# Mathematical Programming II

## Chapter 1: Goal Programming

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1. Consider the following instance of a goal programming problem:

$$\begin{aligned} \text{Minimize } & P_1(s_1^- + s_1^+) + P_2(s_2^-) \\ \text{subject to: } & -x_1 + x_2 + s_1^- - s_1^+ = 2 \\ & x_1 + s_2^- - s_2^+ = 3 \\ & x_2 \leq 3 \\ & x_1 + x_2 \leq 5 \\ & x_1, x_2, s_1^-, s_1^+, s_2^-, s_2^+ \geq 0 \end{aligned}$$

   (a) Solve the problem graphically. If the order of the priorities is reversed, does the solution change?

   (b) Solve the problem using the Simplex method.

2. Consider the set of constraints defined by:

$$\begin{aligned} x_1 + x_2 &\leq 10 \\ x_1 - x_2 &\leq 2 \\ 3x_1 + x_2 &\leq 15 \\ x_1, x_2 &\geq 0 \end{aligned}$$

   Suppose that the following goals are to be achieved, in order of priority:

   1. $x_1 \leq 3$.
   2. $x_1 \geq x_2$.
   3. $x_2 \leq 6$.

   (a) Formulate the problem as a goal programming problem.

   (b) Solve the problem graphically.

3. Consider the set of constraints defined by:

$$\begin{aligned} x_1 + x_2 &\leq 10 \\ 2x_1 + x_2 &\geq 5 \\ -x_1 + x_2 &\geq -5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

   Suppose that the following goals are to be achieved, in order of priority:

   1. $x_2 \geq 2x_1$ and $2x_1 + x_2 \leq 10$.
   2. $x_1 \geq 2$ and $x_2 \geq 8$. Assign a weight of 2 to the first condition and a weight of 1 to the second.
   3. $x_2 = 4x_1$.

   (a) Formulate the problem as a goal programming problem.

   (b) Solve the problem graphically.

4. Consider the subset of $\mathbb{R}^2$ defined by:

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 &\leq 4 \\ 2x_1 + x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Suppose that the following goals are to be achieved, in order of priority:

1. $x_1 \geq 2$
2. $x_2 = x_1$
3. $x_1 + x_2 = 4$
4. $x_2 \geq 5$

Formulate and solve the problem using the Simplex method. Consider $(x_1, x_2) = (1, 0)$ as the initial BFS.

5. Consider the set of constraints defined by:

$$\begin{aligned} x_1 + x_2 &\geq 5 \\ 2x_1 + x_2 &\leq 20 \\ x_1 - x_2 &\geq -5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Suppose that the following goals are to be achieved, in order of priority:

1. $-2x_1 + x_2 \leq 2$ and $x_1 \leq 5$. Assign a weight of 2 to the first condition and a weight of 3 to the second.
2. $4x_1 + x_2 \geq 20$ and $4x_1 - x_2 \geq 10$.
3. $x_2 \geq 6$ and $x_2 \leq 2x_1 - 6$.

(a) Formulate the problem as a goal programming problem.

(b) Solve the problem graphically.

6. A bank has decided to close some of its branches while retaining the employees, who will be reassigned to support the diversification of its activities. To make use of the resulting 460 surplus labour hours per week, the bank has decided to launch two new services: N1) credit intermediation and N2) tax advisory services. The new services are expected to generate weekly profits of 8 and 10 m.u., respectively, per new client, with each client requiring 14 and 16 hours per week in N1 and N2, respectively. The bank intends to launch these new services while ensuring a profit of at least 340 and attracting, in average, at least 20 new clients per week in each service.

(a) Formulate the problem as a goal programming problem and solve it. Briefly discuss the solution, highlighting the relevant values and recommendations.

(b) Assume that failing to attract an average of 20 clients for the credit intermediation service is considered particularly serious by the bank, carrying a penalty six times as high as that associated with failing to attract an average of 20 clients for the tax advisory service. Solve the problem under these new assumptions.

(c) Assume that failing to meet the demand from new clients is also considered particularly serious by the bank, carrying penalties 20 and 15 times as high as those associated with failing to achieve the profit target for N1 and N2, respectively. The bank is therefore considering using additional labour to support these services. With the additional objective of minimizing the use of this extra labour, propose a solution.

7. A company sells two food products, P1 and P2, which generate unit profits of 5 and 3 m.u., respectively. Each day, the company has 100 units of P1 and 180 units of P2 available for sale. The company's store has enough space to hold at most 200 units of P2, or the equivalent amount of P1. Each unit of P1 requires half the space occupied by one unit of P2. The company aims, first, to achieve a profit of at least 1500 m.u. and, second, to make the best possible use of the available storage space. Formulate and solve the problem.

8. A school is considering making its facilities available for birthday parties and recreational activities during the weekend to generate additional revenue. Allocating the facilities to these activities generates a profit of 6 m.u./h and 5 m.u./h, respectively. Birthday parties occupy the facilities for 5 hours, while recreational activities last 4 hours each. To support these activities, the school can hire the services of a security company, which charges 4 m.u. per hour for supporting a birthday party and 3 m.u. per hour for supporting a recreational activity. The school has 580 m.u. available for this purpose. At most four activities can be hosted per day. The school aims to achieve a minimum monthly profit of 900 m.u. (over four weekends), while remaining within the budget for security services and organizing as many birthday parties as recreational activities. Formulate and solve the problem.

9. A faculty has a budget of 66 thousand m.u. to finance an advertising campaign for the launch of a new degree programme. The campaign will use posters and social media advertisements, for which the information displayed in the following table has been collected.

| | Cost | Potential students reached |
|---|---|---|
| Social media advertisements | 2000 | 1200 |
| Poster | 100 | 100 |

   The faculty aims to achieve the following goals, in decreasing order of importance:

   1. Not exceed the budget;
   2. Produce at least 300 posters; and
   3. Reach at least 22 thousand potential students through social media advertisements.

   Propose an advertising campaign and determine the maximum number of potential students that can be reached through each advertising medium.

10. A farm has 100 hectares (ha) of land available for growing corn and/or potatoes. Unit margins of 90 and 60 m.u. per ha have been estimated for corn and potatoes, respectively. The farmer has committed to planting at least 45 ha of potatoes. During the summer, labour for harvesting is limited to 480 labour hours. On average, harvesting requires 6 labour hours per ha of corn and 3 labour hours per ha of potatoes.
    (a) Formulate the linear programming problem that maximizes the total margin.
    (b) The farmer was not satisfied with the solution obtained in part (a) and stated: "The constraints are correct, but maximizing the margin does not reflect my actual goals. As a first priority, I would be satisfied with a margin of 500. Second, I want to allocate labour equally between the two crops. Third, I want to grow at least 50 ha of corn." Formulate and solve this problem.

11. Mantel produces toy cars, each of which requires four wheels and two seats. The factory that manufactures these parts operates three shifts per day. The following table shows the quantities of each part produced during each shift:

    Ideally, the number of wheels produced should be exactly twice the number of seats. However, since production rates vary across shifts, it may not be possible to achieve an exact balance in production. Mantel wants to determine the number of production cycles to run in each shift so as to minimize the imbalance in the production of the two parts.

| Shift | Wheels | Seats |
|---|---|---|
| 1 | 500 | 300 |
| 2 | 600 | 280 |
| 3 | 640 | 360 |

Capacity limitations restrict the number of production cycles to between 4 and 5 in shift 1, between 10 and 20 in shift 2, and between 3 and 5 in shift 3. Formulate the problem as a goal programming problem and solve it.

12. *Camyo Manufacturing* produces four parts that require the use of a lathe and a press, both of which operate 10 hours per day. The following table gives the processing time, in minutes, required by each part on each machine.

| | **Part** | | | |
|---|---|---|---|---|
| **Machine** | **1** | **2** | **3** | **4** |
| Lathe | 10 | 5 | 6 | 8 |
| Press | 5 | 12 | 8 | 4 |

It is desirable to balance the utilization of the two machines, limiting the difference between their total operating times to 30 minutes. In addition, the demand for each part is at least 25 units. The number of units of part 1 cannot exceed the number of units of part 2. Considering that satisfying demand is twice as important as balancing the operating times of the two machines, and that no overtime is allowed on either machine, formulate the problem as a goal programming problem and solve it.

13. The admissions office at Ozark University is processing freshman applications for the upcoming academic year. The applications fall into three categories: in-state, out-of-state, and international. The male-to-female ratios for in-state and out-of-state applicants are 1:1 and 3:2, respectively. For international applicants, the corresponding ratio is 8:1. The American College Test (ACT) score is an important factor in the admission of new students. Statistics collected by the university indicate that the average ACT scores for in-state, out-of-state, and international students are 27, 26, and 23, respectively. The admissions office has established the following desirable goals for the incoming freshman class:

    - The incoming class consists of at least 1200 students.
    - The average ACT score for all incoming students is at least 25.
    - International students constitute at least 10% of the incoming class.
    - The female-to-male ratio is at least 3:4.
    - Out-of-state students constitute at least 20% of the incoming class.

    Formulate the problem as a goal programming problem and solve it.

14. Two products are manufactured on two machines. The following table gives the machining times, in minutes per unit, for the two products:

| | **Product** | |
|---|---|---|
| **Machine** | **1** | **2** |
| 1 | 5 | 3 |
| 2 | 6 | 2 |

The daily production quotas for products 1 and 2 are 80 and 60 units, respectively. Each machine runs 8 hours a day. Overtime, though not desirable, may be used if necessary to meet the production quotas. Formulate the problem as a goal programming problem and solve it. Determine the amount of overtime required to meet the daily production quotas.

15. A government agency plans to send aid to three developing countries, $P_1$, $P_2$, and $P_3$. The purchase prices of blood and antibiotics are 5 US$ and 3 US$ per unit, respectively. The agency has 2 000 000 US$ available and aims to deliver 180 000 units of blood and 200 000 units of antibiotics, if possible.

    A daily loss of 5% of the blood is expected during transportation. The following table gives, for each country, the transportation times and costs, which are the same for blood and antibiotics, as well as the minimum blood demand that must be satisfied.

| | $P_1$ | $P_2$ | $P_3$ |
|---|---|---|---|
| Transportation time | 3 days | 5 days | 2 days |
| Unit transportation cost | 2 $ | 5 $ | 1.5 $ |
| Blood demand | 50 000 | 30 000 | 0 |

    The agency has established the following priority levels:

    1. Satisfy the demand for blood and antibiotics, with blood considered three times as important as antibiotics.
    2. Ensure that the average transportation time does not exceed 3.2 days.
    3. Stay within the available budget.

    Formulate the problem as a goal programming problem and solve it.