Mathematics 2

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Lecture 4: Sequences in $\mathbb{R}^d$ (Introduction to Limits).

Sequences play a central role in Calculus, providing a natural and rigorous framework for defining and describing “limiting” processes.

In the scalar, i.e. in the 1–dimensional case, a sequence is simply an ordered (infinite) list of numbers.

Accordingly, a sequence n higher dimensions consists of vectors rather than real numbers.

We will now study:

- sequences in $\mathbb{R}^d$;
- coordinate sequences;
- convergence;
- uniqueness of limits;
- coordinate-wise convergence;
- subsequences.

Remark 1 (On the notation)

When referring to an arbitrary element $x \in \mathbb{R}^d$, we mean a n–tuple of real numbers $x_1, \ldots, x_n$.

For the sake of readability, we use round parentheses when listing the elements of x as in–line text; e.g. $x = (x_1, \ldots, x_n)$.

Otherwise, we use column-vector notation with square brackets; e.g.

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Definition 2

A ***sequence in*** $\mathbb{R}^d$ is a function on the positive integers with values in $\mathbb{R}^d$,

$$a : \{1, 2, 3, \ldots\} \longrightarrow \mathbb{R}^d.$$

We then write the ***value of*** a ***at*** n ***as*** $a(n)$:

$$n \longmapsto a(n) = \begin{bmatrix} a_1(n) \\ a_2(n) \\ \vdots \\ a_d(n) \end{bmatrix}.$$

The vector $a(n)$ is thus called the ***general term*** of the sequence.

When referring to the entire sequence, we sometimes write $\big(a(n)\big)_{n=1}^{\infty}$, $\big(a(n)\big)_{n\geq 1}$, or simply $\big(a(n)\big)$ when context is clear.

That is to say, a **is a rule that assigns an exact** d**–tuple of numbers to every positive whole number**:

- if $n = 1$, then the value assign is $a(1) = \big(a_1(1),\ a_2(1),\ \ldots,\ a_d(1)\big)$;
- if $n = 2$, then the value assign is $a(2) = \big(a_1(2),\ a_2(2),\ \ldots,\ a_d(2)\big)$;
- if $n = 345$, then the value assign is $a(345) = \big(a_1(345),\ a_2(345),\ \ldots,\ a_d(345)\big)$.
- if $n = 678\ 910\ 111\ 213$, then the value assign is $a(678\ 910\ 111\ 213) = \big(a_1(678\ 910\ 111\ 213),\ \ldots,\ a_d(678\ 910\ 111\ 213)\big)$.

Observe: A sequence in $\mathbb{R}^d$ can be viewed as d real-valued (i.e. 1–dimensional) sequences evolving simultaneously:

$$n \longmapsto a(n) = \begin{bmatrix} a_1(n) \\ a_2(n) \\ \vdots \\ a_d(n) \end{bmatrix} \qquad \Longleftrightarrow \qquad \begin{array}{c} n \longmapsto a_1(n), \\ n \longmapsto a_2(n), \\ \vdots \\ n \longmapsto a_d(n). \end{array}$$

Example 3

Let $a : \{1, 2, \ldots\} \to \mathbb{R}$ be the sequence defined as

$$n \longmapsto a(n) = 2.10.$$

In other words, the sequence that assigns the same number, 2.10, to every positive whole number.

Sequences of the type that assign the same value to every input are often referred to as **constant**; e.g. *a is the constant sequence* 2.10.

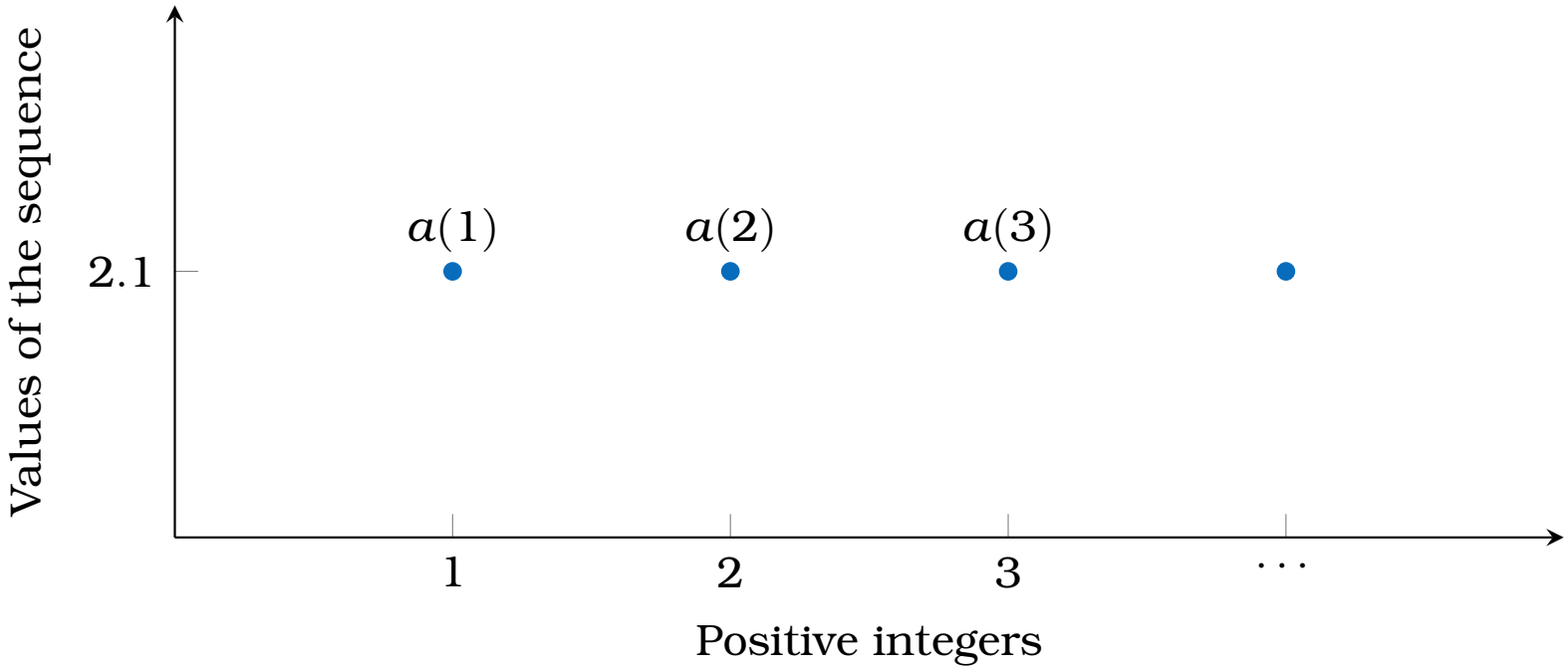


Figure: Plot of the constant sequence $\big(a(n) = 2.10\big)_{n\geq 1}$.

Example 4

Let $a : \{1, 2, \ldots\} \to \mathbb{R}^2$ be the sequence defined as

$$n \longmapsto a(n) = \begin{bmatrix} \sin\left(\frac{2\pi}{3} n\right) \\ \mathrm{e}^n \end{bmatrix}.$$

Then,

$$\begin{aligned} a(1) &= \begin{bmatrix} \sin\left(\frac{2\pi}{3}\right) \\ \mathrm{e} \end{bmatrix} \approx \begin{bmatrix} -0.5 \\ 2.72 \end{bmatrix}, \\ a(2) &= \begin{bmatrix} \sin\left(\frac{4\pi}{3}\right) \\ \mathrm{e}^2 \end{bmatrix} \approx \begin{bmatrix} 0.5 \\ 7.39 \end{bmatrix}, \\ a(3) &= \begin{bmatrix} \sin\left(2\pi\right) \\ \mathrm{e}^3 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 20.09 \end{bmatrix}, \\ &\vdots \end{aligned}$$

Equivalently, a is the sequence that is the sine of multiples of two-thirds of π radians (or $120°$) in the the first coordinate, and is the exponential function in the second:

$$n \longmapsto a_1(n) = \sin\left(\frac{2\pi}{3}n\right) \qquad \text{and} \qquad n \longmapsto a_2(n) = \mathrm{e}^n.$$

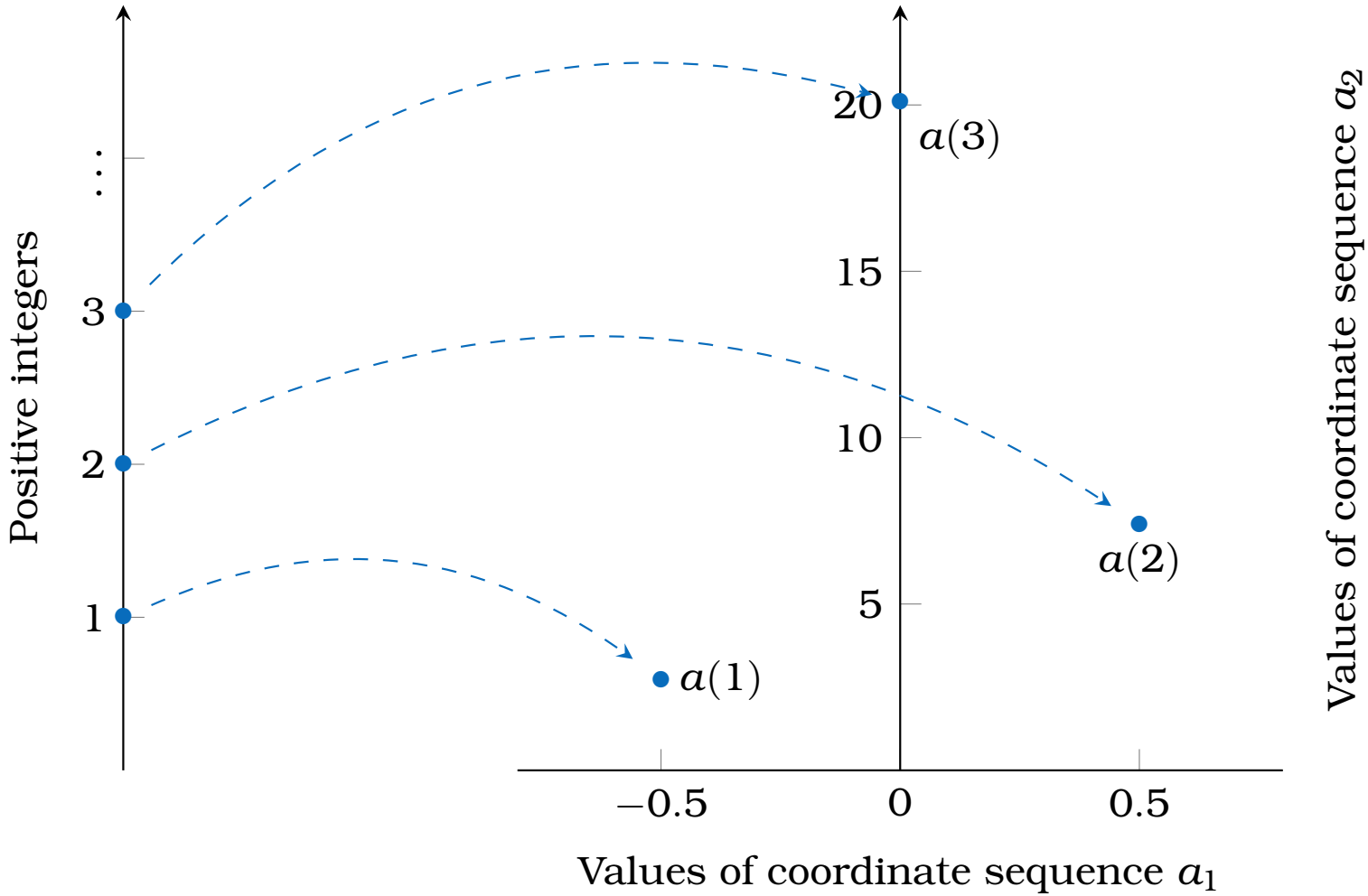


Figure: Plot of the sequence $\{a(n) = (\sin(\frac{2\pi n}{3}), e^n)\}$

Example 5

Let $a : \{1, 2, \ldots\} \to \mathbb{R}^4$ be the sequence defined as

$$n \longmapsto a(n) = \begin{bmatrix} n^{-1} \\ \sqrt{2n-1} \\ \ln \pi \\ (-1)^n \end{bmatrix}.$$

Then, a is a sequence that lists the reciprocals of n, the square-roots of the odd numbers, is constant, and alternates between -1 and 1.

From the previous examples we can see the behavior of a sequence can vary, not only as n grows, but also between its own coordinates.

The notion of **limit** seeks to capture the behavior in a sequence as the list grows.

In the multidimensional case, this requires us to have a way to measure the "nearness" between vectors. **This is why we introduced *distances* and *norms* before.**

Definition 6

Let $\big(a(n)\big)$ be a sequence in $\mathbb{R}^d$ and let $\ell \in \mathbb{R}^d$ a given point.

We say that $\big(a(n)\big)$ ***converges to*** ℓ ***as*** n ***grows***,

$$a(n) \underset{n\to\infty}{\longrightarrow} \ell,$$

or equivalently, that ℓ ***is the limit of the sequence*** $\big(a(n)\big)$,

$$\ell = \lim_{n\to\infty} a(n),$$

if, for every $\varepsilon > 0$, there exists a number $N \in \mathbb{N}$ (possibly dependent on ε) such that

$$\boxed{\forall n > N \qquad \Longrightarrow \qquad a(n) \in B_\varepsilon(\ell).} \tag{1}$$

Observe: Since

$$a(n) \in B_\varepsilon(\ell) \qquad \Longleftrightarrow \qquad d(a(n), \ell) < \varepsilon,$$

the convergence condition (1) can be expressed as

$$\boxed{n > N \qquad \Longrightarrow \qquad d(a(n), \ell) < \varepsilon.}$$

Equivalently, in terms of the Euclidean norm:

$$\boxed{n > N \qquad \Longrightarrow \qquad \|a(n) - \ell\| < \varepsilon.}$$

Definition 6 is interpreted as follows:

No matter how arbitrarily small we want to choose an *error term* for the sequence:

"for **every** $\varepsilon > \mathbf{0}$";

there *will always be a threshold* point (possibly very large but always a *finite* number):

"there **exists** a $\boldsymbol{N} \geq \mathbf{1}$";

such that for *all the terms coming after the threshold*

"for **all** $\boldsymbol{a}(\boldsymbol{n})$ with $\boldsymbol{n} > \boldsymbol{N}$";

their *distance to the limit is less than our prescribed error*:

"$\boldsymbol{d}(\boldsymbol{a}(\boldsymbol{n}), \ell) < \varepsilon$".

Proposition 7

If a sequence $(a(n))$ in $\mathbb{R}^d$ converges, then its limit is unique.

Proof.

Suppose that

$$a(n) \to \ell \qquad \text{and} \qquad a(n) \to \ell'.$$

From Definition 6, this means that for every $\varepsilon > 0$

$$d(a(n), \ell) < \frac{\varepsilon}{2} \qquad \text{and} \qquad d(a(n), \ell') < \frac{\varepsilon}{2}$$

for all n greater than a sufficiently large N.

Proof (cont.)

By the triangle inequality,

$$d(\ell, \ell') \leq d(\ell, a(n)) + d(a(n), \ell') < \varepsilon.$$

In other words, $0 \leq d(\ell, \ell') < \varepsilon$ for every $\varepsilon > 0$. Hence,

$$\ell = \ell'.$$

□

Proposition 8

Let $(a(n))$ be a sequence in $\mathbb{R}^d$, and let $\ell \in \mathbb{R}^d$. Then, $a(n) \longrightarrow \ell$ if and only if each of the coordinates converge. Equivalently:

$$\lim_{n\to\infty} \underbrace{\begin{bmatrix} a_1(n) \\ a_2(n) \\ \vdots \\ a_d(n) \end{bmatrix}}_{=a(n)} = \underbrace{\begin{bmatrix} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_d \end{bmatrix}}_{=\ell} \iff \begin{array}{l} \lim\limits_{n\to\infty} a_1(n) = \ell_1, \\ \lim\limits_{n\to\infty} a_2(n) = \ell_2, \\ \quad \vdots \\ \lim\limits_{n\to\infty} a_d(n) = \ell_d. \end{array}$$

That is to say, convergence is a coordinate-wise notion.

Proof.

Suggested exercise.[1] □

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[1] Hint: use the fact that for the euclidean distance in $\mathbb{R}^d$, the following inequality holds component-wise:

$$|x_i - \ell_i| \leq d(x, \ell) \leq \sum_{j=1}^{d} |x_j - \ell_j| \qquad \forall i = 1, \ldots, d.$$

Example 9

Consider

$$a(n) = \begin{bmatrix} \frac{1}{n} \\ \ln\left(\mathrm{e} + \frac{1}{n}\right) \end{bmatrix}.$$

Since

$$a_1(n) = \frac{1}{n} \to 0 \qquad \text{and} \qquad a_2(n) = \ln\left(\mathrm{e} + \frac{1}{n}\right) \to 1,$$

coordinate-wise convergence gives

$$\boxed{\lim_{n\to\infty} a(n) \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}.}$$

Example 10

Consider

$$a(n) = \begin{bmatrix} \frac{1}{n} \\ \ln\left(\mathrm{e} + \frac{1}{n}\right) \\ \frac{4n^5+5n^3}{8n^5+7n^2} \end{bmatrix}.$$

From the same arguments as above, it remains to to be seen that

$$\frac{4n^5 + 5n^3}{8n^5 + 7n^2} \to \frac{1}{2}.$$

Therefore,

$$\boxed{\lim_{n\to\infty} a(n) \to \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}.}$$

Example 11 (A non-convergent sequence)

Consider

$$a(n) = \begin{bmatrix} \left(\frac{n+3}{n}\right)^{2n} \\ \frac{(-1)^n n^2}{5n^2+1} \end{bmatrix}.$$

For the first coordinate,

$$a_1(n) = \left(\frac{n+3}{n}\right)^{2n} = \left[\left(1+\frac{3}{n}\right)^{n}\right]^2 \to \mathrm{e}^6.$$

Example 11 (cont.)

However, observe that

$$a_2(n) = \frac{(-1)^n n^2}{5n^2 + 1}$$

does not converge. Thus, from the coordinate-wise convergence criterion,

$$\boxed{\big(a(n)\big) \text{ is not convergent.}}$$

Definition 12

Let $a : \{1, 2, \ldots\} \longrightarrow \mathbb{R}^d$, and let $n : \{1, 2, \ldots\} \longrightarrow \{1, 2, \ldots\}$ be a given **strictly increasing sequence of indices**:

$$k \longmapsto n(k) \qquad \text{such that} \qquad n(1) < n(2) < n(3) < \cdots$$

A ***subsequence*** of a is then obtained by considering the composition of a with the increasing sequence of indices n:

$$k \longmapsto a(n(k)). \tag{2}$$

Observe: Since the sequence $(n(k))$ is strictly increasing, the indices of the subsequence (2) remain in their original order.

Example 13

Let

$$a(n) = \begin{bmatrix} n \\ n+2 \end{bmatrix}.$$

In other words, $a(n)$ lists the positive integers starting from 1 in the first coordinate, and starting from 3 in the second: $a(1) = (1, 3)$, $a(2) = (2, 4)$, $a(3) = (3, 5)$, $a(4) = (4, 6)$, ...

Example 13 (cont.)

Now, we consider the *increasing* sequence of even numbers:

$$k \longmapsto 2k, \qquad\qquad k = 1, 2, 3, \ldots$$

Thus, the resulting subsequence is given by

$$k \longmapsto a(2k) = \begin{bmatrix} 2k \\ 2k+2 \end{bmatrix} = \begin{bmatrix} 2k \\ 2(k+1) \end{bmatrix}.$$

E.g., $a(2) = (2, 4)$, $a(4) = (4, 6)$, $a(6) = (6, 8)$, $a(8) = (8, 10)$, $\ldots$

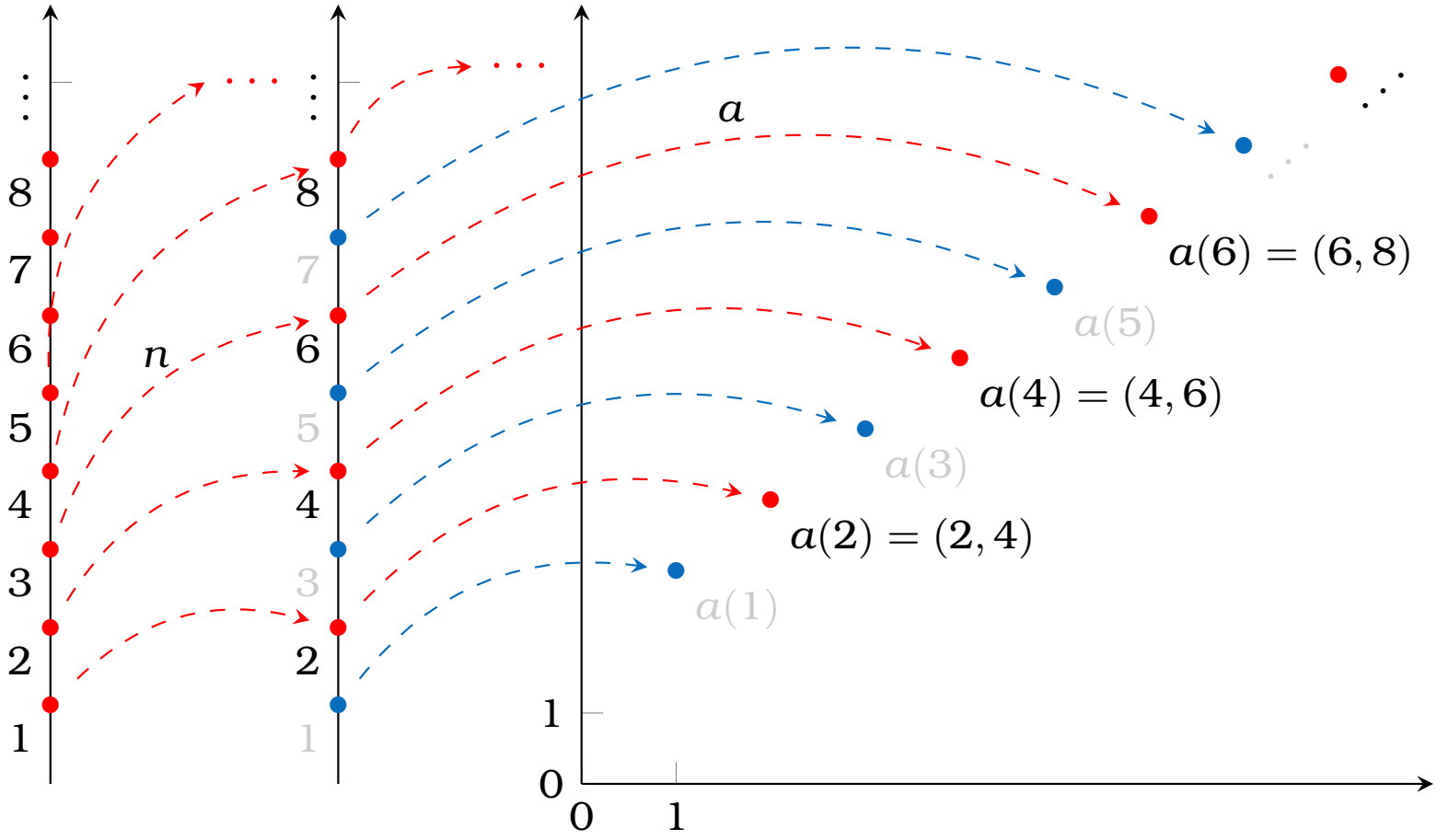


Figure: Plot of the subsequence $\{a(2k) = (2k, 2(k+1))\}$

Proposition 14

Every subsequence of a convergent sequence in $\mathbb{R}^d$ converges to the same limit.

Remark 15

It is often found in literature the notation a_n instead of $a(n)$ whenever the multivariate context is clear and no ambiguity from the coordinate sequence arises.[2]

Similarly, for the case of subsequences it is common to find the simplified notation a_{n_k} instead.

For illustrative purposes, we shall keep the function notation $a(n)$ for the rest of the notes.

[2] See [1], for example.

Summary

- A sequence in $\mathbb{R}^d$ is a sequence of vectors.
- Every vector sequence gives rise to d coordinate sequences.
- Convergence is defined through the Euclidean distance:

$$d(a(n), \ell) \to 0.$$

- The limit of a convergent sequence is unique.
- Convergence in $\mathbb{R}^d$ is equivalent to coordinatewise convergence.
- Every subsequence of a convergent sequence converges to the same limit.

References

[1] Robert Gardner Bartle and Donald R. Sherbert. *Introduction to real analysis*. 4th ed. Hoboken, NJ: Wiley, 2011.