Chapter 7 exercises and solutions

Slide 72: Utility maximization problem

Consider the following utility maximization problem:

$$\max_{x_1,x_2} x_1^{\alpha} x_2^{1-\alpha}, \qquad \text{such that } p_1 x_1 + p_2 x_2 = m.$$

1. Find the Marshallian demand functions.
2. Find the indirect utility function.
3. Find $\lambda(\mathbf{p}, m)$. Show that the derivative of the indirect utility function towards m is equal to $\lambda(\mathbf{p}, m)$. Use this to provide an economic interpretation of $\lambda(\mathbf{p}, m)$.
4. Show Roy's identity for $x_1(\mathbf{p}, m)$.

1. Marshallian demand

$$\max_{x_1,x_2} x_1^{\alpha} x_2^{1-\alpha} \text{ s.t. } p_1x_1 + p_2x_2 = m \quad \iff \quad \max_{x_1,x_2} \alpha \ln x_1 + (1-\alpha)\ln x_2 \text{ s.t. } p_1x_1 + p_2x_2 = m$$

(utility is invariant to positive monotonic transformations)

Lagrangian:

$$\mathcal{L} = \alpha \ln x_1 + (1-\alpha)\ln x_2 - \lambda\left(p_1x_1 + p_2x_2 - m\right)$$

FOCs:

$$\text{wrt } x_1: \quad \alpha\frac{1}{x_1} - \lambda p_1 = 0 \tag{1}$$

$$\text{wrt } x_2: \quad (1-\alpha)\frac{1}{x_2} - \lambda p_2 = 0 \tag{2}$$

$$\text{wrt } \lambda: \quad p_1x_1 + p_2x_2 = m \tag{3}$$

(1)/(2) is the optimality condition:

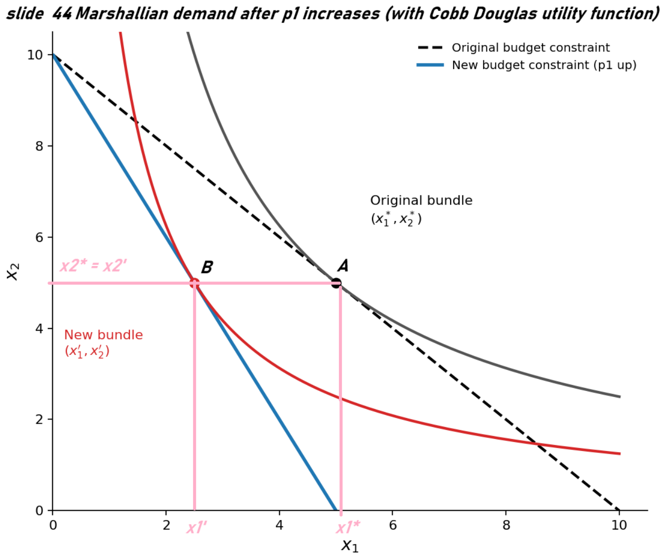
$$\frac{\alpha}{x_1}\frac{x_2}{(1-\alpha)} = \frac{\lambda p_1}{\lambda p_2} \quad \iff \quad \alpha p_2 x_2 = p_1 x_1 (1-\alpha)$$

Use (3), $p_1x_1 = m - p_2x_2$:

$$\begin{aligned}
\alpha p_2 x_2 &= (m - p_2x_2)(1-\alpha)\\
\alpha p_2 x_2 &= m(1-\alpha) - p_2x_2(1-\alpha)\\
\alpha p_2 x_2 + p_2x_2(1-\alpha) &= m(1-\alpha)\\
p_2x_2 &= m(1-\alpha)\\
\iff \quad x_2(\mathbf{p}, m) &= \frac{m(1-\alpha)}{p_2}
\end{aligned}$$

Now we get x_1:

$$p_1x_1 + p_2\frac{m(1-\alpha)}{p_2} = m \qquad p_1x_1 = \alpha m \quad \iff \quad x_1(\mathbf{p}, m) = \frac{\alpha m}{p_1}$$



Side note on slide 45, which was asking to draw the new Marshallian demand upon a price increase. Note that with the Cobb Douglas utility function we use in this exercise, x_2 does not depend on p_1, but only on m and p_2. Therefore, x_2 will not change upon a price increase for good 1. Graphically:

For completeness, we also check the SOCs. The SOC for utility maximization is that the utility function is concave (its Hessian is negative semidefinite). With two goods, we check: (i) $u_{11} < 0$, (ii) $u_{22} < 0$, (iii) $u_{11}u_{22} - u_{12}^2 > 0$.

For $u(x_1, x_2) = \alpha \ln x_1 + (1-\alpha)\ln x_2$:

$$u_{11} = -\frac{\alpha}{x_1^2} < 0, \quad u_{22} = -\frac{1-\alpha}{x_2^2} < 0, \quad u_{12} = u_{21} = 0.$$

Hence

$$H = \begin{pmatrix} -\dfrac{\alpha}{x_1^2} & 0 \\ 0 & -\dfrac{1-\alpha}{x_2^2} \end{pmatrix}, \qquad u_{11}u_{22} - u_{12}^2 = \frac{\alpha(1-\alpha)}{x_1^2 x_2^2} > 0 \quad \text{for } 0 < \alpha < 1,\ x_1, x_2 > 0.$$

All three conditions hold, so u is concave and the bundle $\big(x_1(\mathbf{p}, m), x_2(\mathbf{p}, m)\big)$ found from the FOCs is a maximum.

2. Indirect utility

Plug x^* into the (original) utility to get the indirect utility fct:

$$v(\mathbf{p}, m) = \left(\frac{\alpha m}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)m}{p_2}\right)^{1-\alpha} = m\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}$$

If we plug x^* into the modified (log) utility instead, we get the log *of the indirect utility, not v itself, let's call the transformed indirect utility $\tilde{v}(\mathbf{p}, m)$. We will use $\ln v$ in part 4.*

$$\begin{aligned}\tilde{v}(\mathbf{p}, m) = \ln v(\mathbf{p}, m) &= \alpha \ln\left(\frac{\alpha m}{p_1}\right) + (1-\alpha)\ln\left(\frac{(1-\alpha)m}{p_2}\right) \\ &= \alpha \ln \alpha + (1-\alpha)\ln(1-\alpha) - \alpha \ln p_1 - (1-\alpha)\ln p_2 + \ln m\end{aligned}$$

Taking the exponential gives back $v(\mathbf{p}, m)$ above.

3. Lagrange Multiplier

Note: We cannot use the monotonic transformation to get λ, which tells us utils per unit of m. Why not? A positive monotonic transformation does not change the choice *(the Marshallian demands), but it does change the* utility *numbers. The multiplier λ measures how much the maximized objective increases when the budget constraint is relaxed by one unit of m, so it is measured in utils per unit of m and depends on which utility function we use.*

To see this, call $\tilde{\lambda}$ the multiplier of the modified (log) problem. From FOC (1):

$$\alpha \frac{1}{x_1} = \tilde{\lambda} p_1 \quad \Longleftrightarrow \quad \tilde{\lambda} = \frac{\alpha}{p_1 x_1} = \frac{\alpha}{p_1} \cdot \frac{p_1}{\alpha m} = \frac{1}{m}$$

This is the marginal log-*utility of income, not the marginal utility of income.*

Note also that from 2. we would obtain $\frac{\partial \tilde{v}(\mathbf{p},m)}{\partial m} = \frac{1}{m} = \tilde{\lambda}$, so the envelope result also holds for the modified problem. The intuition is similar though not the same: $\tilde{\lambda} = \frac{1}{m}$ is the proportional *change in utility from one more unit of m (a 1% increase in income raises utility by 1%), while λ is the change in utils.*

Demand depends only on the ordinal ranking (it is invariant to monotonic transformations), while λ depends on the specific utility numbers, i.e. on the cardinal representation of preferences. Therefore, to find $\lambda(\mathbf{p}, m)$ we go back to the original problem.

FOC from original problem, for x_1 to obtain an expression for the λ:

$$\begin{aligned} & \alpha x_1^{\alpha-1} x_2^{1-\alpha} = \lambda p_1 \\ \Longleftrightarrow \quad & \frac{\alpha}{x_1} \underbrace{x_1^{\alpha} x_2^{1-\alpha}}_{\text{utility}} = \lambda p_1 \\ \Longleftrightarrow \quad & \lambda = \frac{\alpha u}{x_1 p_1} \end{aligned}$$

Using the optimal bundle $x(\mathbf{p}, m)$:

$$\lambda = \frac{\alpha u}{p_1} \cdot \frac{p_1}{\alpha m} = \frac{u}{m}$$

At the optimum: $u = v(\mathbf{p}, m)$, so

$$\lambda = \frac{v(\mathbf{p}, m)}{m}$$

Using $v(\mathbf{p}, m) = m \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}$ from 2 and taking the derivative with respect to m:

$$\frac{\partial v(\mathbf{p}, m)}{\partial m} = \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} \quad \Longrightarrow \quad \frac{\partial v(\mathbf{p}, m)}{\partial m} = \frac{v(\mathbf{p}, m)}{m} = \lambda(\mathbf{p}, m)$$

$\hookrightarrow$ We have shown that the derivative of the indirect utility fct toward m is equal to $\lambda(\mathbf{p}, m)$.

Economic interpretation: The $\lambda(\mathbf{p}, m)$ tells us how maximized utility changes when income m increases, i.e., when relaxing the budget constraint. In other words, $\lambda(\mathbf{p}, m)$ is the marginal utility of income, i.e. the extra utility from one more unit of m.

4. Roy's identity for $x_1(\mathbf{p}, m)$

Roy's identity:

$$-\frac{\frac{\partial v(\mathbf{p}, m)}{\partial p_1}}{\frac{\partial v(\mathbf{p}, m)}{\partial m}} = x_1(\mathbf{p}, m)$$

$\hookrightarrow$ *It works also with the modified problem since it's a ratio: by the chain rule,* $\frac{\partial \ln v}{\partial z} = \frac{1}{v}\frac{\partial v}{\partial z}$*, and the* $\frac{1}{v}$ *cancels:*

$$-\frac{\frac{\partial \tilde{v}(\mathbf{p}, m)}{\partial p_1}}{\frac{\partial \tilde{v}(\mathbf{p}, m)}{\partial m}} = -\frac{\frac{1}{v}\frac{\partial v(\mathbf{p}, m)}{\partial p_1}}{\frac{1}{v}\frac{\partial v(\mathbf{p}, m)}{\partial m}} = -\frac{\frac{\partial v(\mathbf{p}, m)}{\partial p_1}}{\frac{\partial v(\mathbf{p}, m)}{\partial m}}$$

Using $\tilde{v}(\mathbf{p}, m)$ from part 2:

$$\frac{\partial \tilde{v}(\mathbf{p}, m)}{\partial p_1} = -\frac{\alpha}{p_1}, \qquad \frac{\partial \tilde{v}(\mathbf{p}, m)}{\partial m} = \frac{1}{m}$$

Putting the two pieces together:

$$\Rightarrow \quad -\frac{-\frac{\alpha}{p_1}}{\frac{1}{m}} = \frac{\alpha m}{p_1} = x_1(\mathbf{p}, m)$$

This is exactly what we had to show.

Slide 73: Expenditure minimization problem

Consider the following expenditure minimization problem:

$$\min_{x_1, x_2} \; p_1 x_1 + p_2 x_2, \qquad \text{such that } x_1^{\alpha} x_2^{1-\alpha} = u.$$

1. Find the Hicksian demand functions.
2. Find the expenditure function.
3. Plug the indirect utility function into the Hicksian demand function. What do you find? Plug the expenditure function into the Marshallian demand function. What do you find?

Note: you can take a log-transformation of the utility function here as well to simplify math. To show different ways to solve the problems, we keep the original problem here.

1. Hicksian demand

$$\mathcal{L} = p_1 x_1 + p_2 x_2 - \lambda \left(x_1^{\alpha} x_2^{1-\alpha} - u \right)$$

FOCs:

$$\text{wrt } x_1: \quad p_1 - \lambda \alpha x_1^{\alpha-1} x_2^{1-\alpha} = 0 \tag{1}$$
$$\text{wrt } x_2: \quad p_2 - \lambda (1-\alpha) x_1^{\alpha} x_2^{-\alpha} = 0 \tag{2}$$
$$\text{wrt } \lambda: \quad x_1^{\alpha} x_2^{1-\alpha} = u \tag{3}$$

Optimality (1)/(2):

$$\frac{p_1}{p_2} = \frac{\lambda\alpha x_1^{\alpha-1}x_2^{1-\alpha}}{\lambda(1-\alpha)x_1^{\alpha}x_2^{-\alpha}} \iff \frac{p_1}{p_2} = \frac{\alpha}{1-\alpha}\cdot\frac{x_2}{x_1} \iff x_2 = \underbrace{\frac{(1-\alpha)p_1}{\alpha p_2}}_{c} x_1 \quad (4)$$

We call the expression c to simplify the math, so we don't have to carry that term around.

$\hookrightarrow$ Substitute x_2 into (3):

$$\begin{aligned} & x_1^{\alpha}(cx_1)^{1-\alpha} = u \\ \iff\ & x_1 = \frac{u}{c^{1-\alpha}} \\ \iff\ & h_1(\mathbf{p}, u) = u\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha} \end{aligned}$$

Substitute now h_1 into (4):

$$h_2(\mathbf{p}, u) = \underbrace{\frac{(1-\alpha)p_1}{\alpha p_2}}_{c}\underbrace{u\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha}}_{h_1} = u\left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha}$$

2. Expenditure function

$$\begin{aligned} e(\mathbf{p}, u) &= p_1 h_1(\mathbf{p}, u) + p_2 h_2(\mathbf{p}, u) \\ &= p_1\, u\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha} + p_2\, u\left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} \\ &= u\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} p_1^{\alpha}p_2^{1-\alpha} + u\left(\frac{1-\alpha}{\alpha}\right)^{\alpha} p_1^{\alpha}p_2^{1-\alpha} \\ &= u\left(\frac{1-\alpha}{\alpha}\right)^{\alpha} p_1^{\alpha}p_2^{1-\alpha}\left(\frac{\alpha}{1-\alpha}+1\right) \\ &= u\left(\frac{1-\alpha}{\alpha}\right)^{\alpha} p_1^{\alpha}p_2^{1-\alpha}\cdot\frac{1}{1-\alpha} \\ &= u\,\frac{(1-\alpha)^{\alpha-1}}{\alpha^{\alpha}}\, p_1^{\alpha}p_2^{1-\alpha} = u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha} \end{aligned}$$

3. Plug $v(\mathbf{p}, m)$ into the Hicksian

Recall that $h_1(\mathbf{p}, u) = u\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha}$ and $v(\mathbf{p}, m) = m\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}$

For good 1, we plug in $v(\mathbf{p}, m)$ which is u at the optimum:

$$h_1(\mathbf{p}, v(\mathbf{p}, m)) = m\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha} = m\cdot\frac{\alpha}{p_1} = x_1(\mathbf{p}, m)$$

For good 2, we do the same:

$$h_2(\mathbf{p}, v(\mathbf{p}, m)) = m\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}\left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} = \frac{m(1-\alpha)}{p_2} = x_2(\mathbf{p}, m)$$

$\hookrightarrow$ As we had learned, if we plug the indirect utility fct into the Hicksian demand, we get the Marshallian demand.

Now, if we plug the expenditure function $e(\mathbf{p}, u) = u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha}$ into the Marshallian demand $x_1 = \alpha m/p_1$ in place of m:

$$x_1 = \frac{\alpha}{p_1} \cdot u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha} = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha} u = h_1(\mathbf{p}, u)$$

$$x_2 = u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha}\frac{(1-\alpha)}{p_2} = u\left(\frac{p_1(1-\alpha)}{\alpha\, p_2}\right)^{\alpha} = h_2(\mathbf{p}, u)$$

$\hookrightarrow$ We find the Hicksian.

At the point where $m = e(\mathbf{p}, u)$, the consumer is solving the UMP and the EMP with the same optimal bundle.

7.4 (exercise from the Varian)

Consider the indirect utility function:

$$v(p_1, p_2, m) = \frac{m}{p_1 + p_2} = m\,(p_1 + p_2)^{-1}$$

a) What are the demand functions?

$$-\frac{\frac{\partial v(\mathbf{p}, m)}{\partial p_1}}{\frac{\partial v(\mathbf{p}, m)}{\partial m}} \;\Rightarrow\; x_1(\mathbf{p}, m) = -\frac{-m\,(p_1 + p_2)^{-2}}{(p_1 + p_2)^{-1}} = \frac{m}{p_1 + p_2}$$

$\Rightarrow$ x_2 will be the same, the problem is symmetric.

b) What is the expenditure function?

We know that:

$$v(\mathbf{p}, e(\mathbf{p}, u)) = u \tag{1}$$

We can also obtain $v(\mathbf{p}, e(\mathbf{p}, u))$ from substituting $e(\mathbf{p}, u)$ in the place of m into $v(\mathbf{p}, m)$

$$v(\mathbf{p}, e(\mathbf{p}, u)) = \frac{e(\mathbf{p}, u)}{p_1 + p_2} \tag{2}$$

$\hookrightarrow$ Intuition: $e(\mathbf{p}, u)$ is the expenditure level needed to achieve u. We require the consumer to reach utility u to get the minimization problem.

Combining (1) and (2), we get:

$$u = \frac{e(\mathbf{p}, u)}{p_1 + p_2} \quad \Longleftrightarrow \quad e(\mathbf{p}, u) = u(p_1 + p_2)$$

8.5 (exercise from the Varian)

Find the demanded bundle for a consumer whose utility fct is

$$u(x_1, x_2) = x_1^{3/2} x_2 \qquad \text{s.t. } 3x_1 + 4x_2 = 100$$

We can use the log-transformed utility function:

$$\tilde{u}(x_1, x_2) = \frac{3}{2} \ln x_1 + \ln x_2 \qquad \text{s.t. } 3x_1 + 4x_2 = 100$$

Lagrangian:

$$\mathcal{L} = \frac{3}{2} \ln x_1 + \ln x_2 - \lambda \left(3x_1 + 4x_2 - 100\right)$$

FOCs:

$$\text{wrt } x_1: \quad \frac{3}{2} \frac{1}{x_1} = 3\lambda \tag{1}$$

$$\text{wrt } x_2: \quad \frac{1}{x_2} = 4\lambda \tag{2}$$

$$\text{wrt } \lambda: \quad 3x_1 + 4x_2 - 100 = 0 \tag{3}$$

Optimality condition, dividing (1) by (2):

$$\frac{x_2}{2x_1} = \frac{1}{4} \qquad\qquad x_2 = \frac{1}{2} x_1$$

$$3x_1 + 4 \cdot \frac{1}{2} x_1 = 100$$

$$x_1 = 20 \qquad x_2 = 10$$

$$(x_1^*, x_2^*) = (20, 10)$$

For completeness, we also check the SOCs. The SOC requires the (transformed) utility function to be concave.

$$\tilde{u}_{11} = -\frac{3}{2x_1^2} < 0, \quad u_{22} = -\frac{1}{x_2^2} < 0, \quad \tilde{u}_{12} = \tilde{u}_{21} = 0,$$

$$\tilde{u}_{11}\tilde{u}_{22} - \tilde{u}_{12}^2 = \frac{3}{2x_1^2 x_2^2} > 0 \qquad \text{for all } x_1, x_2 > 0.$$

The three conditions are satisfied. Hence $\tilde{u}$ is strictly concave and, with the linear budget constraint, $(x_1^*, x_2^*) = (20, 10)$ is the utility-maximizing bundle.