discuss its continuity at $(0,0)$.

(c) Study the differentiability of f at $(0,0)$.

(d) Show that $\left(\dfrac{\partial f}{\partial x}(0,0),\dfrac{\partial f}{\partial y}(0,0)\right)\cdot\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\neq\partial_{\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)}f(0,0)$. Comment on the result.

Solution

(a) f is not continuous at $(0,0)$.

(b) $f'_x(x,y)=\dfrac{x^2+2xy-y^2}{(x+y)^2}$ if $x+y\neq 0$ and $f'_x(0,0)=1$ (it does not exist $f'_x(a,-a), a\neq 0$); $f'_x(x,y)$ is not continuous at $(0,0)$.

(c) f is not differentiable at $(0,0)$.

(d) The two values would only be equal if f was differentiable at $(0,0)$.

Exercise 35.

Compute $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^4 f}{\partial x^2\partial z\partial y}$, for $f(x,y,z)=z^2x^2y+xye^z$.

Solution

$\dfrac{\partial^2 f}{\partial x^2}=2yz^2$, $\dfrac{\partial^4 f}{\partial x^2\partial z\partial y}=4z$.

Exercise 36.

Compute f''_{x^2}, f''_{xy} and f'''_{xyx} for each of the following functions, indicating the corresponding domain of definition:

(a) $f(x,y)=x\sin(x+y)$,

(b) $f(x,y)=\begin{cases} y\sin x, & y\neq 0,\\ 2, & y=0.\end{cases}$

Solution

(a) $f''_{x^2}=2\cos(x+y)-x\sin(x+y)$, $f''_{xy}=\cos(x+y)-x\sin(x+y)$ and $f'''_{xyx}=-2\sin(x+y)-x\cos(x+y)$.

(b) $f''_{x^2}=-y\sin x$, $f''_{xy}=\cos x$ and $f'''_{xyx}=-\sin x$.

Exercise 37.

Compute the differential of order 2, 3 and 4 of the function $f(x,y)=\sqrt{xy}$ at $(1,1)$.

Solution

$$D^2f(1,1)(\mathbf{h}^2)=-\tfrac{1}{4}h_1^2+\tfrac{1}{2}h_1h_2-\tfrac{1}{4}h_2^2,$$

$$D^3f(1,1)(\mathbf{h}^3)=\tfrac{3}{8}h_1^3-\tfrac{3}{8}h_1^2h_2-\tfrac{3}{8}h_1h_2^2+\tfrac{3}{8}h_2^3,$$

$$D^4f(1,1)(\mathbf{h}^4)=-\tfrac{15}{16}h_1^4+4\tfrac{3}{16}h_1^3h_2+6\tfrac{1}{16}h_1^2h_2^2+4\tfrac{3}{16}h_1h_2^3-\tfrac{15}{16}h_2^4.$$

Exercise 38.

Determine the differential of order n of the function $f(x, y) = \sin(x + y)$, at the point $(0, 0)$.

Solution

$D^n f(0,0)(\mathbf{h}) = \sin(\frac{n\pi}{2}) \sum_{i=0}^{n} \binom{n}{i} h_1^i h_2^{n-i}.$

Exercise 39.

Show that $f(x, y) = \log(e^x + e^y)$ satisfies the (differential) equation

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0$$

everywhere in $\mathbb{R}^2$.

Exercise 40.

Let $f \in C^2(\mathbb{R}^2)$ be a real function such that $\frac{\partial f}{\partial u}(0, 0) = \frac{\partial f}{\partial v}(0, 0) = 1$. Also, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$g(x, y) = f(y \sin x, y^2).$$

Show that the Hessian matrix of g at $(0, 0)$ is given by $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$.

Exercise 41.

Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y) = xy^2 + g(u, v, w), \text{ with } u = \sin y^2, \; v = \ln x \text{ and } w = ye^x.$$

Assuming that g is of class $C^2(\mathbb{R}^3)$, compute $\dfrac{\partial^2 f}{\partial y \partial x}(1, 0)$.

Solution

$\dfrac{\partial^2 f}{\partial y \partial x}(1, 0) = e\left(\dfrac{\partial^2 g}{\partial u \partial v}(0, 0, 0) + \dfrac{\partial g}{\partial w}(0, 0, 0)\right).$

Exercise 42.

Show that the following functions are homogeneous or positively homogeneous. Determine in each case the degree of homogeneity and verify Euler's identity.

(a) $f(x, y) = \log \frac{(x+y)^2}{xy}$,

(b) $f(x, y, z) = \frac{\sqrt{x^2+y^2}}{z^2}$,

(c) $f(x, y) = \begin{cases} (x + y) \sin\left(\frac{xy}{x^2+y^2}\right), & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$

Solution

(a) f is homogeneous with degree 0;

(b) f is positively homogeneous with degree -1;

(c) f is homogeneous with degree 1.

Exercise 43.

Study the function $g(x,y,z)=x^2+x^\alpha y^{\beta-3}-z^{3\alpha}y^\beta$ and $h(x,y)=\dfrac{x^3y^\alpha+x^{\beta-1}}{y^{3-\beta}}$, with respect to its homogeneity in terms of the parameters $\alpha,\beta\in\mathbb{R}$,

(a) Using the definition.

(b) Using Euler's identity.

Solution

g is homogeneous with degree 2 for $\alpha=-\frac{3}{2}$ and $\beta=\frac{13}{2}$.

h is homogeneous with degree $\alpha+\beta$ for $\beta=\alpha+4, \alpha\in\mathbb{R}$.

Exercise 44.

Assuming that $g(u,v)$ is differentiable at $\left(\frac{x}{y},\frac{z}{x}\right)$, with $x,y\neq 0$, show that

$$f(x,y,z)=x^2\cdot g\left(\frac{x}{y},\frac{z}{x}\right),$$

satisfies the identity $x\,f'_x+y\,f'_y+z\,f'_z=2.f$. Interpret this result in terms of homogeneity.

Solution

f is positively homogeneous with degree 2.

Exercise 45.

Let $f(\mathbf{x}):\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ be an homogeneous, non constant function with degree 0. Show that $\lim\limits_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ does not exist.

Exercise 46.

Consider the function $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ defined by

$$f(x,y)=\ln\left(\frac{xy}{x+y}\right).$$

Write down Taylor's formula with degree 2, around (1,1).

Solution

$\ln\dfrac{(1+h)(1+k)}{2+h+k}\approx-\ln 2+\dfrac{1}{2}h+\dfrac{1}{2}k+\dfrac{1}{2}\left(-\dfrac{3}{4}h^2+\dfrac{1}{2}hk-\dfrac{3}{4}k^2\right)+r_3(h,k).$

Exercise 47.

Let $f(x,y)=\sin\left(x+y^2\right)$. Using the Taylor's formula, aproximate f by a polynomial of degree 3 for (x,y) nearby $(0,0)$.

Solution

$\sin\left(x+y^2\right)\approx x+y^2-\dfrac{x^3}{3!}.$

5. Otpimization

Exercise 48.

Determine and classify the critical points of the following functions from $\mathbb{R}^2$ to $\mathbb{R}$.

(a) $x^2 + y^2$

(b) $x^2 - y^2$

(c) $x^3 + y^3$

(d) $x^3 - y^3$

(e) $x^4 + y^4$

(f) $x^4 - y^4$

(g) $3xy - x^3 - y^3$

(h) $x \ln x + y \ln y$

(i) $x^3 + ye^y$

(j) $2x^3 + xy^2 + 5x^2 + y^2$

(k) $x^4 + y^4 - 4xy + 1$

(l) x^2y^2

Solution

(a) $(0, 0)$ is a minimum point (b) $(0, 0)$ is a saddle point (c) $(0, 0)$ is a saddle point;

(d) $(0, 0)$ is a saddle point (e) $(0, 0)$ is a minimum point (f) $(0, 0)$ is a saddle point;

(g) $(0, 0)$ is a saddle point and $(1, 1)$ is a maximum point;

(h) $(1/e, 1/e)$ is a minimum point (i) $(0, -1)$ is a saddle point;

(j) $(0, 0)$ is a minimum point, $(-5/3, 0)$ is a maximum point, $(-1, 2)$ and $(-1, -2)$ are saddle points;

(k) $(0, 0)$ is a saddle point, $(1, 1)$ and $(-1, -1)$ are minimum points;

(l) $(0, b)$ and $(a, 0)$ $\forall a, b \in \mathbb{R}$ are minimum points.

Exercise 49.

Determine and classify the critical points of the following functions, in terms of the parameter $a \in \mathbb{R} \setminus \{0\}$.

(a) $f(x, y) = e^{x^2 - ay^2}$

(b) $f(x, y) = ax^2 - y^2$

(c) $f(x, y) = x^3 - ax^2 - 3y^2$

(d) $f(x, y) = \frac{16}{5}x^5 + ay^2 - x$

Solution

(a) Critical point: $(0, 0)$. If $a < 0$, minimum point; if $a > 0$, saddle point.

(b) Critical point: $(0, 0)$. If $a > 0$, $(0, 0)$ is a saddle point; if $a < 0$, $(0, 0)$ is a maximum point.

(c) Critical points: $(0, 0)$ and $(\frac{2a}{3}, 0)$. If $a > 0$, $(0, 0)$ is a maximum point and $(\frac{2a}{3}, 0)$ is a saddle point; if $a < 0$, $(0, 0)$ is a saddle point and $(\frac{2a}{3}, 0)$ is a maximum point.

(d) Critical points: $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$. If $a < 0$, $(-\frac{1}{2}, 0)$ is a maximum point and $(\frac{1}{2}, 0)$ is a saddle point; if $a > 0$, $(-\frac{1}{2}, 0)$ is a saddle point and $(\frac{1}{2}, 0)$ is a minimum point.

Exercise 50.

Consider the function $f(x, y) = (y - \alpha)\, xe^x$.

(a) Knowing that $(0, 1)$ is a critical point, determine α and classify this critical point.

(b) Show that f is unbounded.

Solution

(a) $\alpha = 1$. The critical point is a saddle point.

Exercise 51.

Let function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(x, y) = 4\alpha(y-2)^2 + (\beta^2 - 1)(2x-2)^2,$$

where $\alpha \neq 0$, $\beta \neq 1$, $\beta \neq -1$. Show that $(1, 2)$ is the only critical point and classify it in terms of all possible values of α and β.

Solution

If $|\beta| < 1$ and $\alpha < 0$ then $(1, 2)$ is a local maximum; if $|\beta| > 1$ and $\alpha > 0$ then $(1, 2)$ is a local minimum; in all other cases it is a saddle point.

Exercise 52.

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x, y) = x^2 e^{y^3 - 3y}$.

(a) Determine all critical points of function f.

(b) Show that f attains its global minimum at points of the form $(0, b)$.

(c) Justify that

  (i) f is unbounded over $\mathbb{R}^2$;

  (ii) f has a maximum and minimum over $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$.

Solution

(a) Critical points: $(0, b)$ with $b \in \mathbb{R}$.

Exercise 53.

Determine the global extrema of f over the set M, where:

(a) $f(x, y, z) = x - 2y + 2z$, $\quad M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$

(b) $f(x, y) = 4x^2 + y^2$, $\quad M = \{(x, y) \in \mathbb{R}^2 : 2x^2 + y^2 = 1\}$

(c) $f(x, y) = xy$, $\quad M = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{8} + \frac{y^2}{2} = 1\right\}$

(d) $f(x, y, z) = x^2 + 2y - 2z$ $\quad M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 8\}$

(e) $f(x, y) = x^2 + 2xy + y^2$, $\quad M = \{(x, y) \in \mathbb{R}^2 : (x-3)^2 + y^2 = 2\}$

(f) $f(x, y, z) = 2x + 2y^2 + z^2$, $\quad M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 2\}$

(g) $f(x, y, z) = e^{-x^2 - y^2}$, $\quad M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$

(h) $f(x, y) = 4xy - 2x^2 - 2y^2$, $\quad M = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

(i) $f(x, y) = x^2 + 2xy + y^2$, $\quad M = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 8\}$

Solution

(a) max = 3, min = -3;

(b) max = 2, min = 1;

(c) max = 2, min = -2;

(d) max = 10, min = -8;

(e) max = 25, min = 1;

(f) max = 9/2, min = -2$\sqrt{2}$;

(g) max = 1, min = 1/e;

(h) max = 0, min = -4;

(i) max = 16, min = 0.

Exercise 54.

Determine the global extrema of f over the set A, where

(a) $f(x,y,z) = x - 2y + 2z$ $\qquad A = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$
(note: compare with a) from previous exercise)

(b) $f(x,y) = 4x^2 + y^2$ $\qquad A = \{(x,y) \in \mathbb{R}^2 : 2x^2+y^2 \leq 1\}$
(note: compare with b) from previous exercise)

(c) $f(x,y) = x^2 + 2xy + y^2$ $\qquad A = \{(x,y) \in \mathbb{R}^2 : (x-3)^2+y^2 \leq 2\}$
(note: compare with e) from previous exercise)

Solution

(a) max = 3, min = -3; (b) max = 2, min = 0; (c) max = 25, min = 1.

Exercise 55.

Determine the maximum and minimum distance to the origin of the points in the ellipse $5x^2 + 6xy + 5y^2 = 8$.

Solution

The maximum distance is 2 and the minimum distance is 1.

Exercise 56.

Solve the optimization problem $\min{(x+4y+3z)}$ subject to the condition $x^2 + 2y^2 + \frac{1}{3}z^2 = b$, $(b > 0)$.

Solution

The minimum value is $-6\sqrt{b}$, attained at $(-\frac{\sqrt{b}}{6}, -\frac{\sqrt{b}}{3}, -\frac{3\sqrt{b}}{2})$.

Exercise 57.

Determine the point in the ellipse $x^2 + 2xy + 2y^2 = 2$ with smallest x coordinate.

Solution

$(-2, 1)$.

Exercise 58.

Determine the global extrema of $f(x,y,z) = e^{x^2+y^2+z^2}$ over the set $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1,\ z = 2-y\}$.

Solution

The maximum value is e^{10}, attained at $(0,-1,3)$; the minimum is e^2, attained at $(0,1,1)$.

6. Multiple Integrals

Exercise 59.

Compute the following integrals.

(a) $\displaystyle\int_0^1\int_{-1}^1\int_1^2 x\,dx\,dy\,dz$

(b) $\displaystyle\int_{-1}^1\int_0^1 ye^{xy}\,dx\,dy$

(c) $\displaystyle\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} e^{-x-y-z}\,dx\,dy\,dz$

(d) $\displaystyle\int_0^5\int_0^{+\infty}\left(x^2e^{-2yx}+3ye^{-y^2}\right)dy\,dx$

(e) $\displaystyle\int_1^2\int_1^2\left(1+x+\frac{y}{2}\right)dx\,dy$

(f) $\displaystyle\int_0^1\int_0^1\int_0^{\frac{\pi}{2}} y\cos x\,dx\,dy\,dz$

Solution

(a) 3 (b) $e-\dfrac{1}{e}-2$ (c) 1 (d) $\dfrac{55}{4}$ (e) $\dfrac{13}{4}$ (f) $\dfrac{1}{2}$

Exercise 60.

Compute $\displaystyle\iint_A f(x,y)\,dx\,dy$, where:

a) $f(x,y)=\dfrac{2x}{y^6}$, $\quad A=\{(x,y)\in\mathbb{R}^2 : 1\le y\le 3,\ 0\le x\le y^4\}$

b) $f(x,y)=e^{\frac{y}{x}}$, $\quad A=\{(x,y)\in\mathbb{R}^2 : 1\le x\le 2,\ x\le y\le x^3\}$

c) $f(x,y)=x^2y^5$, $\quad A=\{(x,y)\in\mathbb{R}^2 : 0\le x\le 1,\ x^2\le y\le 1\}$

d) $f(x,y)=ye^x+x^2y$, $\quad A=\{(x,y)\in\mathbb{R}^2 : 0\le y\le 1,\ 0\le x\le y^2\}$

e) $f(x,y)=xe^{-xy}$, $\quad A=\{(x,y)\in\mathbb{R}^2 : 1\le x\le 2,\ \frac{1}{x}\le y<+\infty\}$

f) $f(x,y)=x^3+4y$, $\quad A$ is the region bounded by the lines $y=x^2$ and $y=2x$.

Solution

(a) $\dfrac{26}{3}$ (b) $\dfrac{e^4}{2}-2e$ (c) $\dfrac{2}{45}$ (d) $\dfrac{e}{2}-\dfrac{23}{24}$ (e) $\dfrac{1}{e}$ (f) $\dfrac{32}{3}$

Exercise 61.

Let $A=\{(x,y)\in\mathbb{R}^2 : y\le 1-x,\ y\le 1+x,\ y\ge 0\}$, and compute

a) $\displaystyle\iint_A (x-1)y\,dx\,dy$ b) $\displaystyle\iint_A (y-2y^2)e^{xy}\,dx\,dy$ c) $\displaystyle\iint_A (x+y)\,dx\,dy$

Solution

(a) $-\dfrac{1}{3}$ (b) 0 (c) $\dfrac{1}{3}$

Exercise 62.

Compute $\displaystyle\int_{-1}^1\int_{-1}^1 f(x,y)\,dx\,dy$, with $f(x,y)=\begin{cases} xy, & x\ge 0 \text{ and } y\ge 0,\\ 1-x-y, & \text{otherwise.}\end{cases}$ **Solution** $\dfrac{17}{4}$

Exercise 63.

Using a double integral compute the area of set A, where

(a) $A = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \wedge x^2 \leq y \leq 1\}$;

(b) $A = \{(x, y) \in \mathbb{R}^2 : 1 - x \leq y \leq 1 \wedge x \leq 1\}$;

(c) $A = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y \leq 2 - x^2\}$;

(d) $A = \{(x, y) \in \mathbb{R}^2 : x + y + 2 \geq 0 \wedge x + y^2 \leq 0\}$;

(e) $A = \{(x, y) \in \mathbb{R}^2 : y \leq x^2 \wedge y - x \geq 0 \wedge 2y - x \leq 3 \wedge x \geq 0\}$;

(f) $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \wedge x^2 + y^2 \leq 1\}$;

(g) $A = \{(x, y) \in \mathbb{R}^2 : y^2 + x \leq 2 \wedge x - y \geq 0\}$;

(h) $A = \{(x, y) \in \mathbb{R}^2 : y^2 \leq x \wedge x \leq \frac{y^2}{4} + 3\}$;

Solution

(a) $\dfrac{4}{3}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{8}{3}$ (d) $\dfrac{9}{2}$ (e) $\dfrac{35}{48}$ (f) $\dfrac{7}{2}$ (g) $\dfrac{9}{2}$ (h) 8

Exercise 64.

Compute $\displaystyle\int_0^4 \int_{2x}^8 \sin(y^2)\, dy\, dx$ (suggestion: Draw the integration region and invert the integration order).

Solution

$\dfrac{1 - \cos 64}{4}$

Exercise 65.

Compute $\displaystyle\iint_A g(x, y)\, dx\, dy$, where $A = [0, 2] \times [0, 4]$ and $g(x, y) = \begin{cases} x - y, & x \leq y \leq 2x, \\ 0, & \text{otherwise.} \end{cases}$

Solution

$-\dfrac{4}{3}$

7. Ordinary Differential Equations

Exercise 66.

Determine the general solution of the following differential equations with separable variables.

(a) $y' = xy - x$

(b) $dx\, e^y = dy(x+1) - dx$

(c) $\dfrac{dy}{dx} + \dfrac{1+y^3}{xy^2(1+x^2)} = 0$

(d) $\sqrt{1-x^2}\, dy - \sqrt{1-y^2}\, dx = 0$

(e) $e^{x^4} yy' = x^3\left(9 + y^4\right)$

(f) $e^y(4+x^2)y' = x\,(2+e^y)$

(g) $e^{3x}dy + \left(4+y^2\right) dx = 0$

(h) $4xe^y dx + \left(x^4+4\right) dy = 0$

Solution

(a) $y(x) = 1 + e^{\frac{1}{2}x^2} C$

(b) $\ln\left(e^{y(x)} + 1\right) - y(x) + \ln(x+1) = C$

(c) $\dfrac{1}{3}\ln\left|1+y^3\right| + \ln|x| - \dfrac{1}{2}\ln(1+x^2) = C$

(d) $\arcsin(y(x)) - \arcsin x = C$

(e) $\dfrac{1}{6}\arctan\left(\dfrac{1}{3}y^2(x)\right) + \dfrac{1}{4}e^{-x^4} = C$

(f) $\ln\left(2+e^{y(x)}\right) - \dfrac{1}{2}\ln\left(4+x^2\right) = C$

(g) $\dfrac{1}{2}\arctan\left(\dfrac{1}{2}y(x)\right) - \dfrac{1}{3}e^{-3x} = C$

(h) $-e^{-y(x)} + \arctan\dfrac{1}{2}x^2 = C$

Exercise 67.

Determine the general solution of the following first-order linear differential equations.

(a) $y' - y = 2xe^{x^2+x}$

(b) $4x^2y' + 8xy = -12\sin(3x)$

(c) $x^2y' + 3xy = 4e^{2x}$

(d) $xy' = 2x + 2y$

Solution

(a) $y(x) = e^x\left(C + e^{x^2}\right)$

(b) $y(x) = \frac{\cos(3x)+C}{x^2}$

(c) $y(x) = \frac{e^{2x}(2x-1)+C}{x^3}$

(d) $y(x) = -2x + Cx^2$

Exercise 68.

Solve the following initial value problems.

(a) $y' + 4y = 0, \quad y(0) = 6$

(b) $\dfrac{dy}{dt} + y\sin t = 0, \quad y\left(\frac{\pi}{3}\right) = \frac{3}{2}$

(c) $y'\cos x + 2y\sin x = \sin x, \quad y(0) = 0$

(d) $(1+x^2)y' = -2xy + 2, \quad y(0) = 1$

(e) $(1+x^2)y' + y = 0, \quad y(1) = 1$

(f) $2y' + 4xy = 4x, \quad y(0) = -2$

(g) $y' + y\sin x = \sin x\cos x, \quad y\left(\frac{\pi}{2}\right) = 0$

(h) $(1+x^2)y' = 4x(1+y), \quad y(0) = 1$

Solution

(a) $y(x) = 6e^{-4x}$

(b) $y(t) = \dfrac{3}{2}e^{\cos t - \frac{1}{2}}$

(c) $y(x) = \frac{1-\cos^2 x}{2}$

(d) $y(x) = \frac{2x+1}{1+x^2}$

(e) $y(x) = e^{\frac{\pi}{4} - \arctan(x)}$

(f) $y(x) = 1 - 3e^{-x^2}$

(g) $y(x) = \cos x + 1 - e^{\cos x}$

(h) $y(x) = 2(1+x^2)^2 - 1$

Exercise 69.

Show that any homogeneous first-order linear differential equation can be written as a separable variables equation.

Exercise 70.

Determine the general solution of the following differential equations.

(a) $y'' - 7y' + 12y = 0$

(b) $y'' + 4y = 0$

(c) $y'' - 4y' + 4y = 0$

(d) $y'' + 2y' + 10y = 0$

(e) $y'' + y' - 6y = 8$

(f) $y'' + 3y' + 2y = e^{5x}$

(g) $y'' - y = \sin x$

(h) $y'' - y = e^{-x}$

(i) $y'' - 6y = 36(x-1)$

(j) $y'' - 9y = 9x^2$

(k) $y'' + 3y' + 2y = \sin x$

(l) $y'' + 3y' + 2y = e^{-x}$

(m) $y'' - 4y' + 4y = 6e^{2x}$

Solution

(a) $y(x) = C_1e^{3x} + C_2e^{4x}$

(b) $y(x) = C_1\cos 2x + C_2\sin 2x$

(c) $y(x) = (C_1 + C_2x)e^{2x}$

(d) $y(x) = (C_1\cos 3x + C_2\sin 3x)e^{-x}$

(e) $y(x) = C_1e^{-3x} + C_2e^{2x} - \dfrac{4}{3}$

(f) $y(x) = C_1e^{-2x} + C_2e^{-x} + \dfrac{1}{42}e^{5x}$

(g) $y(x) = C_1e^{-x} + C_2e^{x} - \dfrac{1}{2}\sin x$

(h) $y(x) = C_1e^{x} + C_2e^{-x} - \dfrac{1}{2}xe^{-x}$

(i) $y(x) = C_1e^{\sqrt{6}x} + C_2e^{-\sqrt{6}x} - 6x + 6$

(j) $y(x) = C_1e^{3x} + C_2e^{-3x} - x^2 - \dfrac{2}{9}$

(k) $y(x) = C_1e^{-2x} + C_2e^{-x} - \dfrac{3}{10}\cos x + \dfrac{1}{10}\sin x$

(l) $y(x) = C_1e^{-2x} + C_2e^{-x} + xe^{-x}$

(m) $y(x) = C_1e^{2x} + C_2xe^{2x} + 3x^2e^{2x}$

Exercise 71.

Solve the following initial value problems.

(a) $\begin{cases} y'' + y' - 2y = 0 \\ y(0) = -1,\ y'(0) = 0 \end{cases}$

(b) $\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 0,\ y'(0) = 1 \end{cases}$

(c) $\begin{cases} y'' + 2y' + y = x^2 \\ y(0) = 0,\ y'(0) = 1 \end{cases}$

(d) $\begin{cases} y'' + 4y = 4x + 1 \\ y(\frac{\pi}{2}) = 0,\ y'(\frac{\pi}{2}) = 0 \end{cases}$

(e) $\begin{cases} 9y'' + y = 0 \\ y\left(\frac{3\pi}{2}\right) = 2,\ y'\left(\frac{3\pi}{2}\right) = 0 \end{cases}$

(f) $\begin{cases} 2y'' - 4y' + 2y = 0 \\ y(0) = -1,\ y'(0) = 1 \end{cases}$

(g) $\begin{cases} y'' - 2y' + 10y = 10x^2 \\ y(0) = 0,\ y'(0) = 3 \end{cases}$

Solution

(a) $y(x) = -\dfrac{2}{3}e^{-2x} - \dfrac{1}{3}e^{x}$

(b) $y(x) = \dfrac{1}{2}\sin(2x)e^{-x}$

(c) $y(x) = -(6+x)e^{-x} + x^2 - 4x + 6$

(d) $y(x) = \dfrac{2\pi+1}{3}\cos 2x + \dfrac{1}{3}\sin 2x + x + \dfrac{1}{4}$

(e) $y(x) = 2\sin\left(\dfrac{x}{3}\right)$

(f) $y(x) = (-1+2x)e^{x}$

(g) $y(x) = e^{x}\left(\dfrac{3}{25}\cos 3x + \dfrac{62}{75}\sin 3x\right) + x^2 + \dfrac{2}{5}x - \dfrac{3}{25}$

Exercise 72.

Solve the following boundary value problems:

(a) $\begin{cases} y'' - 6y' + 9y = e^{3x} \\ y(0) = 0,\ y(1) = 0 \end{cases}$

(b) $\begin{cases} y'' + 4y = 0 \\ y(0) = 0,\ y(\pi) = 0 \end{cases}$

(c) $\begin{cases} y'' - 10y' + 25y = 50 \\ y(0) = 0,\ y(2) = 2 \end{cases}$

(d) $\begin{cases} y'' - 8y' + 16y = 0 \\ y(0) = 1,\ y(1) = e^4 \end{cases}$

Solution

(a) $y(x) = \left(-\frac{1}{2}x + \frac{1}{2}x^2\right) e^{3x}$

(b) $y(x) = C \sin 2x$

(c) $y(x) = (-2 + x)e^{5x} + 2$

(d) $y(x) = e^{4x}$

Exercise 73.

Knowing that $y = e^{2x}$ is a solution of the differential equation

$$y'' - \alpha y' + 10y = 0, \quad \alpha \in \mathbb{R},$$

determine α and the general solution of this equation.

Solution

$\alpha = 7$; $y(x) = C_1 e^{2x} + C_2 e^{5x}$.

Exercise 74.

Knowing that $y(x) = xe^{2x}$ is a solution of the differential equation $2y'' - \alpha y' + 8y = 0$, with $\alpha \in \mathbb{R}$, solve the boundary value problem

$$\begin{cases} 2y'' - \alpha y' + 8y = 16 \\ y(0) = 1, \quad y(1) = 2 \end{cases}.$$

Solution

$y(x) = -e^{2x} + xe^{2x} + 2$.

Exercise 75.

Solve the following problem with periodic conditions, $\begin{cases} y'' + 4y = 4, \\ y(0) = y(\pi/2), \quad y'(0) = y'(\pi/2) \end{cases}$.

Solution

$y(x) = 1$.

Exercise 76.

Determine the general solution of the following differential equations.

(a) $y' + y^2 \sin x = 0$

(b) $yy' + x = 0$

(c) $y'' - 2y' = 0$

(d) $y'y - x(2y^2+1)e^{x^2} = 0$

(e) $\dfrac{dy}{dx}\cos y = -x\dfrac{\sin y}{1+x^2}$

(f) $y' + 6yx^5 - x^5 = 0$

Solution

(a) $y^{-1}(x) = -\cos x + C$

(b) $y^2(x) = -x^2 + C$

(c) $y(x) = C_1 + C_2e^{2x}$

(d) $\dfrac{1}{4}\ln(2y^2+1) - \dfrac{1}{2}e^{x^2} = C$

(e) $\ln|\sin y| + \dfrac{1}{2}\ln(1+x^2) = C$

(f) $y(x) = \dfrac{1}{6} + Ce^{-x^6}$

Exercise 77.

Determine the values of a and b for which e^{2x} and e^{-2x} are solutions to the differential equation $y''+ay'+by=0$. For those values of a and b, compute the general solution to the differential equation.

Solution

$a = 0$ and $b = -4$; $y_h(x) = Ae^{2x} + Be^{-2x}, \quad A, B \in \mathbb{R}$.

Exercise 78. (Malthusian population growth)

(a) Compute the time evolution of a population level $y(t)$ knowing that:

  i) at each time t, the growth rate of the population, $\dfrac{dy/y}{dt}$, is equal to r (with $r > 0$);

  ii) at time $t = 0$ there are y_0 (millions of individuals).

(b) Estimate the value of the Portuguese population in the year 2020, knowing that in the year 2013 (by hypothesis $t = 0$) there is a record of $y = 10.457$ (million individuals) and that $r = -0.00476$.

(c) Comment on the Malthusian hypothesis, by studying $\lim_{t\to\infty} y(t)$.

Solution

(a) $y(t) = y_0e^{rt}$

(b) $y(8) \approx 10.0663$

(c) $+\infty$ if $r > 0$, y_0 if $r = 0$ and 0 if $r < 0$.

Exercise 79. (Population growth according to Verhulst)

(a) Compute the time evolution of a population level $y(t)$ knowing that:

  i) at each time t, the rate of growth of the population, $\dfrac{dy/y}{dt}$, is equal to r minus ay (with $r > 0$ and $a > 0$);

  ii) at time $t = 0$ there is a record of y_0 million individuals.

(b) Estimate the value of the Portuguese population in the year 2020, knowing that in the year 2013 (by hypothesis $t = 0$) there is a record of $y = 10.457$ million individuals and that $r = -0.00229$, $a = 0.00033$.

(c) Study $\lim_{t\to\infty} y(t)$.

Exercise 80. (Domar's growth model)
Consider an economical model where:

i) the aggregated demand y_d varies on time according to the equation $\dfrac{dy_d}{dt} = \dfrac{dI}{dt}\dfrac{1}{s}$, where $I = I(t)$ is the investment and s the marginal propensity to saving ($1/s$ is the Keynesian multiplier);

ii) the productive capacity $y_c = \rho K$ follows the differential equation $\dfrac{dy_c}{dt} = \rho\dfrac{dK}{dt}$, where $K = K(t)$ is the stock of capital of the economy ($\dfrac{dK}{dt} = I(t)$).

Obtain the time trajectory of the investment, $I(t)$, satisfying the equilibrium of Domar's model ? the variation of the aggregated demand = variation of the productive capacity.

Exercise 81.
Consider the demand and supply functions of a given commodity:

$$Q_d = a - bP, \quad Q_s = -c + dP.$$

(a) Determine the time evolution of the price level $P(t)$ knowing that at each time t, the variation rate of $P(t)$ is proportional to the excess demand, i.e.

$$\frac{dP}{dt} = \alpha(Q_d - Q_s),$$

and that $P(0) = P_0 \neq P_e = \dfrac{a+c}{b+d}$ (equilibrium price).

(b) Verify under which conditions we have $\lim\limits_{t\to\infty} P(t) = P_e$.