

Microeconomics

Chapter 7

Utility maximization

Fall 2026

Consumer preferences: Roadmap

Recall that the goal of economic theory is to account for behavior based on the assumption that actors have **stable preferences**, attempt to do **as well as possible** given those preferences and the constraints placed on their resources, and that changes in behavior are due to **changes in these constraints**.

In this section, we use this approach to develop a theory of consumer behavior based on the **simplest assumptions** possible.

Along the way, we develop the tool of **comparative statics analysis**, which attempts to characterize how economic agents (i.e. consumers, firms, governments, etc.) react to **changes in the constraints** they face.

Consumer preferences

We need a description of the goods and services that a consumer may consume. To start, we can define **consumption bundles** and a **consumption set**.

Consumption bundle: a list of consumption goods, described by the vector $\mathbf{x}$ in R_+^k , where k is the number of different goods, and element $x_i \geq 0$ reflects the specific consumption for good $i = 1, \dots, k$. Thus if the consumer is given bundle $x = (x_1, x_2, \dots, x_k)$, she is given x_1 units of good 1, x_2 units of good 2, and so on

Consumption set: the set of all possible consumption bundles $\mathbf{x}$ that a consumer can hypothetically choose. This set is denoted by X ,

$$X = \{\mathbf{x} \text{ in } R_+^k : \mathbf{x} \text{ can be hypothetically chosen}\}.$$

Later, when we study the theory of the firm, we will see the equivalent concepts: the production plan and the production possibility set.

Consumer preferences

The consumer is assumed to have **preferences** on $\mathbf{x}$ in X . We assume that those preferences are represented by a binary relationship.

Let $\mathbf{x} \neq \mathbf{y}$ be two bundles in X , then the binary relationships are:

$$\mathbf{x} \succ \mathbf{y}$$

$$\mathbf{x} \succeq \mathbf{y}$$

$$\mathbf{x} \sim \mathbf{y}$$

$$\mathbf{x} \preceq \mathbf{y}$$

$$\mathbf{x} \prec \mathbf{y}$$

Consumer preferences

The consumer is assumed to have **preferences** on $\mathbf{x}$ in X . We assume that those preferences are represented by a binary relationship.

Let $\mathbf{x} \neq \mathbf{y}$ be two bundles in X , then the binary relationships are:

$\mathbf{x} \succ \mathbf{y}$ $\mathbf{x}$ is preferred to $\mathbf{y}$

$\mathbf{x} \succeq \mathbf{y}$ $\mathbf{x}$ is weakly preferred to $\mathbf{y}$

$\mathbf{x} \sim \mathbf{y}$ indifferent between $\mathbf{x}$ and $\mathbf{y}$

$\mathbf{x} \preceq \mathbf{y}$ $\mathbf{y}$ is weakly preferred to $\mathbf{x}$

$\mathbf{x} \prec \mathbf{y}$ $\mathbf{y}$ is preferred to $\mathbf{x}$

Assumptions on preferences

1. Completeness: for all $\mathbf{x}$ and $\mathbf{y}$ in X , either $\mathbf{x} \succeq \mathbf{y}$ or $\mathbf{y} \succeq \mathbf{x}$ or both.

This ensures that the consumer can make comparisons: The consumer has the ability and information to evaluate alternatives.

2. Transitivity: for all $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in X , if $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{z}$, then $\mathbf{x} \succeq \mathbf{z}$.

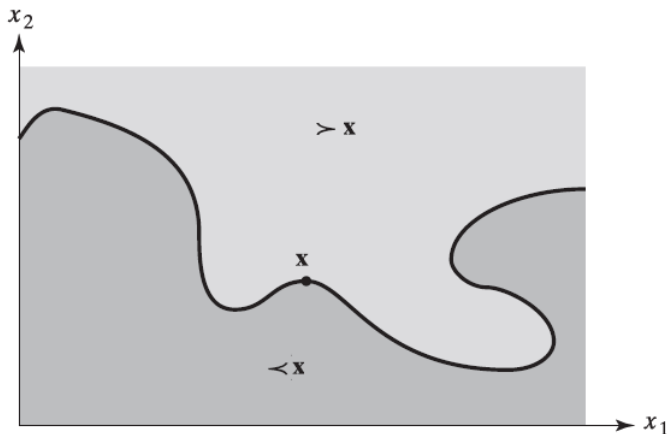
This ensures that choices are consistent: pairwise comparisons are linked together in a consistent way.

Completeness and transitivity imply that the consumer can completely rank any finite number of $\mathbf{x}$ in X , from best to worst, possibly with ties. A preference relation is called **rational** if it is complete and transitive.

3. Continuity: for all $\mathbf{y}$ in X , the sets $\{\mathbf{x} : \mathbf{x} \succeq \mathbf{y}\}$ and $\{\mathbf{x} : \mathbf{x} \preceq \mathbf{y}\}$ are closed sets.

This ensures the absence of discontinuous consumption behavior. Continuity mostly speaks to the mathematical aspects of representing preferences by a utility function.

Consumer preferences in a graph



The graph above reflects a hypothetical set of preferences over bundles $\mathbf{x} = (x_1, x_2)$ that satisfy the three assumptions: Completeness, transitivity, and continuity (and local non-satiation). What about convexity?

Additional assumptions on preferences

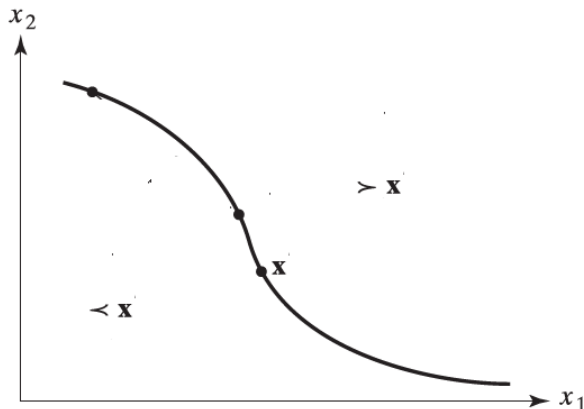
4. Monotonicity: if $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ then $\mathbf{x} \succ \mathbf{y}$.

This implies that at least as much of every good, and strictly more of some good(s), is strictly preferred. If free disposal of unwanted goods is allowed, then this assumption seems quite harmless.

5. Convexity: given $\mathbf{x} \neq \mathbf{y} \neq \mathbf{z}$ in X , if $\mathbf{x} \succeq \mathbf{z}$ and $\mathbf{y} \succeq \mathbf{z}$, then $t\mathbf{x} + (1 - t)\mathbf{y} \succ \mathbf{z}$ for all $0 < t < 1$.

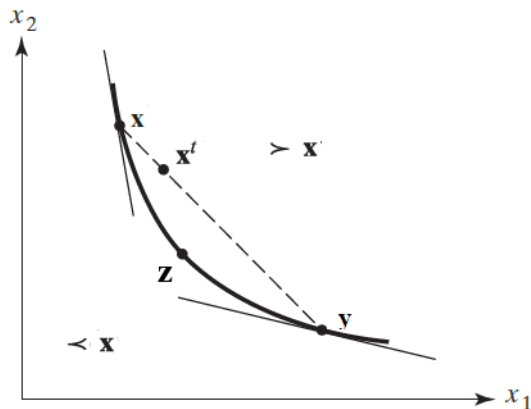
This implies that a consumer prefers a balanced consumption bundle instead of an extreme bundle: The weighted average (or mixture) of bundles $t\mathbf{x} + (1 - t)\mathbf{y}$ is preferred to the extreme bundle $\mathbf{z}$.

Consumer preferences in a graph



The graph above reflects a hypothetical set of preferences over bundles $\mathbf{x} = (x_1, x_2)$ that satisfy the four assumptions: Completeness, transitivity, continuity, and monotonicity. What about convexity?

Consumer preferences in a graph



The graph above reflects a hypothetical set of preferences over bundles $\mathbf{x} = (x_1, x_2)$ that satisfy the five assumptions: Completeness, transitivity, continuity, monotonicity, and convexity.

From preferences to utility function

Under the three assumptions completeness, transitivity and continuity, there exists a continuous **utility function** that represent those preferences.

Existence of a utility function: if preferences are complete, transitive, and continuous, there exists a utility function $u(\mathbf{x}) : R_+^k \rightarrow R$ that represents those preferences. That is, there exists a utility function $u(\mathbf{x})$ that satisfies $u(\mathbf{x}) \geq u(\mathbf{y}) \leftrightarrow \mathbf{x} \succeq \mathbf{y}$.

A utility function $u(\mathbf{x})$ is a convenient way to describe preferences. In particular, now we can aim to find the consumption bundle $\mathbf{x}$ that maximizes utility subject to economic constraints.

The additional two assumptions of monotonicity and convexity guarantee that the Second Order Condition (SOC) of such a constrained maximization problem are met.

What is the SOC? What is function maximization?

Short intro on function maximization

First-order condition (FOC)

- Consider the function $y = f(x)$.
- The necessary condition for a relative extremum (maximum or minimum) is that the first-order derivative be zero, i.e. $f'(x) = 0$.

Interpretation of the FOC

At the highest and lowest points of a curve, the tangent to the curve at such points is horizontal. The slope of the curve is zero.

Question

Why is $f'(x) = 0$ not a sufficient condition for a local maximum or minimum?

Answer: Because $f'(x) = 0$ at some inflexion points.

⇒ The first-order condition does not distinguish between a maximum and a minimum.

Short intro on function maximization (continues)

Second-order condition (SOC): If the first-order condition is satisfied at $x = x_0$,

- $f(x_0)$ is a local maximum if $f''(x_0) < 0$
- $f(x_0)$ is a local minimum if $f''(x_0) > 0$

Interpretation of the SOC - Maximum

As you move up a curve from the left, leading to a maximum, the curve gets increasingly flatter, i.e. the slope gets smaller and smaller. This means that $f'' < 0$. For example, if f' goes from 6 to 2, it means that $f'' < 0$.

Interpretation of the SOC - Minimum

As you move down a curve from the left, leading to a minimum, the curve gets increasingly flatter. However, since the slope is negative, a flattening of the curve implies that $f'' > 0$. For example, if f' goes from -3 to -2, it means that $f'' > 0$.

Exercise 3.a

$$f(x) = a - 2bx - x^2 \quad (1)$$

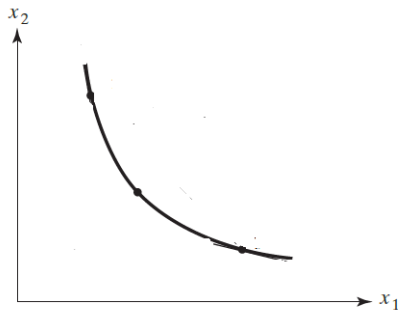
- $f'(x) = -2b - 2x = 0 \iff x = -b$
- Check the second order condition: $f''(x) = -2 < 0 \Rightarrow$ It is a maximum
- The local maximum is at $x = -b$ and $f(-b) = a + b^2$

Indifference curve

A utility function $u(\mathbf{x})$ is often represented by an indifference curve. This is simply a **level set** for the utility function: for a function $y = f(\mathbf{x})$, the level set $L(\bar{y}) = \{\mathbf{x} : f(\mathbf{x}) = \bar{y}\}$ collects all points where the function takes the fixed value $\bar{y}$.

Indifference curve: all consumption bundles that give utility level u ,

$$I(u) = \{\mathbf{x} : u(\mathbf{x}) = u\}.$$



The ordinal character of utility functions

The only relevant feature of a utility function is its **ordinal character**: if some function $u(\mathbf{x})$ represents a consumer's preferences, then so will the function $u'(\mathbf{x}) = u(\mathbf{x}) + 5$. That is, if $u(\mathbf{x})$ satisfies $u(\mathbf{x}) \geq u(\mathbf{y}) \leftrightarrow \mathbf{x} \succeq \mathbf{y}$ then $u'(\mathbf{x})$ also satisfies $u'(\mathbf{x}) \geq u'(\mathbf{y}) \leftrightarrow \mathbf{x} \succeq \mathbf{y}$.

No interpretation should be given to the actual numbers that are given by $u(\mathbf{x})$, only to the ordering of those numbers.

The above implies that the **utility function is invariant to positive monotonic transformations**: let $u(\mathbf{x})$ be a utility function that represents a consumer's preferences. Then $g(u(\mathbf{x}))$ also represents that consumer's preferences if $g : R \rightarrow R$ is a positive monotonic transformation: $u(\mathbf{x}) > u(\mathbf{y})$ implies $g(u(\mathbf{x})) > g(u(\mathbf{y}))$.

Marginal utility

Consider a setting with two goods, so that $u(\mathbf{x}) = u(x_1, x_2)$.

Marginal utility of good 1 or 2: how much does utility change if we change the consumption of good 1 or 2.

$$MU_i = \frac{\partial u(\mathbf{x})}{\partial x_i}, \quad \text{for } i = 1, 2.$$

Marginal rate of substitution

Marginal rate of substitution: The trade-off that a person is willing to make between two goods.

Let $x_2(x_1)$ be the indifference curve at utility level $u = \bar{u}$, then:

$$MRS = \frac{\partial x_2(x_1)}{\partial x_1}.$$

By definition, $x_2(x_1)$ traces all bundles such that $u(x_1, x_2) = \bar{u}$. Hence, $x_2(x_1)$ satisfies the identity $u(x_1, x_2(x_1)) = \bar{u}$.

Taking the total derivative towards x_1 , we obtain:

$$\frac{\partial u(\mathbf{x})}{\partial x_1} + \frac{\partial u(\mathbf{x})}{\partial x_2} \frac{\partial x_2(x_1)}{\partial x_1} = 0.$$

By rearranging, we can get an expression for the MRS without having to find $x_2(x_1)$:

$$MRS = \frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\frac{\partial u(\mathbf{x})}{\partial x_1}}{\frac{\partial u(\mathbf{x})}{\partial x_2}} = -\frac{MU_1}{MU_2}.$$

Partial and total derivative

Notice that the variable x_1 enters $f(x_1, x_2(x_1))$ twice. So there are two types of derivatives:

- **Partial derivative:** How does $f(x_1, x_2(x_1))$ change when x_1 changes while keeping x_2 fixed.

$$\frac{\partial f(\mathbf{x})}{\partial x_1}$$

- **Total derivative:** How does $f(x_1, x_2(x_1))$ change when x_1 changes while also allowing x_2 to change.

$$\frac{df(\mathbf{x})}{dx_1} = \frac{\partial f(\mathbf{x})}{\partial x_1} + \frac{\partial f(\mathbf{x})}{\partial x_2} \frac{\partial x_2(x_1)}{\partial x_1}$$

Two observations:

- 1 If $x_2(x_1)$ is constant, so that $x_2(x_1) = x_2$, then the partial derivative is equal to the total derivative.
- 2 If $x_2(x_1)$ is not constant, but you substituted for $x_2(x_1)$ in $f(\cdot)$, so that $f(x_1, x_2(x_1)) = f(x_1)$, taking the partial derivative gives you the total derivative.

Marginal rate of substitution

We can write that:

$$MRS = -\frac{MU_1}{MU_2}.$$

Imagine that $MU_1 = 2$ and $MU_2 = 1$. Then $MRS = -\frac{MU_1}{MU_2} = -\frac{2}{1} = -2$. Note that we can also reason this intuitively from $MRS = \frac{\partial x_2(x_1)}{\partial x_1}$:

- If consumer increases x_1 by 1, then utility increases by 2: $MU_1 = \frac{\Delta u}{\Delta x_1} = 2 \rightarrow \Delta u = 2 \times \Delta x_1 \rightarrow$ and $\Delta x_1 = 1$, so $\Delta u = 2$.
- Consumer needs to decrease x_2 by 2 as to keep utility constant:
 $MU_2 = \frac{\Delta u}{\Delta x_2} = 1 \rightarrow \Delta u = 1 \times \Delta x_2 \rightarrow$ and if $\Delta x_2 = -2$, then $\Delta u = -2$.
- Hence, $MRS = \frac{\Delta x_2}{\Delta x_1} = \frac{-2}{1} = -2$.

Exercise

1. Suppose $MU_1 = 9$ and $MU_2 = 3$ at utility level $u = \bar{u}$. The consumer increases x_1 by 2. How much does the consumer need to decrease x_2 as to keep utility at level $\bar{u}$?
2. Confirm that in the example above we have:

$$-\frac{MU_1}{MU_2} = \frac{\Delta x_2}{\Delta x_1}.$$

Exercise

Assume completeness, transitivity, continuity, monotonicity, and convexity.

- ① Show that indifference curves must be downwards sloping.
- ② Show that two indifference curves cannot cross.
- ③ Show that indifference curves become less steep as we move downward and to the right along them.

Consumer behavior

We will generally assume that consumers aim to maximize utility. More specifically, we will assume that the consumer will want to **choose the bundle x** from the **set of affordable alternatives** to **maximize utility**.

Budget constraint

Budget constraint: Let m be the budget of a consumer, and let $\mathbf{p}$ be the vector of fixed prices for the k goods. Then we can define the budget constraint, or the set of affordable alternatives, as:

$$B = \{\mathbf{x} : \mathbf{p}\mathbf{x} = m\}.$$

Consider a setting with two goods, then we can write B as:

$$B = \{(x_1, x_2) : p_1 x_1 + p_2 x_2 = m\}.$$

Note that for any fixed m , we can think of B as a level set, and of the budget constraint as an **isobudget** line. The constraint gives us all bundles (x_1, x_2) that cost m :

$$B = \{(x_1, x_2) : x_2 = \left(\frac{m}{p_2}\right) - \left(\frac{p_1}{p_2}\right)x_1\}.$$

The intercept of the budget constraint $\left(\frac{m}{p_2}\right)$ gives the budget in terms of the price of x_2 .

The slope of the budget line $\left(\frac{\partial x_2(x_1)}{\partial x_1} = -\frac{p_1}{p_2}\right)$ gives the **economic rate of substitution**: when x_1 increases, how much does x_2 need to decrease as to keep spending budget m .

Utility maximization

Now that we have introduced the utility function and the budget constraint, we can write the problem of utility maximization as:

$$\begin{aligned} \max_{\mathbf{x}} u(\mathbf{x}), \\ \text{such that } \mathbf{p}\mathbf{x} = m. \end{aligned}$$

Since the only relevant feature about the utility function is its ordinal character, the problem of utility maximization should be interpreted as the problem of finding the most preferred bundle.

Constrained optimization

The utility maximization problem is an example of **constrained optimization**: we want to maximize (or minimize) a function subject to a constraint. Written generally,

$$\begin{aligned} \max_{\mathbf{x}} f(\mathbf{x}), \\ \text{such that } g(\mathbf{x}) = c. \end{aligned}$$

Before applying this to the consumer's problem, let's recall how to solve it: **the method of Lagrange**.

The method of Lagrange

First, write down the Lagrangian,

$$\mathcal{L} = f(\mathbf{x}) - \lambda(g(\mathbf{x}) - c).$$

Second, differentiate $\mathcal{L}$ wrt each endogenous variable: x_i for $i = 1, \dots, n$ and λ . The FOCs for an interior solution $\mathbf{x}^*$ set these derivatives to zero,

$$\frac{\partial f(\mathbf{x})}{\partial x_1} - \lambda \frac{\partial g(\mathbf{x})}{\partial x_1} = 0,$$

$$\frac{\partial f(\mathbf{x})}{\partial x_2} - \lambda \frac{\partial g(\mathbf{x})}{\partial x_2} = 0,$$

$$\vdots$$

$$\frac{\partial f(\mathbf{x})}{\partial x_n} - \lambda \frac{\partial g(\mathbf{x})}{\partial x_n} = 0,$$

$$g(\mathbf{x}) - c = 0.$$

Third, since we have $n + 1$ unknown endogenous variables (x_i for $i = 1, \dots, n$ and λ) and $n + 1$ FOCs, we can solve for the endogenous variables in terms of the exogenous variables of the problem.

The method of Lagrange with two goods

First, write down the Lagrangian,

$$\mathcal{L} = u(\mathbf{x}) - \lambda(p_1x_1 + p_2x_2 - m).$$

Second, differentiate $\mathcal{L}$ wrt each endogenous variable: x_1 , x_2 and λ . The FOCs for an interior solution $\mathbf{x}^*$ set these derivatives to zero,

$$\frac{\partial u(\mathbf{x})}{\partial x_1} - \lambda p_1 = 0,$$

$$\frac{\partial u(\mathbf{x})}{\partial x_2} - \lambda p_2 = 0,$$

$$p_1x_1 + p_2x_2 - m = 0.$$

Third, since we have 3 unknown endogenous variables (x_1 , x_2 and λ) and 3 FOCs, we can solve for the endogenous variables in terms of the exogenous variables (p_1 , p_2 and m).

The method of Lagrange with two goods

Dividing the first two FOCs by each other gives us the following optimality condition:

$$\underbrace{\frac{p_1}{p_2}}_{?} = \underbrace{\frac{\partial u(\mathbf{x})/\partial x_1}{\partial u(\mathbf{x})/\partial x_2}}_{?}$$

We will combine this optimality condition with a graphical analysis of the Lagrange method to introduce the SOC and develop an economic intuition for the method more generally (again).

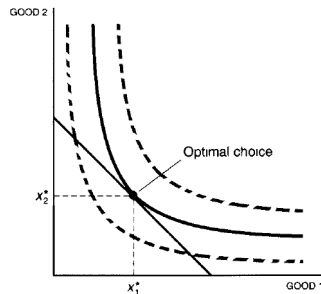
The method of Lagrange with two goods

Dividing the first two FOCs by each other gives us the following optimality condition:

$$\underbrace{\frac{p_1}{p_2}}_{\text{Economic rate of substitution}} = \underbrace{\frac{\partial u(\mathbf{x})/\partial x_1}{\partial u(\mathbf{x})/\partial x_2}}_{\text{Marginal rate of substitution (MRS)}}$$

We will combine this optimality condition with a graphical analysis of the Lagrange method to introduce the SOC and develop an economic intuition for the method more generally.

The method of Lagrange graphically



The utility-maximizing consumer wants to find a **point on the budget constraint with maximal utility**: this is a point where the indifference curve is furthest to the northeast. This point $\mathbf{x}^*$ is characterized by the slopes of the two lines being equal, which is the optimality condition we have seen before:

$$\frac{p_1}{p_2} = \frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2}$$

A refresher: concave and convex functions, and convex sets

Before stating the second-order condition, let's recall these definitions.

Let $\mathbf{x}^1$ and $\mathbf{x}^2$ be two points in the domain of a function f . Define the following convex combinations:

$$\mathbf{x}^t = t\mathbf{x}^1 + (1 - t)\mathbf{x}^2$$

$$y^t = tf(\mathbf{x}^1) + (1 - t)f(\mathbf{x}^2)$$

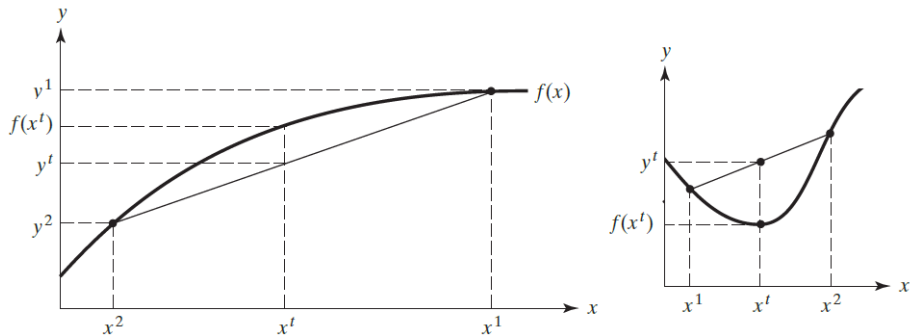
Concave function: f is a concave function if for all $\mathbf{x}^1$ and $\mathbf{x}^2$ we have that for $0 \leq t \leq 1$

$$f(\mathbf{x}^t) \geq y^t$$

Convex function: f is a convex function if for all $\mathbf{x}^1$ and $\mathbf{x}^2$ we have that for $0 \leq t \leq 1$

$$f(\mathbf{x}^t) \leq y^t$$

A refresher: concave and convex functions, and convex sets



Relationship between concave and convex functions and convex sets:

- f is concave (left) $\leftrightarrow$ points on and below the graph form a convex set
- f is convex (right) $\leftrightarrow$ points on and above the graph form a convex set

A refresher: upper and lower contour sets

Upper contour set for function f : let $f(\bar{\mathbf{x}}) = \bar{y}$ for some $\mathbf{x} = \bar{\mathbf{x}}$, then

$$\{\mathbf{x}: f(\mathbf{x}) \geq \bar{y}\}$$

Lower contour set for function f : let $f(\bar{\mathbf{x}}) = \bar{y}$ for some $\mathbf{x} = \bar{\mathbf{x}}$, then

$$\{\mathbf{x}: f(\mathbf{x}) \leq \bar{y}\}$$

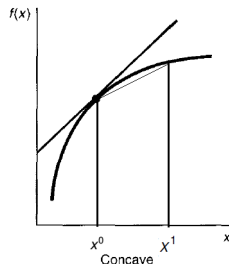
Useful relationships between concave and convex functions and convex sets:

- f is concave $\leftrightarrow$ upper contour set is a convex set
- f is convex $\leftrightarrow$ lower contour set is a convex set

Recall that for a utility function $u(\mathbf{x})$, the upper contour set at $\bar{\mathbf{x}}$ is exactly the set of bundles weakly preferred to $\bar{\mathbf{x}}$. A **concave utility function** therefore has a **convex upper contour set** – consistent with the convexity of preferences assumed above.

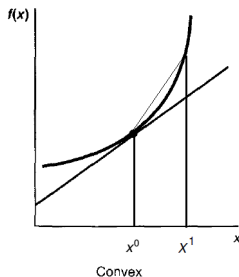
Concavity: Equivalent Definitions

- f is concave $\leftrightarrow f(x^t) \geq y^t$ for $0 \leq t \leq 1$.
- f is concave $\leftrightarrow$ points on and below the graph form a convex set.
- f is concave $\leftrightarrow$ upper contour set is a convex set.
- f is concave $\leftrightarrow f''(x) \leq 0$.
- f is concave $\leftrightarrow f(x) \leq f(x^0) + f'(x^0)(x - x^0)$.
- If $f(x)$ is concave, and $f'(x^*) = 0$, then x^* maximizes the function.



Convexity: Equivalent Definitions

- f is convex $\leftrightarrow f(x^t) \leq y^t$ for $0 \leq t \leq 1$.
- f is convex $\leftrightarrow$ points on and above the graph form a convex set.
- f is convex $\leftrightarrow$ lower contour set is a convex set.
- f is convex $\leftrightarrow f''(x) \geq 0$.
- f is convex $\leftrightarrow f(x) \geq f(x^0) + f'(x^0)(x - x^0)$.
- If f is convex, and $f'(x^*) = 0$, then x^* minimizes the function.



Second-order condition with $b > 1^*$

With $n > 1$ goods, so that we have $(y, \mathbf{x})$, we can define the **hessian** $\mathbf{D}^2 f(\mathbf{x})$: a matrix of second partial derivatives of f with respect to each of its inputs,

$$\mathbf{D}^2 f(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2}, \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2}, \dots, \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_1}, \frac{\partial^2 f(\mathbf{x})}{\partial x_2^2}, \dots, \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_n} \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1}, \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_2}, \dots, \frac{\partial^2 f(\mathbf{x})}{\partial x_n^2} \end{pmatrix}.$$

In this case, the SOC for profit maximization is that the hessian of the utility function must be **negative semidefinite** at the optimal point:

$$\mathbf{h}^T \mathbf{D}^2 f(\mathbf{x}) \mathbf{h} \leq 0 \quad \text{for all vectors } \mathbf{h} = (h_1, \dots, h_n) \neq 0.$$

Second Order Conditions for Two-Variable Functions

In the 1-variable case, one can check for the second order derivatives. With two variables, this is not enough. For finding a max, one needs to check:

- ① $f_{xx} < 0$
- ② $f_{yy} < 0$
- ③ $f_{xx}f_{yy} - f_{xy}^2 > 0$ (determinant of the matrix > 0)

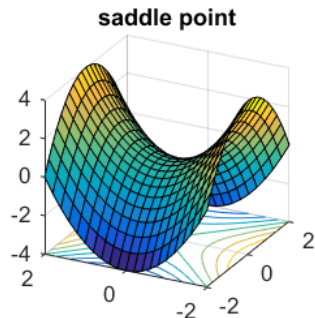
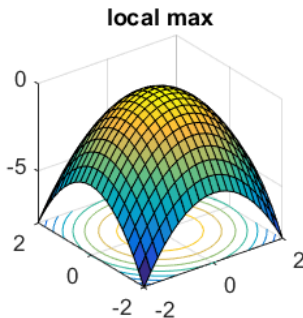
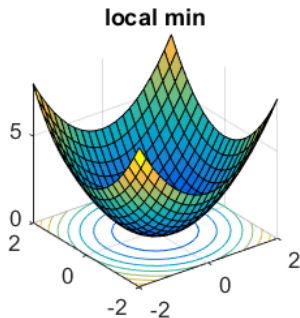
Question: Why do you need the third condition?

Answer: if you only check the sign second-order derivatives f_{xx} and f_{yy} , you cannot be sure whether you have found a max or a saddle point.

Second Order Conditions for Two-Variable Functions (continues)

- $f_{xx}f_{yy} > f_{xy}^2$: minimum (if $f_{xx} > 0$ and $f_{yy} > 0$)
- $f_{xx}f_{yy} < f_{xy}^2$: saddle point
- $f_{xx}f_{yy} = f_{xy}^2$: the test is inconclusive (beyond the scope of this course)

Visually



At home (optional): Check the three conditions for the following equation $f(x, y) = x^2 + y^2 + 2x$. You can also plot it on WolframAlpha to see it visually and to better understand why the second order conditions alone are not enough.

Exercise

Consider the following function:

$$f(x_1, x_2) = x_1(20 - 5x_1) + x_2(4 - 2x_2) - 2x_1x_2 - 4$$

Maximize $f(x_1, x_2)$ with respect to x_1 and x_2 .

1. Compute the first-order conditions (FOCs).
2. Solve FOCs to find the candidate optimum (x_1^*, x_2^*) .
3. Check second-order conditions.

Second-order condition

The SOC for utility-maximization is that:

$$V(u) \text{ is convex} \leftrightarrow u(\mathbf{x}) \text{ is concave} \leftrightarrow \mathbf{h}^T \mathbf{D}^2 u(\mathbf{x}) \mathbf{h} \leq 0.$$

With this condition we can be certain that the indifference curve is always weakly above the budget line. This is what we need for $\mathbf{x}^*$ to be utility-maximizing.

This guarantees that any change in the bundle $\mathbf{x}$ that keeps spending constant (that is, a change along the budget line) must result in weakly lower utility (that is, an indifference curve related to a weakly lower u).

Graphically:

The method of Lagrange: economic intuition

Recall that the optimality condition is:

$$\frac{p_1}{p_2} = \frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2}.$$

The LHS is the **economic rate of substitution** ($\frac{\partial x_2(x_1)}{\partial x_1} = -\frac{p_1}{p_2}$): when x_1 increases, how much does x_2 need to decrease as to keep **spending the same**.

The RHS is the **marginal rate of substitution** ($\frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2}$): when x_1 increases, how much does x_2 need to decrease as to keep **utility constant**.

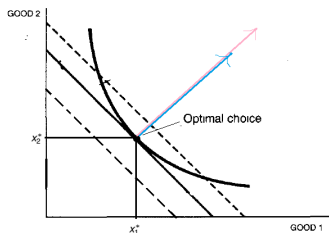
The method of Lagrange: economic intuition (continues)

Hence, the optimality condition tells us that at $\mathbf{x}^*$ the economic and marginal rate of substitution need to be equal. Imagine they are not:

$$\frac{p_1}{p_2} = \frac{2}{1} \neq \frac{1}{1} = \frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2},$$

then we can consume one unit less of x_1 and one unit more of x_2 , so that utility stays constant, but we still have an additional euro to spend. This cannot be $\mathbf{x}^*$.

The method of Lagrange: What about λ ?



Then why does the method require λ ? **The gradient of a function is perpendicular to the level set of that function.** Hence, $\mathbf{D}u(\mathbf{x}^*)$ is perpendicular to the indifference curve and $\mathbf{p}$ is perpendicular to the isobudget line.

Since the slopes of the indifference curve and the isobudget line must be equal at $\mathbf{x}^*$, both gradients must “point in the same direction” at $\mathbf{x}^*$. However, they are not necessarily of “equal length” at $\mathbf{x}^*$. Hence, we multiply $\mathbf{D}u(\mathbf{x})$ by a scalar λ :

$$\mathbf{p} = \lambda \mathbf{D}u(\mathbf{x}).$$

These are simply the FOCs of the Lagrangian. It turns out that λ also has a useful economic interpretation. More on this later.

Marshallian demand and indirect utility

Marshallian demand function $\mathbf{x}(\mathbf{p}, m)$: a function that gives us the optimal choice of consumption goods as a function of prices $\mathbf{p}$ and budget m .

How to get this function? From the FOCs of the Lagrangian we can write $\mathbf{x}$ in terms of $(\mathbf{p}, m)$.

The Marshallian demand function tells us how demand changes when prices change, while income is kept constant. This demand function can in principle be estimated with data on consumption, prices, and income.

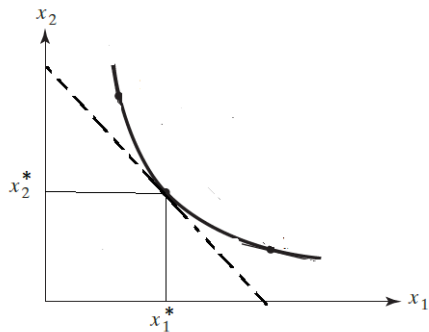
Indirect utility function $v(\mathbf{p}, m)$: a function that gives us the maximum utility achievable given prices $\mathbf{p}$ and budget m .

How to get this function? Substitute $\mathbf{x}(\mathbf{p}, m)$ into $u(\mathbf{x}) = u(\mathbf{x}(\mathbf{p}, m)) = v(\mathbf{p}, m)$

Exercise

The graph below indicates the optimal consumption bundle. Imagine p_1 increases.

1. Draw the new budget constraint.
2. Show the new Marshallian demand.



The Lagrange multiplier

The **Lagrange multiplier** λ measures how the optimal solution to the constrained optimization problem changes when the constraint is relaxed.

When we apply this interpretation of λ to the utility maximization problem, the optimal solution is the indirect utility function $v(\mathbf{p}, m)$ and the constraint is relaxed if we increase the budget m .

Hence, in this setting the Lagrange multiplier measures how utility changes, $\Delta v(\mathbf{p}, m)$, when we increase spending, Δm , so $\lambda = \frac{\Delta v(\mathbf{p}, m)}{\Delta m}$.

The envelope theorem (1/3)

To prove this interpretation of λ , we will use a general result called the **envelope theorem**.

Envelope theorem: if you want to know how an optimized function (e.g., $v(\mathbf{p}, m)$) changes when an exogenous variable changes (e.g., m), only the *direct* effect of this exogenous variable needs to be considered, even if the exogenous variable also enters the optimized function *indirectly* as part of the solution to the endogenous choice variables (e.g., $\mathbf{x}(\mathbf{p}, m)$).

The envelope theorem (2/3)

Consider an arbitrary maximization problem

$$\max_x f(a, x),$$

where x is the endogenous choice variable and a is an exogenous variable. The solution is a function $x(a)$: the optimal x as a function of a .

Substituting $x(a)$ into $f(a, x)$ gives the maximum value of f as a function of a : $f(a, x(a)) = f(a)$.

We are often interested in how this maximum value changes when a changes:

$$\frac{\partial f(a, x(a))}{\partial a}.$$

This is exactly the question we are asking about λ above: how does the maximized objective ($v(\mathbf{p}, m)$) change when the exogenous variable (m) changes?

The envelope theorem (3/3)

Partially differentiating the objective function, after substituting $x(a)$, is like taking a total derivative:

$$\frac{\partial f(a, x(a))}{\partial a} = \underbrace{\frac{\partial f(a, x)}{\partial a}}_{\text{direct effect}} + \underbrace{\frac{\partial f(a, x)}{\partial x} \frac{\partial x(a)}{\partial a}}_{\text{indirect effect}} = \frac{df(a, x)}{da}.$$

The FOC of maximizing $f(a, x)$ is $\frac{\partial f(a, x)}{\partial x} = 0$, so the indirect effect is zero at $x = x^* = x(a)$, and only the direct effect remains:

$$\frac{\partial f(a, x(a))}{\partial a} = \frac{\partial f(a, x)}{\partial a}.$$

Intuitively, the partial derivative of $f(a, x(a))$ with respect to a is given by the partial derivative of f with respect to a “holding x fixed at the optimal choice $x(a)$ ”. Indeed, since $x(a)$ is chosen optimally, f will not change when $x(a)$ changes slightly because of a change in a .

The Lagrange multiplier

The proof for this interpretation of λ follows from the envelope theorem.

Consider the Lagrangian with two goods,

$$\mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda) = u(\mathbf{x}) - \lambda(p_1 x_1 + p_2 x_2 - m).$$

First, note that:

$$\frac{\partial \mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda)}{\partial m} = \lambda.$$

Second, substitute the Marshallian demand functions $\mathbf{x}(\mathbf{p}, m)$ and the Lagrange multiplier $\lambda(\mathbf{p}, m)$ into the Lagrangian to obtain the Lagrangian evaluated at the optimal point:

$\mathcal{L}(\mathbf{p}, m, \mathbf{x}(\mathbf{p}, m), \lambda(\mathbf{p}, m)) = \mathcal{L}(\mathbf{p}, m)$. It turns out, this is equal to:

$$\begin{aligned}\mathcal{L}(\mathbf{p}, m) &= u(\mathbf{x}(\mathbf{p}, m)) - \lambda(\mathbf{p}, m)(p_1 x_1(\mathbf{p}, m) + p_2 x_2(\mathbf{p}, m) - m), \\ &= u(\mathbf{x}(\mathbf{p}, m)), \\ &= v(\mathbf{p}, m).\end{aligned}$$

The Lagrange multiplier

Third, use the logic of the envelope theorem to show that at the optimal point:

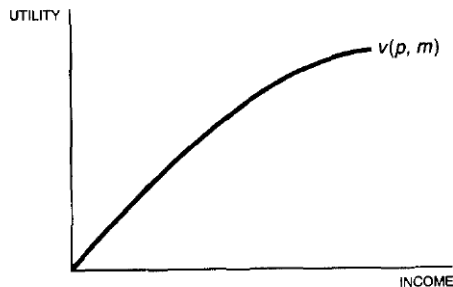
$$\begin{aligned}\frac{\partial \mathcal{L}(\mathbf{p}, m)}{\partial m} &= \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial m}}_{\text{direct effect}} + \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial x_1} \frac{\partial x_1(\cdot)}{\partial m} + \frac{\partial \mathcal{L}(\cdot)}{\partial x_2} \frac{\partial x_2(\cdot)}{\partial m} + \frac{\partial \mathcal{L}(\cdot)}{\partial \lambda} \frac{\partial \lambda(\cdot)}{\partial m}}_{\text{indirect effect}}, \\ &= \frac{\partial \mathcal{L}(\cdot)}{\partial m}, \\ &= \lambda(\mathbf{p}, m),\end{aligned}$$

as the indirect effects are zero because of the FOCs of the Lagrangian.

Since $\mathcal{L}(\mathbf{p}, m) = v(\mathbf{p}, m)$, we conclude that:

$$\frac{\partial \mathcal{L}(\mathbf{p}, m)}{\partial m} = \frac{\partial v(\mathbf{p}, m)}{\partial m} = \lambda(\mathbf{p}, m).$$

The indirect utility function



Indirect utility function $v(\mathbf{p}, m)$: maximum utility for each income m .

Since preferences satisfy monotonicity, indirect utility is increasing in m : If m increases, x increases, and so u increases. We can take the inverse of the indirect utility function $u = v(\mathbf{p}, m)$, written as $m = v^{-1}(\mathbf{p}, u) = e(\mathbf{p}, u)$.

Expenditure function $e(\mathbf{p}, u)$: minimum income required to achieve utility u .

Expenditure minimization

You can also get the expenditure function $e(\mathbf{p}, u)$ by solving the consumers' choice problem via expenditure minimization.

Expenditure minimization problem (EMP): find $\mathbf{x}$ that minimizes expenditure $\mathbf{p}\mathbf{x}$ subject to the utility constraint $u(\mathbf{x}) = u$.

$$\begin{aligned} \min_{\mathbf{x}} \mathbf{p}\mathbf{x}, \\ \text{such that } u(\mathbf{x}) = u. \end{aligned}$$

So far we discussed the **utility maximization problem (UMP)**: find $\mathbf{x}$ that maximizes utility $u(\mathbf{x})$ subject to the budget constraint $\mathbf{p}\mathbf{x} = m$.

$$\begin{aligned} \max_{\mathbf{x}} u(\mathbf{x}), \\ \text{such that } \mathbf{p}\mathbf{x} = m. \end{aligned}$$

The method of Lagrange for the EMP

The upcoming slides discuss the method of Lagrange for the **EMP**.

First, write down the Lagrangian,

$$\mathcal{L} = p_1 x_1 + p_2 x_2 - \lambda(u(\mathbf{x}) - u)$$

Second, differentiate $\mathcal{L}$ wrt each endogenous variable: x_1 , x_2 and λ . The FOCs for an interior solution $\mathbf{x}^*$ set these derivatives to zero,

$$p_1 - \lambda \frac{\partial u(\mathbf{x})}{\partial x_1} = 0,$$

$$p_2 - \lambda \frac{\partial u(\mathbf{x})}{\partial x_2} = 0,$$

$$u(\mathbf{x}) - u = 0.$$

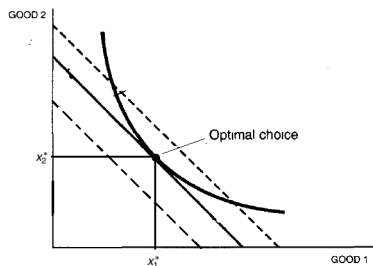
Third, since we have 3 unknown endogenous variables (x_1 , x_2 and λ) and 3 FOCs, we can solve for the endogenous variables in terms of the exogenous variables (p_1 , p_2 and u).

The method of Lagrange for the EMP

Dividing the first two FOCs by each other gives us the following optimality condition (again):

$$\frac{p_1}{p_2} = \frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2}.$$

The method of Lagrange for the EMP



The expenditure-minimizing consumer wants to find a **point on the indifference curve with minimal expenditures**: this is a point where the intercept of the budget line ($\frac{m}{p_2}$) is minimal. This point $\mathbf{x}^*$ is characterized by the slopes of the two lines being equal, which is the optimality condition we have seen before:

$$\frac{p_1}{p_2} = \frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2}$$

Second-order condition for the EMP

The SOC for expenditure-minimization is that:

$$V(u) \text{ is convex} \leftrightarrow u(\mathbf{x}) \text{ is concave} \leftrightarrow \mathbf{h}^T \mathbf{D}^2 u(\mathbf{x}) \mathbf{h} \leq 0.$$

With this condition we can be certain that the indifference curve is always weakly above the budget line. This is what we need for $\mathbf{x}^*$ to be expenditure-minimizing.

This guarantees that any change in the bundle $\mathbf{x}$ that keeps utility constant (that is, a change along the indifference curve) must result in weakly higher spending (that is, a budget line related to a weakly higher m).

Hicksian demand and expenditure function

Hicksian demand function $h(\mathbf{p}, u)$: a function that gives us the optimal choice of consumption goods as a function of prices $\mathbf{p}$ and utility u .

How to get this function? From the FOCs of the Lagrangian for the EMP we can write $\mathbf{x}$ in terms of $(\mathbf{p}, u)$.

The Hicksian demand function tells us how demand changes when prices change, while utility is kept constant. However, if one wants to keep utility constant while prices change, one needs to change/compensate income. This is a theoretical concept and often referred to as **compensated demand**. It is also difficult to estimate since we do not have data on utility.

Expenditure function $e(\mathbf{p}, u)$: a function that gives us the minimum income required to achieve utility u at prices $\mathbf{p}$.

How to get this function? Substitute $h(\mathbf{p}, u)$ into $\mathbf{p}\mathbf{x} = \mathbf{p}h(\mathbf{p}, u) = e(\mathbf{p}, u)$.

The Lagrange multiplier

Recall that the **Lagrange multiplier** λ measures how the optimal solution to the constrained optimization problem changes when the constraint is relaxed.

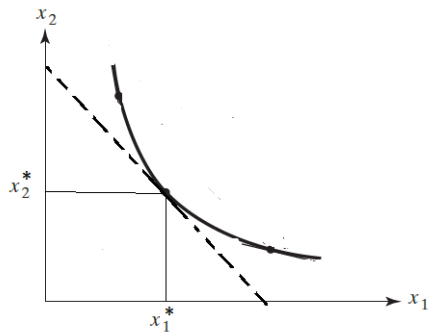
When we apply this interpretation of λ to the expenditure minimization problem, the optimal solution is the expenditure function $e(\mathbf{p}, m)$ and the constraint is relaxed if we increase utility u .

Hence, in this setting the Lagrange multiplier measures how expenditures change, $\Delta e(\mathbf{p}, m)$, when we increase utility, Δu , so $\lambda = \frac{\Delta e(\mathbf{p}, m)}{\Delta u}$.

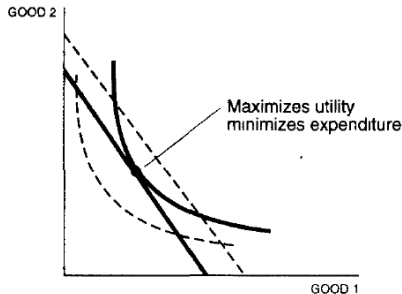
Exercise

The graph below indicates the optimal consumption bundle. Imagine p_1 increases.

1. Draw the new budget constraint.
2. Show the new Hicksian demand.



Duality: $EMP = UMP$

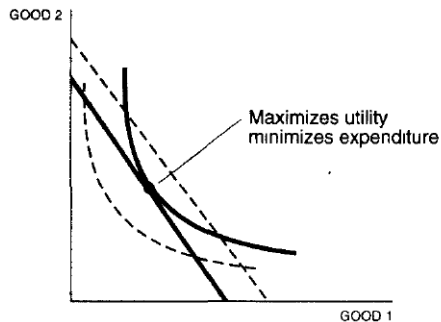


EMP: Given the indifference curve, slide the budget constraint to the south-west region until it just touches the indifference curve.

UMP: Given the budget constraint, slide the indifference curve to the north-east region until it just touches the budget constraint.

Under the assumptions we made, EMP and UMP will find the same optimal point $\mathbf{x}^*$. This is referred to as **duality**.

Two important identities



UMP: Let $\mathbf{x}(\mathbf{p}, m)$ and $v(\mathbf{p}, m)$ be the solutions.

EMP: Let $\mathbf{h}(\mathbf{p}, u)$ and $e(\mathbf{p}, u)$ be the solutions.

Duality ensures that (1) $\mathbf{x}(\mathbf{p}, m) = \mathbf{h}(\mathbf{p}, u)$ if $u = v(\mathbf{p}, m)$, and
(2) $\mathbf{h}(\mathbf{p}, u) = \mathbf{x}(\mathbf{p}, m)$ if $m = e(\mathbf{p}, u)$.

Two important identities

(1) $x_i(\mathbf{p}, m) = h_i(\mathbf{p}, v(\mathbf{p}, m))$: The Marshallian demand at income m is the same as the Hicksian demand at utility $v(\mathbf{p}, m)$.

(2) $h_i(\mathbf{p}, u) = x_i(\mathbf{p}, e(\mathbf{p}, u))$: The Hicksian demand at utility u is the same as the Marshallian demand at income $e(\mathbf{p}, u)$.

It is identity (2) that gives rise to the term compensated demand for the Hicksian demand: It is the Marshallian demand when income changes are arranged as to achieve some target level of utility.

Marshallian demand from indirect utility

If you were given the Marshallian demand $\mathbf{x}(\mathbf{p}, m)$, finding the indirect utility function is easy: just substitute the Marshallian demand into $u(\mathbf{x})$,

$$u(\mathbf{x}(\mathbf{p}, m)) = v(\mathbf{p}, m).$$

It turns out that if you know the indirect utility function, it also easy to find the Marshallian demand. This is what **Roy's identity** shows us.

Roy's identity

Roy's identity shows that we can find the Marshallian demand function from the indirect utility function as follows:

$$-\frac{\partial v(\mathbf{p}, m) / \partial p_i}{\partial v(\mathbf{p}, m) / \partial m} = x_i(\mathbf{p}, m).$$

In words, **Roy's identity** is that the Marshallian demand function can be found from the fraction that contains the derivative of the indirect utility function towards the price and income. This is similar to Hotelling's lemma (which you will see for the firm's profit function in Chapter 3) and Shephard's lemma below, but then applied to utility maximization.

Proof Roy's identity

Consider the Lagrangian with two goods,

$$\mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda) = u(\mathbf{x}) - \lambda(p_1 x_1 + p_2 x_2 - m).$$

First, note that:

$$\frac{\partial \mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda)}{\partial p_1} = -\lambda x_1,$$

$$\frac{\partial \mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda)}{\partial m} = \lambda.$$

Second, substitute the Marshallian demand functions $\mathbf{x}(\mathbf{p}, m)$ and the Lagrange multiplier $\lambda(\mathbf{p}, m)$ into the Lagrangian to obtain the Lagrangian evaluated at the optimal point:

$\mathcal{L}(\mathbf{p}, m, \mathbf{x}(\mathbf{p}, m), \lambda(\mathbf{p}, m)) = \mathcal{L}(\mathbf{p}, m)$. It turns out, this is equal to:

$$\begin{aligned}\mathcal{L}(\mathbf{p}, m) &= u(\mathbf{x}(\mathbf{p}, m)) - \lambda(\mathbf{p}, m)(p_1 x_1(\mathbf{p}, m) + p_2 x_2(\mathbf{p}, m) - m), \\ &= u(\mathbf{x}(\mathbf{p}, m)), \\ &= v(\mathbf{p}, m).\end{aligned}$$

Proof Roy's identity

Third, use the logic of the envelope theorem to show that at the optimal point:

$$\begin{aligned}\frac{\partial \mathcal{L}(\mathbf{p}, m)}{\partial p_1} &= \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial p_1}}_{\text{direct effect}} + \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial x_1} \frac{\partial x_1(\cdot)}{\partial p_1} + \frac{\partial \mathcal{L}(\cdot)}{\partial x_2} \frac{\partial x_2(\cdot)}{\partial p_1} + \frac{\partial \mathcal{L}(\cdot)}{\partial \lambda} \frac{\partial \lambda(\cdot)}{\partial p_1}}_{\text{indirect effect}} \\ &= \frac{\partial \mathcal{L}(\cdot)}{\partial p_1} = -\lambda(\mathbf{p}, m)x_1(\mathbf{p}, m), \\ \frac{\partial \mathcal{L}(\mathbf{p}, m)}{\partial m} &= \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial m}}_{\text{direct effect}} + \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial x_1} \frac{\partial x_1(\cdot)}{\partial m} + \frac{\partial \mathcal{L}(\cdot)}{\partial x_2} \frac{\partial x_2(\cdot)}{\partial m} + \frac{\partial \mathcal{L}(\cdot)}{\partial \lambda} \frac{\partial \lambda(\cdot)}{\partial m}}_{\text{indirect effect}} \\ &= \frac{\partial \mathcal{L}(\cdot)}{\partial m} = \lambda(\mathbf{p}, m),\end{aligned}$$

as the indirect effects are zero because of the FOCs of the Lagrangian.

Since $\mathcal{L}(\mathbf{p}, m) = v(\mathbf{p}, m)$, we conclude that:

$$-\frac{\partial \mathcal{L}(\mathbf{p}, m) / \partial p_i}{\partial \mathcal{L}(\mathbf{p}, m) / \partial m} = -\frac{\partial v(\mathbf{p}, m) / \partial p_i}{\partial v(\mathbf{p}, m) / \partial m} = x_i(\mathbf{p}, m).$$

Hicksian demand from the expenditure function

If you were given the Hicksian demand $\mathbf{h}(\mathbf{p}, u)$, finding the expenditure function is easy: just substitute the Hicksian demand into $\mathbf{p}\mathbf{x}$,

$$\mathbf{p}\mathbf{h}(\mathbf{p}, u) = e(\mathbf{p}, u).$$

It turns out that if you know the expenditure function, it is also easy to find the Hicksian demand. This is what **Shephard's lemma** shows us.

Shephard's lemma

Shephard's lemma shows that we can find the Hicksian demand function from the expenditure function as follows:

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_i} = h_i(\mathbf{p}, u).$$

In words, **Shephard's lemma** is that the derivative of the expenditure function towards a price gives us the Hicksian demand function. This is similar to Hotelling's lemma (Chapter 3) and Roy's identity above, but then applied to expenditure minimization. You will see the analogous result for the firm's cost function in Chapter 4.

Proof Shephard's lemma

Consider the Lagrangian for the EMP with two goods,

$$\mathcal{L}(\mathbf{p}, u, \mathbf{x}, \lambda) = p_1 x_1 + p_2 x_2 - \lambda(u(\mathbf{x}) - u).$$

First, note that:

$$\frac{\partial \mathcal{L}(\mathbf{p}, u, \mathbf{x}, \lambda)}{\partial p_1} = x_1.$$

Second, substitute the Hicksian demand functions $\mathbf{h}(\mathbf{p}, u)$ and the Lagrange multiplier $\lambda(\mathbf{p}, u)$ into the Lagrangian to obtain the Lagrangian evaluated at the optimal point:

$\mathcal{L}(\mathbf{p}, u, \mathbf{h}(\mathbf{p}, u), \lambda(\mathbf{p}, u)) = \mathcal{L}(\mathbf{p}, u)$. It turns out, this is equal to:

$$\begin{aligned}\mathcal{L}(\mathbf{p}, u) &= p_1 h_1(\mathbf{p}, u) + p_2 h_2(\mathbf{p}, u) - \lambda(\mathbf{p}, u)(u(\mathbf{h}(\mathbf{p}, u)) - u), \\ &= p_1 h_1(\mathbf{p}, u) + p_2 h_2(\mathbf{p}, u), \\ &= e(\mathbf{p}, u).\end{aligned}$$

Proof Shephard's lemma

Third, use the logic of the envelope theorem to show that at the optimal point:

$$\begin{aligned}\frac{\partial \mathcal{L}(\mathbf{p}, u)}{\partial p_1} &= \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial p_1}}_{\text{direct effect}} + \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial x_1} \frac{\partial h_1(\cdot)}{\partial p_1} + \frac{\partial \mathcal{L}(\cdot)}{\partial x_2} \frac{\partial h_2(\cdot)}{\partial p_1} + \frac{\partial \mathcal{L}(\cdot)}{\partial \lambda} \frac{\partial \lambda(\cdot)}{\partial p_1}}_{\text{indirect effect}}, \\ &= \frac{\partial \mathcal{L}(\cdot)}{\partial p_1} = h_1(\mathbf{p}, u),\end{aligned}$$

as the indirect effects are zero because of the FOCs of the Lagrangian.

Since $\mathcal{L}(\mathbf{p}, u) = e(\mathbf{p}, u)$, we conclude that:

$$\frac{\partial \mathcal{L}(\mathbf{p}, u)}{\partial p_1} = \frac{\partial e(\mathbf{p}, u)}{\partial p_1} = h_1(\mathbf{p}, u).$$

Exercise

Consider the following utility maximization problem:

$$\max_{x_1, x_2} x_1^\alpha x_2^{1-\alpha},$$

such that $p_1 x_1 + p_2 x_2 = m$.

1. Find the Marshallian demand functions.
2. Find the indirect utility function.
3. Find $\lambda(\mathbf{p}, m)$. Show that the derivative of the indirect utility function towards m is equal to $\lambda(\mathbf{p}, m)$. Use this to provide an economic interpretation of $\lambda(\mathbf{p}, m)$.
4. Show Roy's identity for $x_1(\mathbf{p}, m)$.

Exercise

Consider the following expenditure minimization problem:

$$\begin{aligned} \min_{x_1, x_2} & p_1 x_1 + p_2 x_2, \\ \text{such that } & x_1^\alpha x_2^{1-\alpha} = u. \end{aligned}$$

1. Find the Hicksian demand functions.
2. Find the expenditure function.
3. Plug the indirect utility function into the Hicksian demand function. What do you find? Plug the expenditure function into the Marshallian demand function. What do you find?

Homework exercises

Exercises: 7.4(a)-(b), 8.5, and exercises on the slides